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CONTENTS OF VOLUME XXVI

Memoirs

	PAGE
I. On the P_{λ_n} Test for Randomness: Remarks, Further Illustration, and Table for P_{λ_n} . By FLORENCE N. DAVID	1
II. On the Corrections for the Moment Coefficients of Frequency Distributions when the Start of the Frequency is one of the Characteristics to be determined. By E. S. MARTIN	12
III. On Asymptotic Formulae for the Hypergeometric Series. Part II. By O. L. DAVIES	59
IV. Die Statistik der Seltenen Ereignisse. Von ROLF LÜDERS	108
V. On Certain Non-normal Symmetrical Frequency Distributions. By G. H. HANSMANN	129
VI. A Biometric Study of the "Flatness" of the Facial Skeleton in Man. By T. L. WOO and G. M. MORANT	196
VII. The Wilkinson Head of Oliver Cromwell and its Relationship to Busts, Masks, and Painted Portraits. By KARL PEARSON and G. M. MORANT	269
VIII. Contribution à l'Étude de la Théorie de la Corrélation. Par Professeur CARLOS E. DIEULEFAIT	379
IX. The Use of Confidence or Fiducial Limits illustrated in the Case of the Binomial. By C. J. CLOPPER and EGON S. PEARSON	404
X. The Roumanian Silhouette. By MARIOARA PERTIA and OTHERS	414
XI. On a New Method of Determining "Goodness of Fit." By KARL PEARSON	425
XII. A Statistical Study of the <i>Daucus Carota</i> L. (Second Article.) By WILLIAM DOWELL BATEN	443

Miscellanea

(i) On Certain Measures of Dependence between Statistical Variables. By Professor J. F. STEFFENSEN	251
(ii) Remarks on Professor Steffensen's Measure of Contingency. By KARL PEARSON	255

	PAGE
(iii) Spearman's General Factor Again. By Professor BURTON H. CAMP	260
(iv) Note on the Recurrence Formulae for the Moments of the Point Binomial. By A. A. KRISHNASWAMI AYYANGAR	262
(v) Note on the Incomplete Moments of the Hypergeometrical Series. By A. A. KRISHNASWAMI AYYANGAR	264
(vi) On Simometers and their Handling. By KARL PEARSON	265
(vii) Paul Harzer's <i>Tabellen für alle statistische Zwecke in Wissen- schaft und Praxis</i> . Review by F. GARWOOD	469

List of Plates, etc.

Pearson and Morant: The Cromwell Head.

The Wilkinson Head in Right Profile, showing the oak pole and
the corroded tip of the iron prong, and the Cincture marking
the removal of the skull-cap to take out the Brain. Note flowing
moustache and hair on chin *Frontispiece*

Plate I. Photograph of Egyptian Criminal and of his Skull *to face page* 270

Plate II. Drawing of the Skull of Egyptian Criminal fitted to the
Drawing of his Face. To show the manner in which the real
Skull fits its actual Face " " "

Plate III. J. T. Smith's Engraving of John Cranch, "an impe-
cunious person of poetic nature," Writer of the first account of
the Head discussed in this Memoir " " 279

Plate IV. Title-page of John Cranch's account of the Cromwell
Head with an Engraving of it by him, 1799. [The Head has
been reversed in the Engraving] " " ,

Plate V. John Cranch's Oil Painting of the Head. The Painting
is in the possession of Canon Horace Wilkinson. Painted in
1799 or thereabouts " " 292

Plate VI. Sepia Drawing of the Wilkinson Head from a grangerised
copy of Pennant's *London*, 1790, in the Print Room, British
Museum. Probably from the last quarter of the 18th Century " " "

Plate VII. The Lord Protector lying in state at Somerset House.
From an Engraving by Jas. Caldwell of an original Print in the
Collection of John Towneley, Esq. The original Print must have
been contemporary, see Plate VIII " " 300

Contents

v

Plate VIII. His Highness' Effigies standing in state. From an Engraving by Jas. Caldwell of an original Print in the Collection of John Towneley, Esq. The original Print must have been contemporary, for it appears much reduced as Frontispiece to Carrington's <i>History of the Life and Death of... Oliver, late Lord Protector</i> , 1659, the Year following Cromwell's Death	to face page 300
Plate IX. Facsimile of the Inscription on the gilded plate found on Cromwell's Breast when his Coffin was opened at the Exhumation. Copy with Statement by Dr Cromwell Mortimer in the possession of Karl Pearson	" " 298
Plate X. Facsimile of the Resolution of the House of Commons for the Exhumation of the Bodies of the Regicides and their Hanging on the Gallows at Tyburn, with the Lords' Amendment "in their coffins." From the Library of the Speaker, by his kind permission	" " 310
Plate XI. Facsimile of the Mason's Receipt for the Money for digging up the Bodies of Cromwell, Ireton and Bradshaw. From Dr Cromwell Mortimer, in the possession of Karl Pearson	" " "
Plate XII. Interior of Westminster Hall during the Trial of Charles I. Bradshaw presides and Cromwell is on his right at K	" " 314
Plate XIII. The Triple Gallows at Tyburn, with the Pit for Burial below the Gallows. From a contemporary Dutch Print	" " "
Plate XIV. The Scene at Tyburn. The Building seen <i>under</i> and <i>above</i> the Gallows appears to be Westminster Hall. Whether the Building on which the Heads are poled is supposed to be the South End of the Hall we cannot say; it is more like the West Side curtailed. From a contemporary Dutch Print	" " "
Plate XV. Plan of Westminster Hall and Palace in Tudor Times (based on early maps and other sources of information). From William Newton's <i>London, Westminster and Southwark as in the Olden Times...</i> London, 1855. The Plan shows jagged steps on both Gables of the Hall, the Fish Yard to the West of the Hall and the range of Buildings at the South-West Corner and others South of the Hall (the former may have included "Heaven")	" " 316
Plate XVI. View of the City of Westminster. Print showing the Buildings at South End of Westminster Hall, which would not hinder, however, the View from Lambeth and the River of Heads poled at that End	" " "
Plate XVII. Print showing View of Westminster Hall from the River. The Roofs of Buildings at South-East Corner would not screen Heads poled on the South End of the Hall. Part of the	

Parapet of the Hall, with Windows opening on to it, is visible. Date of Drawing <i>circa</i> 1722	to face page 316
Plate XVIII. View of Westminster from St James's Park. Engraving by Hollar, <i>circa</i> 1660, of an ancient Drawing in the Collection of John Towneley, Esq. 1 is the Abbey, 2 the Hall. It is clear that Heads poled on the South End of the Hall would be distinctly visible from the Park	" " "
Plate XIX. View from New Palace Yard of the North End of Westminster Hall. Coffee Houses built up against front on right, then Court of Exchequer Buildings, with "Hell" and "Paradise" beneath them. To left, Door into Tally Court and Receipt of Exchequer; further to left, Doorway to Speaker's Court	" " 319
Plate XX. View of New Palace Yard, Westminster. This View is of some interest because it seems to indicate in the Wall or Palisade in front of the Tudor Buildings (Court of Exchequer) just above the two-horse Coach in the foreground, the Entrance to a Cellar, from which two Persons are coming out. Was this one of the Entrances to the Taverns under the Exchequer?	" " "
Plate XXI. Interior of Westminster Hall in Pepys' Day. Court of King's Bench to left, Court of Chancery to right of Steps, Court of Common Pleas in right-hand corner of Print. Stalls for Sale of Books and Hosiery, etc.	" " 322
Plate XXII. Leads of St Stephen's, of the White Hall, and of other Buildings at the South End of Westminster Hall, on to any of which the Cromwell Head may have been blown down without falling to the ground. The top-floor of St Stephen's suggests a residence with flowers growing on the flat in front. On the Leads of St Stephen's to the right, behind the ventilator, is a small Pent-house which may have offered shelter to a Guard. Engraving by J. T. Smith, 1806	" " 324
Plate XXIII. J. T. Smith's Plan of the Palace of Westminster, showing the Positions in his opinion of the Taverns "Heaven," "Hell," "Paradise" and "Purgatory." [Were the Taverns and Coffee-Houses encircling St Stephen's the source of the inebriety of 18th Century Politicians or were such houses there to satisfy their alcoholic cravings?]	" " "
Plate XXIV. The Wilkinson Head. Occipital View showing the wedge of flesh cut out by the axe, with at least two axe cuts inside the wedge, also an axe cut above the vertebra where severance has been finally made by two blows of the axe. The fracture of the Occipital may be seen about half an inch to the left of the piece chiselled out	" " 327

Plate XXV. The Wilkinson Head. Enlarged Occipital View showing the hair on back of Head, the chisel and saw marks on the Cincture and the abnormality of the Sutures in this region. <i>to face page</i>	327
Plate XXVI. The Wilkinson Head. Full face, showing broken Nose, shrinkage of Eyelids, the Cincture, hole for Wart, flowing Moustache on right side (cf. Plate LXV) and broken Lips.	" " 330
Plate XXVII. The Wilkinson Head. Full face with the Pole vertical, corroded Point of Iron Spike and the many Pimples	" " "
Plate XXVIII. The Wilkinson Head. Left Profile, showing flesh shrunk back from Cincture, distorted Ear and nature of Hair on Scalp and Chin. The fracture of the occipital is shown by a white line on the extreme right	" " "
Plate XXIX. The Wilkinson Head. Right Profile, showing Ear cut off and wedge cut out of back of Neck	" " "
Plate XXX. The Wilkinson Head. Three-quarter face, showing the Wart-cavity, the Pimples and the flowing Moustache on Right (cf. British Museum, wax death-mask, Plate LXV)	" " "
Plate XXXI. The Wilkinson Head. Three-quarter face looking slightly down	" " "
Plate XXXII. Skiagram of the Wilkinson Head in Profile, showing Iron Spike and its side-pieces, the Mandible, the Cincture, and the place where the Vertebral Column has been severed	" " 332
Plate XXXIII. Skiagram of the Wilkinson Head with the Skull-cap removed, showing the metal spike and the three nails, which still attach it to the Oak pole. On the border of the Cincture to the right may be seen, as a nearly vertical pale bar, the start of the Occipital Fracture	" " "
Plate XXXIV. Skiagram of the Wilkinson Head, showing a further view of the Iron Spike, the Cincture and Mandible	" " "
Plate XXXV. Skiagrams of Skull-caps. (a) Young Adult, Sutures open. No Pacchionian pits. (b) Adult, internally Sutures largely obliterated, externally beginning to close. A few Pacchionian pits	" " "
Plate XXXVI. Skiagrams of Skull-caps. (c) Oldish Adult, externally Sutures largely closed or obliterated, internally nearly fully obliterated. Many Pacchionian pits. (d) Wilkinson Head, showing Hole where Spike was rammed through fracturing the right parietal. Sutures closing or closed. Numerous Pacchionian pits ("Oldish Adult")	" " "
Plate XXXVII. (a) Scrapings from Linen. Straight Flax Fibres. (b) Modern medical Lint. Twisted Cotton Fibres. (c) Egyptian Mummy. Straight Flax Fibres. (d) Sample from Pad in Nostril of Wilkinson Head. Straight Flax Fibres.	" " 340

Plate XXXVIII. The Walker Portrait of Cromwell in the Possession of the Rev. Paul Cromwell Bush. By kind permission	to face page 344
Plate XXXIX. The Walker Portrait of Cromwell in the National Portrait Gallery, No. 536	" " "
Plate XXXIX (<i>bis</i>). The Walker Portrait of Cromwell in Justitsministeriet in Copenhagen. The details are far more complete than in the National Portrait Gallery, No. 536. See Plate XXXIX	" " "
Plate XL. The Walker Portrait of Cromwell in the Tangye Collection, London Museum	" " "
Plate XL (<i>bis</i>). The Stockholm Portraits of Cromwell (Walker). I. (a) Traditionally the portrait Cromwell sent to Queen Christina with Milton's Latin verses. Gripsholm, No. 1213. (b) Gouache on ivory. Formerly preserved in the Castle of Drottningholm. In gold capsule and back enamelled in turquoise blue. An original by Walker, or a copy of a Walker. It has been associated with Whitelocke's mission to Sweden	" " "
Plate XL (<i>ter</i>). The Stockholm Portraits of Cromwell (Walker). II. (c) Gripsholm, No. 93, after Walker. Chiefly notable for its emblematic frame	" " "
Plate XLI. The Lely Portrait of Cromwell, 1650(?), in the Pitti Gallery at Florence, from a photograph by Anderson	" " 346
Plate XLI (<i>bis</i>). The Lely Portrait of Cromwell, 1653, from the larger mezzotint prepared by Faber in 1735 of the picture owned at that date by William Powlett, Esq.	" " "
Plate XLI (<i>ter</i>). The Lely Portrait of Cromwell, 1653, from the smaller mezzotint prepared by Faber in 1740 of the Powlett-Cavendish painting	" " "
Plate XLII. Cooper Miniatures of Cromwell. (a) and (b) the Duke of Devonshire's Miniatures, (c) the Marquess of Crewe's Miniature	" " 351
Plate XLIII. The Houbraken Engraving of Cromwell (1747 or before), professing to be a copy of the Duke of Devonshire's miniature (Plate XLII (a)), but looking more like a reversal of the Marquess of Crewe's Miniature ((c) on same Plate)	" " "
Plate XLIV. The unfinished Miniature of Cromwell by Cooper, in the possession of the Duke of Buccleuch. Reproduced from their Engraving by kind permission of Messrs P. & D. Colnaghi & Co.	" " "
Plate XLV. Viscount Harcourt's Miniature of Cromwell by Cooper	" " "

Plate XLVI. The smaller Miniatures at Chequers Court. (a) The "cased" Miniature by Cooper. (b) Distant Copy from a Cooper. By Petitot(?). (c) Miniature from Cromwell's later life to illustrate the developed "puffiness." Artist(?)	to face page 353
Plate XLVII. The larger Miniature at Chequers Court, (?) Cooper. The crumple in the collar and spot of reflection on armour below link it curiously with the Lely portrait (Plate XLI), and the National Portrait Gallery, No. 588 (Plate XLIX)	" " "
Plate XLVIII. National Portrait Gallery, No. 514, said to be a copy of the Buccleuch Miniature (Plate XLIV), but the Face has been drawn out vertically and the Head given a tilt to the right, which associate it more with the Harcourt Miniature than with the Buccleuch	" " "
Plate XLIX. National Portrait Gallery, No. 588. Painter uncertain. Crude painting on a small scale in oils. The crumple in the collar and the reflection on armour immediately below indicate that it has some relation to the Lely and the larger Chequers Court Miniature. (See Plates XLI and XLVII).	" " "
Plate L. Two Miniatures, both attributed to Cooper, fitted to each other. Note eyes, chins and foreheads	" " 355
Plate LI. Dunbar Medal (see Plate LXI) fitted to Cooper Miniature. Note receding forehead and position of eye (!) Medal - - - , Miniature —	" " "
Plate LII. Dunbar Medal (see Plate LXI) fitted with the Chequers Court Life Mask. Note chins and noses. Medal — , Life Mask - - -	" " "
Plate LIII. The Walker Portrait in the Tangye Collection, London Museum, fitted with the Wilkinson Head	" " 356
Plate LIV. Sketch of the Walker Portrait at Hinchingsbrooke fitted with the Wilkinson Head. Chief failure, left frontal	" " "
Plate LV. Sketch of the Lely Florence Portrait of Cromwell fitted with the Wilkinson Head	" " "
Plate LVI. The Cooper Profile Miniature (Plate XLII (a)) fitted with the Profile Contour of the Wilkinson Head	" " "
Plate LVII. Sketch of the "Cooper" Portrait of Cromwell at Sidney Sussex College, Cambridge, fitted with the Wilkinson Head	" " "
Plate LVIII. Sketch of the Buccleuch Miniature fitted with the Wilkinson Head. Note position of Wart-cavity and the left Frontal	" " "

Plate LIX. Sketch (by a different Artist) of the Buccleuch Miniature fitted with the facial outlines of a different photograph of the Wilkinson Head	to face page 356
Plate LX. Sketch of the Walker Portrait of Ireton, in the possession of Mrs R. B. Polhill-Drabble, fitted with the Wilkinson Head. The fit is good vertically but the Head is far too broad horizontally to be Ireton's	" " "
Plate LXI. Medals and Coins of Cromwell. (a) Lord Protector Medal, 1653. (b) Dunbar Medal, 1650. (c) Lord General Medal, 1650. (d) Crown Piece, 1658	" " 358
Plate LXII. Life Mask of Cromwell at Chequers Court. Full face (somewhat inclined)	" " 360
Plate LXIII. Life Mask of Cromwell at Chequers Court. Left Profile	" " "
Plate LXIV. Life Mask of Cromwell at Chequers Court. Right Profile	" " "
Plate LXV. Wax Death Mask of Cromwell in the British Museum, showing the bandage over the Cincture. Full face	" " 362
Plate LXVI. Wax Death Mask of Cromwell in the British Museum. Left Profile	" " "
Plate LXVII. Plaster Cast taken at some time from the British Museum Death Mask but indicating more clearly the position of the Wart over the right Eye. Biometric Laboratory, University College, London	" " "
Plate LXVIII. Left Profile of Plaster Cast in Biometric Laboratory, University College, London	" " "
Plate LXIX. Death Mask of Cromwell in the Ashmolean Museum. It shows the thin coating of plaster over the Forehead put on to hide the Cincture. Full face	" " 364
Plate LXX. Death Mask of Cromwell in the Ashmolean Museum. Right Profile	" " "
Plate LXXI. Death Mask of Cromwell in the possession of the Rev. P. Cromwell Bush. Full face	" " "
Plate LXXII. Death Mask of Cromwell in the possession of the Rev. P. Cromwell Bush. Profile	" " "
Plate LXXIII. Death Mask of Cromwell in the possession of Major A. P. Frankland, D.S.O. Original from which the Distorted Ashmolean Type of Death Mask has sprung. Nearly full face. By kind permission of Major Frankland and the Proprietors of the <i>Illustrated London News</i>	" " "

Plate LXXIV. The Distorted Ashmolean Type of Cromwell Death Mask. Full face	<i>to face page 364</i>
Plate LXXV. The Distorted Ashmolean Type of Cromwell Death Mask. Right Profile	" " "
Plate LXXVI. The Distorted Ashmolean Type of Cromwell Death Mask. Left Profile	" " "
Plate LXXVII. Edward Pierce's Bust of Cromwell, Ashmolean Museum. Full face	" " 365
Plate LXXVIII. Edward Pierce's Bust of Cromwell, Ashmolean Museum. Three-quarter face	" " "
Plate LXXIX. Edward Pierce's Bust of Cromwell, Ashmolean Museum. Right Profile	" " "
Plate LXXX. The Bust of Cromwell in the Library, King's College, Cambridge. Full face	" " "
Plate LXXXI. The Bust of Cromwell in the Library, King's College, Cambridge. Right Profile	" " "
Plate LXXXII. The Florence Bust of Cromwell, from a cast in the National Portrait Gallery, London. Full face	" " "
Plate LXXXIII. The Florence Bust of Cromwell, from a cast in the National Portrait Gallery, London. Left Profile	" " "
Plate LXXXIV. The Duke of Grafton's Bust of Cromwell. Three-quarter face. The original or another specimen of the Florence Bust	" " 366
Plate LXXXV. Median Sagittal Section of the Wilkinson Head. $\frac{2}{3}$ natural size	" " "
Plate LXXXVI. Horizontal Section of the Wilkinson Head. $\frac{2}{3}$ natural size	" " "
Plate LXXXVII. Transverse Section of the Wilkinson Head. $\frac{2}{3}$ natural size	" " "
Plate LXXXVIII. The Chequers Life Mask of Cromwell fitted with (a) the full face facial outlines and (b) the median sagittal contour with average flesh allowance in profile of the Wilkinson Head	" " 368
Plate LXXXIX. Full face of the British Museum Wax Death Mask fitted with the facial outlines of the Wilkinson Head	" " "
Plate XC. Profile of the British Museum Wax Death Mask fitted with the profile — and the average flesh allowance — — — of the Wilkinson Head	" " "

Plate XCI. Full face of the Ashmolean Death Mask fitted with the facial outlines of the Wilkinson Head	to face page 368
Plate XCII. Profile of the Ashmolean Death Mask fitted with the profile outline of the Wilkinson Head	" " "
Plate XCIII. Not quite full face of Major A. P. Frankland's Death Mask of Cromwell fitted with the facial outlines of the Wilkinson Head	" " 370
Plate XCIV. Full face of the Distorted Ashmolean Type of Death Mask fitted with the facial outlines of the Wilkinson Head	" " "
Plate XCV. Profile of the Distorted Ashmolean Type of Death Mask fitted with the profile outline of the Wilkinson Head	" " "
Plate XCVI. Full face of the Cast of the Florence Bust fitted with the facial outlines of the Wilkinson Head	" " "
Plate XCVII. Profile of the Cast of the Florence Bust fitted with (a) profile outline — and (b) flesh allowance - - - - of the Wilkinson Head	" " "
Plate XCVIII. Full face of the Ashmolean Bust fitted with the facial outlines of the Wilkinson Head	" " "
Plate XCIX. Profile of the Ashmolean Bust fitted with (a) the median sagittal contour and (b) the flesh allowance — — — — of the Wilkinson Head	" " "
Plate C. Full face of the King's College, Cambridge Bust fitted with the facial outlines of the Wilkinson Head	" " "
Plate CI. Profile of the King's College, Cambridge Bust fitted with (a) bone profile and (b) flesh allowance — — — — of the Wilkinson Head	" " "
<i>Marioara Pertia and others: The Roumanian Silhouette.</i>	
Plate I. Type Silhouette, based on fifty male Students from Old Roumania	" " 417
Plate II. Type Silhouette, based on fifty male Students from Transylvania	" " 417
Two Construction-Silhouettes on tissue in pocket at end of volume.	

BIOMETRIKA

ON THE P_{λ_n} TEST FOR RANDOMNESS: REMARKS, FURTHER ILLUSTRATION, AND TABLE OF P_{λ_n} FOR GIVEN VALUES OF $-\log_{10}\lambda_n$.

By FLORENCE N. DAVID.

DR KARL PEARSON has drawn my attention to the fact that in his recent paper on the P_{λ_n} Test for Randomness in Sampling*, he had judged roughly in which direction the probability integrals should be measured, endeavouring to choose that direction which would lead to the more stringent result, i.e. that in which the probability of a more improbable result occurring would be the smaller. Consider the simple case of an individual drawn at random from a normal parent population. If in the ordinary notation we found $\frac{1}{2}(1 - \alpha) = .005$, we should argue that such an individual was very unlikely to have been drawn in a single random sampling, for it deviates from the mean value by about 2.58 times the standard deviation in excess, but actually the improbability would be as great in the case of an individual deviating from the mean value by 2.58 in defect, in which case the probability integral *measured in the same direction* would be .995. Now, if we have two probability integrals p_1 and p_2 , we may either take $p_1 \times p_2$ or $(1 - p_1)(1 - p_2)$ but we cannot take $p_1(1 - p_2)$ or $(1 - p_1)p_2$, for the proof of the randomness in this test depends on the probability integrals being measured in the *same* direction. Accordingly in this case p_1p_2 will be larger than $(1 - p_1)(1 - p_2)$ if $p_1 + p_2$ be greater than unity. But when we increase the number of probability integrals there is no easy method of at once determining whether

$$p_1p_2\dots p_n \text{ or } (1 - p_1)(1 - p_2)\dots(1 - p_n)$$

will give the more stringent test. These remarks apply equally well when the frequency distribution of the parental population, true or hypothesised, is skew.

In order to ascertain whether any strikingly different results would arise if the probability integrals in Pearson's illustrations had been measured in the opposite sense, I have worked out the values of $-\log_{10}(1 - p_1)(1 - p_2)\dots(1 - p_n)$ for those illustrations and determined the corresponding values of $Q_{\lambda_n} = 1 - P_{\lambda_n}$. This seemed practically as short as comparing the two products

$$p_1p_2\dots p_n \text{ and } (1 - p_1)(1 - p_2)\dots(1 - p_n),$$

for it is actually the value of $I(n - 1, u)$ we require for relative appreciation.

* *Biometrika*, Vol. xxv. pp. 379—410.

P_{λ_n} Test for Randomness

The following Table I gives the value of Q_{λ_n} for the various illustrations, and denotes which method leads to the more stringent results: Q_{λ_n} measures the probability of less probable cases occurring. The smaller it is, the less stringent the test.

TABLE I.

Illustration	Table and Page, <i>Biometrika</i> , Vol. xxv.	A Value of Q_{λ_n} from $p_1 p_2 \dots p_n$ *	B Value of Q_{λ_n} from $(1-p_1)(1-p_2)\dots(1-p_n)$	A or B the more stringent
(i)	Table I, p. 387	·118	·343	A
(ii)	Table III, p. 390	·070	·074	A
(iii)	Table IV, p. 391	·738	·520	B
(iv)	Table V, p. 392	·000,008	·995,351	A
(v)	Table VII, p. 396	·037	·242	A
(vi)	Table VIII, p. 397	·649	·888	A
(viii)	Table IX, p. 400	·125	·345	A
(ix)	Table XI, p. 403	·048	·976	A
(x)	Table XII, p. 405	·436	·393	B
(xi)	Table XIII, p. 406	·524	·431	B
(xii)	Table XIV, p. 409	·221	·653	A
Given below	Table II below	·575	·388	B

Thus in 8 out of 12 illustrations the more stringent direction of the probability integrals was selected by mere inspection. In the other 4 cases B ought to have been taken instead of A, but in none of these four cases was the difference such as to upset the judgment as to randomness deduced from A. The results, however, might modify a judgment as to one hypothesis being more or less probable than a second. The departure from randomness, represented by $Q_{\lambda_n} = \cdot 5$, may be in excess or defect and we must necessarily be as cautious of the approach of Q_{λ_n} to 1 as to 0. Hence in cases of doubt it may be desirable to work out the values of Q_{λ_n} by B as well as A.

A further illustration of the method may be made here, namely to testing whether a series of *Regression Coefficients* may be taken as random samples from a bivariate normal population. We reproduce the theoretical discussion and its application to a particular example which has already been given in the *Annals of Eugenics*, as it is likely to escape many readers of *Biometrika*.

The distribution of the regression coefficient $R_{y:x}$, which we will write R_t for the t th Age Group, is in samples of n_t given by*

$$z = z_0 a^{n_t-1} \frac{1}{\{a^2 + (R_t - b)^2\}^{\frac{1}{2}n_t}} \dots\dots\dots (i),$$

where a and b are constants of the parent population, of which we are in ignorance.

* R.S. Proc. Vol. 112 A, pp. 1—14, or *Tables for Statisticians*, Part II, p. cli.

If we suppose a and b to be the same for all the sampled populations, then we have the combined surface in the case of k samples, if $S(n_t) = N$,

$$z = z_0' a^{N-k} \prod_{t=1}^{t=k} \frac{1}{\{a^2 + (R_t - b)^2\}^{\frac{1}{2}n_t}} \dots\dots\dots(ii).$$

To determine the most probable values of the constants a and b we must make z a maximum, or taking logarithmic differentials of z with regard to a and b :

$$\left. \begin{aligned} \frac{N-k}{a} - \sum_{t=1}^{t=k} \frac{n_t a}{a^2 + (R_t - b)^2} &= 0 \\ \sum_{t=1}^{t=k} \frac{n_t (R_t - b)}{a^2 + (R_t - b)^2} &= 0 \end{aligned} \right\} \dots\dots\dots(iii).$$

Thus to determine a^2 and b we have the equations

$$\sum_{t=1}^{t=k} \frac{n_t}{a^2 + (R_t - b)^2} = \frac{N-k}{a^2} \dots\dots\dots(iv),$$

$$\sum_{t=1}^{t=k} \frac{n_t R_t}{a^2 + (R_t - b)^2} = b \frac{N-k}{a^2} \dots\dots\dots(v).$$

These equations appear to be only capable of solution by approximation, the chief difficulty being to obtain the *first* approximation.

Returning to Equation (i) we see that if n_t be considerable, we may replace it by

$$z = \frac{z_0}{a} e^{-\frac{1}{2} \sum_{t=1}^{t=k} n_t \left(\frac{R_t - b}{a} \right)^2} \dots\dots\dots(vi),$$

and accordingly (ii) by

$$z = \frac{z_0'}{a} e^{-\frac{1}{2} \sum_{t=1}^{t=k} n_t \left(\frac{R_t - b}{a} \right)^2} \dots\dots\dots(vii).$$

Again taking logarithmic differentials we have

$$b = \sum_{t=1}^{t=k} \frac{(n_t R_t)}{N} \dots\dots\dots(viii),$$

$$a^2 = \frac{N}{k} \sum_{t=1}^{t=k} \frac{n_t (R_t - b)^2}{N} \dots\dots\dots(ix).$$

Thus the first approximation to b is given by b_1 = the weighted mean of R_t 's and the first approximation to a^2 is a_1^2 equal N/k times the weighted standard deviation of the R_t 's. If we now put $b = b_1 + \epsilon_1$ and $a^2 = a_1^2 + \epsilon_1'$ in (iv) and (v), where ϵ_1 and ϵ_1' are small quantities, the squares and products of which may be neglected, we can proceed to a second approximation and if needful to higher approximations. Substituting these values of a and b in (i) we obtain by aid of the *Tables of the Incomplete B-Function* the probability integral p_t of each R_t .

Illustration. Regression Coefficients of Sitting Height on Stature. As the method of determining whether a number of regression coefficients may be considered as due to random sampling from one and the same population is of interest from the

TABLE II. To test Regression Coefficients of Sitting Height on Stature.

(i) Age Groups	(ii) n_i	(iii) R_i	(iv) $R_i - \bar{R}_i$	(v) $(R_i - \bar{R}_i)^2$	(vi) $a/\sqrt{n_i}$	(vii) $\frac{R_i - \bar{R}_i}{a/\sqrt{n_i}}$	(viii) p_i	(ix) $\log_{10} p_i$
6-12	99	.6261	-.2101	-.0441, 4201	-.0806, 8345-	-2.61	.99534	1.007, 9715
13-16	331	.8554	+ .0192	-.0003, 6864	-.0440, 7067	+ 0.44	.32997	1.518, 4745-
16	238	.8515	+ .0153	-.0002, 3409	-.0519, 7268	+ 0.29	.38591	1.586, 4860
17	288	.8120	-.0242	-.0005, 8564	-.0472, 4625+	- 0.51	.69497	1.841, 9661
18	319	.9019	+ .0657	-.0043, 1649	-.0448, 9194	+ 1.46	.072145	2.858, 2062
19	310	.7371	-.0991	-.0098, 2081	-.0455, 3891	- 2.18	.98537	1.963, 5993
20	368	.8744	+ .0382	-.0014, 5924	-.0407, 0497	+ 0.94	.17361	1.239, 5747
21	321	.9271	+ .0909	-.0082, 6281	-.0447, 5187	+ 2.03	.02118	2.345, 9280
22	289	.7834	-.0528	-.0027, 8784	-.0471, 6439	- 1.12	.86964	1.938, 8398
23	315	.8515	+ .0153	-.0002, 3409	-.0451, 7602	+ 0.34	.36693	1.564, 5832
24	275	.7743	-.0619	-.0038, 3161	-.0483, 5009	- 1.28	.89973	1.954, 1122
25	224	.8607	+ .0245	-.0006, 0025	-.0535, 7218	+ 0.46	.32276	1.508, 8797
26	224	.8068	-.0204	-.0008, 6436	-.0535, 7218	- 0.55	.70884	1.850, 5482
27	193	.8915	+ .0553	-.0030, 5809	-.0577, 1449	+ 0.96	.16853	1.226, 6772
28	188	.8242	-.0120	-.0001, 4400	-.0584, 7692	- 0.20+	.58121	1.764, 3331
29	186	.8704	+ .0342	-.0011, 6964	-.0587, 9050-	+ 0.38	.28096	1.448, 6445
30	197	.8445	+ .0083	-.0000, 6889	-.0571, 2549	+ 0.14+	.44236	1.645, 7758
31-32	305	.8328	-.0034	-.0000, 1156	-.0459, 1062	- 0.07	.52790	1.722, 5517
33-34	280	.8441	+ .0079	-.0000, 6241	-.0479, 1639	+ 0.16+	.43447	1.637, 9598
35-36	269	.7660	-.0702	-.0049, 2804	-.0488, 8633	- 1.44	.92507	1.966, 1746
37-38	265	.8288	-.0074	-.0000, 5476	-.0492, 5387	- 0.15+	.55962	1.747, 8932
39-41	318	.8361	-.0001	-.0000, 0001	-.0449, 6242	- 0.00	.50000	1.698, 9700.
42-44	267	.8255	-.0107	-.0001, 1449	-.0490, 6906	- 0.22	.58706	1.768, 6825
45-47	196	.8699	+ .0337	-.0011, 3569	-.0572, 7110	+ 0.59	.27760	1.443, 4195
48-51	221	.8099	-.0303	-.0009, 1809	-.0539, 3459	- 0.56	.71226	1.852, 6386
52-55	146	.8732	+ .0370	-.0013, 6900	-.0603, 5704	+ 0.56	.28774	1.459, 0002
56-61	186	.8819	+ .0457	-.0020, 8849	-.0587, 9050-	+ 0.78	.21770	1.337, 8584
62-61	161	.8645	+ .0283	-.0008, 0089	-.0631, 9027	+ 0.45-	.32636	1.513, 6969

$$\log_{10} \lambda_n = -11.5865, 5666$$

$$S n_i (R_i - \bar{R}_i)^2 = 18.0005, 0868$$

$$6999 \quad 5852.6910$$

$$\sqrt{n} \log_{10} e = 2.298, 0705$$

$$a^2 = .0428, 7531$$

$$S \frac{(n_i R_i)}{N} - \bar{R}_i = .8362$$

$$I(27, 5.041, 8630) = .42451$$

$$a = .8017, 9505, 5-$$

practical standpoint, we may illustrate it on the data for the 28 Age Groups in the regression of Sitting Height on Stature. The necessary working is given at length in Table II. The numbers are here sufficiently large to justify for present purposes the use of a normal curve (vii) for the distribution of $R_{sh:st}$. Column (i) indicates the Age Groups, (ii) the number, n_t , of cases in the sample, (iii) the observed value of $R_{sh:st}$, (iv) the difference of this from the weighted mean value $\bar{R}_{sh:st}$, (v) the squares of the entries in (iv), to ascertain the weighted standard deviation of $R_{sh:st}$, (vi) gives this weighted standard deviation a divided by $\sqrt{n_t}^*$, (vii) gives the ratio of the deviation in (iv) to the standard deviation $a/\sqrt{n_t}$ of $R_{sh:st}$ in (vi), (viii) gives the values of p_t and finally (ix) the values of $\log_{10} p_t$. The sum of the values in this column (ix) is $-11.5865,5666$ and since $\sqrt{n} \log_{10} e = \sqrt{28} \log_{10} e = 2.298,0705$, $u = 5.041,8630$ and $I(27, u) = .42451$ from the *Incomplete Γ -Function Table*, using third differences. Thus the probability on our hypothesis of more divergent sets of samples of 28 Age Groups occurring is .57549 or some 58% of sets would exhibit greater divergence. As far as this test is concerned it seems not unreasonable to suppose that the regression coefficient of Sitting Height on Stature is constant for all ages.

We may now examine this problem supposing the probability integrals measured in the reverse direction.

The values of $1 - p_t$ and their logarithms are given in Table III below:

TABLE III.

Probability Integrals of the Regression Coefficients of Sitting Height on Stature.

No.	Value of $1 - p_t$	$\log_{10} (1 - p_t)$	No.	Value of $1 - p_t$	$\log_{10} (1 - p_t)$
1	.00466	3.668,3859	15	.41879	1.621,9963
2	.07003	1.826,0942	16	.71904	1.856,7531
3	.61409	1.788,2320	17	.55764	1.746,3539
4	.30503	1.484,3426	18	.47210	1.674,0340
5	.92785 ⁵	1.967,4801	19	.56553	1.752,4556
6	.01463	2.165,2443	20	.07493	2.874,6557
7	.82639	1.917,1851	21	.44038	1.643,8276
8	.97882	1.990,7028	22	.50000	1.698,9700
9	.13136	1.118,4631	23	.41294	1.615,8870
10	.63307	1.801,4517	24	.72240	1.858,7777
11	.10027	1.001,1710	25	.28774	1.459,0002
12	.67724	1.830,7426	26	.71226	1.852,6386
13	.29116	1.464,1317	27	.78230	1.893,3733
14	.83147	1.919,8466	28	.67364	1.828,4279
$\sqrt{n} \log_{10} e = 2.298,0705$			$-\log_{10} \lambda_n = 12.679,3754$ $u = 5.517,4005$		

* $\frac{1}{\sqrt{n_t}}$ is found at once from the tabled values in Comrie's edition of *Barlow's Tables*.

Hence

$$P_{\lambda_n} = I(27, 5.517, 4005) \\ = .6124,$$

from the *Incomplete Γ -Function Table*. And accordingly we may take $Q_{\lambda_n} = .388$, which is somewhat more stringent than the $Q_{\lambda_n} = .575$ of the A method as noted in Table I, but would not alter our judgment that the hypothesis that all the regression coefficients may be taken as due to random sampling from the same normal population of regression coefficient $R_{gh:st} = .8362$ is not unreasonable. Had we asked whether Case No. 1 of regression coefficient $R_{gh:st} = .6261$ was a reasonable sample from a population of $R_{gh:st} = .8362$ we should have answered in the negative. This indicates how different our judgment may be when we are considering one or twenty-eight samples. Table IV below gives the value of P_{λ_n} for given values of n , the number of probability integrals, and of $-\log_{10} \lambda_n$, i.e. the logarithm of the product of the probability integrals to base 10 with the minus sign before it. This table saves the trouble of computing $u = \frac{-\log_{10} \lambda_n}{\sqrt{n \log_{10} e}}$ and then interpolating into the

Incomplete Γ -Function Table. The latter must still be used for values $n > 30$.

The table will suffice by using $\delta^2 P_{\lambda_n}$ to obtain values of P_{λ_n} to four figures except when n and $-\log_{10} \lambda_n$ are very small, say $n < 6$ and $-\log_{10} \lambda_n < 1.125$, when $\delta^4 P_{\lambda_n}$ may be found from $\delta^2 P_{\lambda_n}$ if we desire four-figure accuracy.

The present Table IV may be used to find without much extra labour $P(\chi^2, n')$. It is true that it will only give the value of $P(\chi^2, n')$ to four figures, but this is usually ample, and it doubles the range of the table in *Tables for Statisticians*, going up to $n' = 61$. All we need to do is to enter the present table with

$$n = \frac{1}{2}(n' - 1), \text{ and } -\log_{10} \lambda_n = \frac{1}{2} \chi^2 \log_{10} e,$$

and then

$$P(\chi^2, n') = 1 - P_{\lambda_n}.$$

For example: What is the value of $P(\chi^2, n')$ for $\chi^2 = 14$, $n' = 17$?

Here $n = 8$, $-\log_{10} \lambda_n = 7 \times \log_{10} e = 7 \times .434,2944 = 3.0406$.

But from our column for $n = 8$, we have

$$-\log_{10} \lambda_n, \text{ for } 3.00, = .3875,$$

$$-\log_{10} \lambda_n, \text{ for } 3.25, = .4729,$$

whence by linear difference the required P_{λ_n}

$$= .3875 + .0854 \times .1624 = .4014,$$

and

$$P(\chi^2, n') = .5986.$$

Palin Elderton's table gives .5987, which is the value actually provided by the present table, if second differences are used.

Of course the *Tables of the Incomplete Γ -Function* carry the $P(\chi^2, n')$ Table to 7 decimal places up to $n' = 101$, but a table to 4 decimal places up to $n' = 61$ will be found not without value.

TABLE IV.

Table of the P_{λ_n} Function.

$-\log_{10} \lambda_n$	n=2		n=3		n=4		n=5		n=6		$-\log_{10} \lambda_n$	n=4 (cont.)		n=5 (cont.)		n=6 (cont.)	
	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$		P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$
.000	.0000	+818	.0000	+7	.0000	-12	.0000	—	.0000	—	7.500	.9999 ⁷	—	.9999 ⁸	—	.9994	—
.125	.0343	+454	.0032	+144	.0002	+24	.0000	+3	.0000	—	7.625	.9999 ⁷	—	.9999 ⁸	—	.9995 ⁸	—
.250	.1139	+206	.0208	+186	.0029	+62	.0003	+13	.0000	+2	7.750	.9999 ⁸	—	.9999 ⁸	—	.9996	—
.375	.2142	+53	.0670	+170	.0117	+91	.0020	+30	.0003	+7	7.875	.9999 ⁸	—	.9999 ⁸	—	.9997	—
.500	.3127	-36	.1101	+128	.0297	+105	.0065 ⁺	+47	.0012	+16	8.000	.9999 ⁸	—	.9999 ⁸	—	.9998	—
.675	.4216	-64	.1760	+80	.0582	+105	.0158	+62	.0037	+25	8.125	.9999 ⁸	—	.9999 ⁸	—	.9998 ⁸	—
.750	.5151	-106	.2499	+36	.0973	+92	.0314	+71	.0086	+30	8.250	.9999 ⁸	—	.9999 ⁸	—	.9999 ⁸	—
.875	.5980	-111	.3273	-9	.1456	+73	.0540	+74	.0171	+45	8.375	.9999 ⁸	—	.9999 ⁷	—	.9999 ⁸	—
1.000	.6697	-107	.4046	-28	.2012	+52	.0841	+71	.0301	+52	8.500	1.0000 ⁻	—	.9999 ⁸	—	.9999 ⁸	—
1.125	.7308	-99	.4792	-47	.2619	+28	.1212	+63	.0483	+56	8.625			.9999 ⁸	—	.9999 ⁸	—
1.250	.7819	-87	.5490	-58	.3255 ⁺	+8	.1647	+52	.0721	+56	8.750			.9999 ⁸	—	.9999 ⁸	—
1.375	.8243	-76	.6130	-64	.3899	-9	.2134	+38	.1016	+53	8.875			.9999 ⁸	—	.9999 ⁸	—
1.500	.8592	-64	.6705 ⁺	-60	.4534	-23	.2659	+24	.1364	+48	9.000			.9999 ⁸	—	.9999 ⁷	—
1.625	.8876	-54	.7216	-64	.5145 ⁺	-34	.3208	+11	.1759	+40	9.125			.9999 ⁸	—	.9999 ⁷	—
1.750	.9106	-45	.7662	-60	.5723	-41	.3709	-1	.2195 ⁺	+31	9.250			.9999 ⁸	—	.9999 ⁸	—
1.875	.9291	-37	.8048	-55	.6260	-45	.4329	-12	.2662	+22	9.375			1.0000 ⁻	—	.9999 ⁸	—
2.000	.9439	-30	.8379	-50	.6751	-47	.4877	-20	.3151	+12	9.500					.9999 ⁸	—
2.125	.9558	-24	.8660	-44	.7196	-47	.5405 ⁺	-27	.3653	+3	9.625					.9999 ⁸	—
2.250	.9652	-19	.8898	-38	.7594	-45	.5906	-32	.4157	-5	9.750					.9999 ⁸	—
2.375	.9727	-16	.9097	-33	.7947	-43	.6376	-36	.4657	-12	9.875					.9999 ⁸	—
2.500	.9786	-12	.9262	-28	.8257	-40	.6810	-36	.5145 ⁻	-18	10.000					.9999 ⁸	—
2.625	.9833	-10	.9400	-24	.8527	-36	.7208	-37	.5614	-23	10.125					.9999 ⁸	—
2.750	.9870	-8	.9513	-20	.8761	-33	.7509	-36	.6061	-26	10.250					.9999 ⁸	—
2.875	.9898	-6	.9606	-17	.8961	-29	.7894	-35	.6481	-29	10.375					1.0000 ⁻	—
3.000	.9921	-5	.9682	-14	.9133	-26	.8184	-33	.6874	-30							
3.125	.9939	-4	.9744	-12	.9279	-23	.8441	-31	.7236	-30							
3.250	.9952	-3	.9795 ⁻	-10	.9402	-20	.8607	-28	.7567	-30							
3.375	.9963	-2	.9836	-8	.9506	-17	.8865 ⁺	-26	.7869	-29							
3.500	.9971	-2	.9869	-6	.9593	-11	.9037	-23	.8141	-28							
3.625	.9978	-1	.9895 ⁺	-5	.9665 ⁺	-12	.9186	-21	.8385 ⁺	-27							
3.750	.9983	-1	.9917	-4	.9726	-9	.9314	-18	.8603	-25							
3.875	.9987	-1	.9934	-3	.9776	-9	.9424	-16	.8795 ⁺	-23							
4.000	.9990	-1	.9947	-3	.9817	-7	.9517	-14	.8965 ⁻	-21							
4.125	.9992	-1	.9958	-2	.9851	-6	.9597	-12	.9114	-19							
4.250	.9994	—	.9967	-2	.9879	-5	.9664	-10	.9244	-17							
4.375	.9995 ⁺	—	.9974	-1	.9902	-4	.9721	-9	.9357	-15							
4.500	.9996	—	.9979	-1	.9921	-3	.9769	-8	.9454	-14							
4.625	.9997	—	.9984	-1	.9936	-3	.9809	-7	.9538	-12							
4.750	.9998	—	.9987	-1	.9948 ⁸	-2	.9812	-6	.9605 ⁺	-10							
4.875	.9998	—	.9990	-1	.9958 ⁸	-2	.9870	-4	.9672	-9							
5.000	.9999	—	.9992	—	.9967	-2	.9894	-4	.9725 ⁻	-8							
5.125	.9999 ⁸	—	.9994	—	.9973	-1	.9913	-3	.9770	-7							
5.250	.9999 ⁸	—	.9995 ⁺	—	.9979	-1	.9929	-3	.9808	-6							
5.375	.9999 ⁸	—	.9996	—	.9983	-1	.9942	-2	.9840	-5							
5.500	.9999 ⁸	—	.9997	—	.9986	-1	.9952	-2	.9867	-4							
5.625	.9999 ⁸	—	.9998	—	.9989	-1	.9961	-2	.9889	-4							
5.750	.9999 ⁷	—	.9998	—	.9991	—	.9968	-1	.9908	-3							
5.875	.9999 ⁸	—	.9999	—	.9993	—	.9974	-1	.9924	-3							
6.000	.9999 ⁸	—	.9999 ⁸	—	.9994 ⁸	—	.9979	-1	.9937	-2							
6.125	1.0000 ⁻	—	.9999 ⁸	—	.9996	—	.9983	-1	.9948	-2							
6.250			.9999 ⁸	—	.9996 ⁸	—	.9986	-1	.9958	-2							
6.375			.9999 ⁸	—	.9997	—	.9989	—	.9965 ⁺	-1							
6.500			.9999 ⁸	—	.9998	—	.9991	—	.9971	-1							
6.625			.9999 ⁸	—	.9998	—	.9993	—	.9977	-1							
6.750			.9999 ⁸	—	.9999	—	.9994	—	.9981	-1							
6.875			.9999 ⁸	—	.9999	—	.9995 ⁺	—	.9984	-1							
7.000			.9999 ⁸	—	.9999 ⁸	—	.9996	—	.9987	-1							
7.125			.9999 ⁸	—	.9999 ⁸	—	.9997	—	.9990	—							
7.250			1.0000 ⁻	—	.9999 ⁸	—	.9998	—	.9992	—							
7.375					.9999 ⁸	—	.9998	—	.9993	—							

P_{λ} Test for Randomness

Table of the P_{λ} Function (cont.)

$-\log_{10} \lambda_{\text{H}}$	$n=7$		$n=8$		$n=9$		$n=10$		$n=11$		$n=12$		$n=13$		$n=14$		$-\log_{10} \lambda_{\text{H}}$
	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	$P_{\lambda_{\text{H}}}$	$\delta^2 P_{\lambda_{\text{H}}}$	
0.00	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	0.00
0.25	-0000	+ 2	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	0.25
0.50	-0002	+ 17	-0000	+ 4	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	0.50
0.75	-0020	+ 55	-0004	+ 18	-0001	+ 4	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	0.75
1.00	-0084	+ 109	-0026	+ 47	-0006	+ 18	-0001	+ 6	-0000	+ 2	-0000	+ 1	-0000	—	-0000	—	1.00
1.25	-0277	+ 157	-0095 ⁺	+ 86	-0029	+ 39	-0008	+ 15	-0002	+ 5	-0001	+ 1	-0000	+ 1	-0000	—	1.25
1.50	-0618	+ 179	-0250 ⁺	+ 124	-0091	+ 69	-0030	+ 33	-0009	+ 14	-0003	+ 5	-0001	+ 1	-0000	—	1.50
1.75	-1138	+ 169	-0629	+ 147	-0223	+ 99	-0085 ⁺	+ 56	-0030	+ 27	-0010	+ 12	-0003	+ 5	-0000	+ 3	1.75
2.00	-1827	+ 132	-0955 ⁺	+ 148	-0453	+ 121	-0196	+ 82	-0078	+ 47	-0029	+ 23	-0010	+ 10	-0003	+ 4	2.00
2.25	-2647	+ 79	-1529	+ 129	-0805 ⁺	+ 129	-0389	+ 100	-0173	+ 68	-0071	+ 38	-0027	+ 20	-0010	+ 9	2.25
2.50	-3546	+ 23	-2232	+ 93	-1287	+ 121	-0682	+ 112	-0334	+ 84	-0151	+ 55	-0064	+ 31	-0025 ⁺	+ 16	2.50
2.75	-4469	- 27	-3029	+ 50	-1889	+ 99	-1067	+ 111	-0579	+ 97	-0287	+ 71	-0132	+ 40	-0057	+ 26	2.75
3.00	-5385 ⁻	- 65	-3875 ⁺	+ 7	-2590	+ 67	-1603	+ 98	-0921	+ 100	-0493	+ 84	-0246	+ 60	-0116	+ 38	3.00
3.25	-6196	- 89	-4729	- 31	-3358	+ 32	-2217	+ 76	-1383	+ 95	-0783	+ 89	-0421	+ 72	-0212	+ 51	3.25
3.50	-6988	-101	-5552	- 59	-4187	- 3	-2907	+ 49	-1900	+ 80	-1162	+ 90	-0687	+ 80	-0390	+ 62	3.50
3.75	-7579	-103	-6316	- 78	-4933	- 33	-3646	+ 18	-2517	+ 58	-1631	+ 80	-0993	+ 82	-0569	+ 71	3.75
4.00	-8117	- 98	-7002	- 88	-5717	- 55	-4403	- 10	-3192	+ 35	-2179	+ 65	-1401	+ 78	-0849	+ 75	4.00
4.25	-8588	- 88	-7599	- 90	-6246	- 70	-5150 ⁺	- 33	-3902	+ 9	-2792	+ 45	-1886	+ 68	-1204	+ 74	4.25
4.50	-8911	- 76	-8106	- 86	-7094	- 78	-5864	- 52	-4621	- 14	-3450 ⁺	+ 24	-2439	+ 53	-1633	+ 68	4.50
4.75	-9188	- 63	-8527	- 79	-7624	- 80	-6526	- 63	-5326	- 34	-4132	+ 2	-3045 ⁺	+ 35	-2129	+ 57	4.75
5.00	-9401	- 51	-8870	- 69	-8104	- 77	-7125 ⁻	- 71	-5997	- 48	-4817	- 17	-3685 ⁺	+ 16	-2683	+ 43	5.00
5.25	-9564	- 41	-9143	- 59	-8507	- 71	-7653	- 73	-6620	- 59	-5485 ⁻	- 34	-4342	- 2	-3278	+ 27	5.25
5.50	-9685 ⁺	- 32	-9357	- 49	-8839	- 63	-8108	- 66	-7184	- 65	-6119	- 46	-4995 ⁺	- 19	-3901	+ 10	5.50
5.75	-9775 ⁺	- 24	-9524	- 39	-9107	- 55	-8495 ⁻	- 66	-7683	- 65	-6707	- 54	-5630	- 33	-4533	- 6	5.75
6.00	-9841	- 18	-9650 ⁺	- 31	-9321	- 46	-8816	- 58	-8117	- 63	-7241	- 59	-6231	- 44	-5159	- 21	6.00
6.25	-9886	- 13	-9746	- 24	-9489	- 38	-9079	- 51	-8488	- 60	-7716	- 60	-6789	- 51	-5764	- 33	6.25
6.50	-9922	- 10	-9817	- 19	-9619	- 31	-9291	- 44	-8799	- 54	-8130	- 59	-7206	- 55	-6336	- 42	6.50
6.75	-9946	- 8	-9899	- 14	-9719	- 24	-9459	- 35	-9056	- 48	-8486	- 55	-7749	- 56	-6867	- 48	6.75
7.00	-9962	- 6	-9907	- 11	-9794	- 19	-9592	- 31	-9265 ⁺	- 41	-8788	- 51	-8146	- 54	-7350 ⁻	- 51	7.00
7.25	-9975	- 3	-9934	- 8	-9850 ⁺	- 15	-9694	- 23	-9434	- 35	-8938	- 43	-8488	- 51	-7782	- 52	7.25
7.50	-9983	- 2	-9954	- 6	-9892	- 11	-9773	- 19	-9507	- 29	-9245 ⁻	- 40	-8780	- 47	-8163	- 50	7.50
7.75	-9988	- 1	-9968	- 4	-9923	- 8	-9833	- 15	-9672	- 23	-9412	- 33	-9026 ⁻	- 42	-8493	- 48	7.75
8.00	-9992	- 1	-9978	- 3	-9945 ⁺	- 6	-9878	- 12	-9754	- 19	-9546	- 28	-9228	- 37	-8776	- 44	8.00
8.25	-9995	—	-9985	- 2	-9961	- 5	-9911	- 8	-9816	- 15	-9653	- 23	-9393	- 32	-9015 ⁻	- 40	8.25
8.50	-9997	—	-9990	- 2	-9973	- 4	-9936	- 7	-9864	- 12	-9736	- 19	-9528	- 27	-9214	- 35	8.50
8.75	-9998	—	-9993	- 1	-9981	- 3	-9954	- 5	-9900	- 9	-9801	- 15	-9635 ⁺	- 22	-9378	- 30	8.75
9.00	-9998	—	-9995 ⁺	—	-9987	- 2	-9967	- 4	-9927	- 7	-9851	- 12	-9720	- 18	-9512	- 26	9.00
9.25	-9999	—	-9997	—	-9991	- 1	-9977	- 3	-9947	- 5	-9889	- 9	-9787	- 15	-9620	- 22	9.25
9.50	-9999 ⁺	—	-9998	—	-9994	- 1	-9984	- 2	-9962	- 4	-9918	- 7	-9839	- 12	-9706	- 18	9.50
9.75	-9999 ⁺	—	-9999 ⁻	—	-9996	- 1	-9989	- 2	-9973	- 3	-9940	- 5	-9879	- 9	-9774	- 14	9.75
10.00	-9999 ⁺	—	-9999 ⁺	—	-9997	—	-9992	- 1	-9981	- 2	-9956	- 3	-9910	- 7	-9828	- 12	10.00
10.25	-9999 ⁺	—	-9999 ⁺	—	-9998	—	-9994	- 1	-9986	- 1	-9968	- 2	-9933	- 5	-9870	- 10	10.25
10.50	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9996	—	-9990	- 1	-9977	- 2	-9951	- 4	-9902	- 8	10.50
10.75	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9997	—	-9993	- 1	-9983	- 1	-9964	- 3	-9927	- 7	10.75
11.00	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	-9995 ⁺	—	-9988	- 1	-9973	- 2	-9945 ⁺	- 5	11.00
11.25	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9997	—	-9992	- 1	-9981	- 1	-9960	- 4	11.25
11.50	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	-9994	—	-9985 ⁺	- 1	-9970	- 3	11.50
11.75	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	-9996	—	-9990	- 1	-9978	- 2	11.75
12.00	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9997	—	-9993	—	-9984	- 1	12.00
12.25	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9998	—	-9995 ⁻	—	-9989	- 1	12.25
12.50	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9999 ⁻	—	-9996	—	-9992	- 1	12.50
12.75	—	—	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9997	—	-9994	- 1	12.75
13.00	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	-9996	—	13.00
13.25	—	—	—	—	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁻	—	-9997	—	13.25
13.50	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	13.50
13.75	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9998	—	13.75
14.00	—	—	—	—	—	—	-9999 ⁺	—	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	14.00
14.25	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	14.25
14.50	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	14.50
14.75	—	—	—	—	—	—	-9999 ⁺	—	1.0000 ⁻	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	14.75
15.00	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	15.00
15.25	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	1.0000 ⁻	—	-9999 ⁺	—	15.25
15.50	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	15.50
15.75	—	—	—	—	—	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	-9999 ⁺	—	15.75
16.00	—	—	—	—	—	—	-9999 ⁺	—	—	—	-9999 ⁺	—	—	—	-9999 ⁺	—	16.00
16.25	—	—	—	—	—	—	1.0000 ⁻	—	—	—	—	—	—	—	-9999 ⁺	—	16.25

Table of the P_{λ_n} Function (cont.)

$-\log_{10} \lambda_n$	$n=15$		$n=16$		$n=17$		$n=18$		$n=19$		$n=20$		$n=21$		$n=22$		$-\log_{10} \lambda_n$
	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	
1.50	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	1.50
1.75	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	1.75
2.00	-0000	+ 1	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	2.00
2.25	-0003	+ 4	-0001	+ 1	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	-0000	—	2.25
2.50	-0009	+ 8	-0003	+ 4	-0001	+ 1	-0000	+ 1	-0000	—	-0000	—	-0000	—	-0000	—	2.50
2.75	-0023	+14	-0009	+ 7	-0003	+ 3	-0001	+ 1	-0000	+ 1	-0000	—	-0000	—	-0000	—	2.75
3.00	-0051	+22	-0021	+12	-0008	+ 6	-0003	+ 3	-0001	+ 1	-0000	+ 1	-0000	—	-0000	—	3.00
3.25	-0101	+32	-0045+	+19	-0019	+10	-0008	+ 4	-0003	+ 2	-0001	+ 1	-0000	+1	-0000	—	3.25
3.50	-0183	+43	-0088	+27	-0040	+16	-0017	+ 8	-0007	+ 4	-0003	+ 2	-0001	+ 1	-0000	+ 1	3.50
3.75	-0308	+54	-0158	+37	-0076	+23	-0035+	+13	-0015+	+ 8	-0007	+ 3	-0003	+ 1	-0001	—	3.75
4.00	-0487	+62	-0284	+46	-0136	+31	-0067	+20	-0031	+11	-0014	+ 6	-0006	+ 3	-0002	+ 2	4.00
4.25	-0728	+68	-0417	+55	-0227	+40	-0117	+27	-0068	+16	-0027	+10	-0012	+ 6	-0005+	+ 3	4.25
4.50	-1037	+71	-0625-	+61	-0358	+48	-0195+	+35	-0101	+24	-0050+	+14	-0024	+ 8	-0011	+ 4	4.50
4.75	-1414	+69	-0893	+64	-0537	+54	-0307	+42	-0188	+29	-0088	+20	-0044	+12	-0021	+ 7	4.75
5.00	-1858	+66	-1225+	+63	-0770	+59	-0461	+49	-0264	+37	-0145-	+26	-0076	+17	-0038	+10	5.00
5.25	-2360	+59	-1621	+59	-1062	+60	-0664	+53	-0397	+44	-0227	+32	-0125-	+22	-0066	+14	5.25
5.50	-2911	+48	-2075-	+51	-1413	+57	-0920	+56	-0574	+47	-0342	+38	-0196	+28	-0108	+19	5.50
5.75	-3496	+34	-2580	+40	-1822	+52	-1232	+55	-0798	+52	-0496	+44	-0295+	+34	-0169	+25	5.75
6.00	-4100	+20	-3125+	+28	-2283	+44	-1599	+52	-1074	+52	-0692	+47	-0428	+39	-0255-	+30	6.00
6.25	-4709	+4	-3698	+14	-2788	+34	-2018	+46	-1402	+52	-0936	+49	-0601	+43	-0371	+35	6.25
6.50	-5309	-21	-4285-	+1	-3327	+22	-2483	+38	-1782	+46	-1229	+49	-0816	+46	-0521	+39	6.50
6.75	-5888	-32	-4873-	-12	-3887	+10	-2986	+28	-2208	+41	-1572	+47	-1077	+47	-0711	+43	6.75
7.00	-6434	-40	-5449	-23	-4458	-2	-3517	+17	-2975+	+33	-1961	+42	-1385+	+46	-0944	+44	7.00
7.25	-6940	-45	-6003	-31	-5025+	-13	-4065+	+ 6	-3175-	+22	-2392	+36	-1739	+43	-1220	+44	7.25
7.50	-7402	-48	-6526	-38	-5580	-23	-4619	-5	-3697	+14	-2859	+28	-2136	+38	-1541	+43	7.50
7.75	-7816	-48	-7010	-42	-6111	-31	-5168	-15	-4233	+ 2	-3354	+19	-2570	+31	-1904	+39	7.75
8.00	-8182	-47	-7452	-45	-6612	-36	-5702	-23	-4771	-7	-3868	+ 9	-3036	+23	-2306	+34	8.00
8.25	-8501	-45	-7850	-45	-7077	-40	-6213	-30	-5302	-16	-4391	-	-3525+	+15	-2742	+27	8.25
8.50	-8775-	-41	-8202	-44	-7501	-42	-6694	-35	-5817	-23	-4914	-9	-4029	+ 6	-3205+	+20	8.50
8.75	-9007	-37	-8510	-42	-7883	-43	-7141	-38	-6309	-29	-5428	-17	-4540	-2	-3688	+12	8.75
9.00	-9203	-33	-8776	-39	-8223	-42	-7549	-40	-6772	-33	-5926	-24	-5049	-10	-4183	+ 4	9.00
9.25	-9364	-29	-9002	-35	-8520	-40	-7910	-40	-7202	-37	-6400	-27	-5547	-17	-4682	-4	9.25
9.50	-9498	-25	-9193	-31	-8778	-37	-8244	-30	-7504	-38	-6847	-33	-6029	-23	-5177	-11	9.50
9.75	-9606	-21	-9353	-28	-9000	-34	-8532	-38	-7949	-38	-7240	-35	-6487	-28	-5681	-17	9.75
10.00	-9693	-18	-9485+	-24	-9187	-30	-8783	-35	-8266	-37	-7639	-36	-6918	-31	-6127	-23	10.00
10.25	-9762	-15	-9593	-20	-9344	-27	-8998	-32	-8545-	-35	-7982	-36	-7317	-33	-6570	-27	10.25
10.50	-9817	-12	-9681	-17	-9475-	-23	-9182	-29	-8789	-34	-8288	-30	-7682	-35	-6986	-30	10.50
10.75	-9860	-10	-9751	-14	-9583	-20	-9337	-26	-8999	-30	-8558	-34	-8014	-35	-7371	-32	10.75
11.00	-9894	-8	-9808	-12	-9670	-17	-9466	-22	-9179	-28	-8795+	-32	-8310	-34	-7725-	-33	11.00
11.25	-9920	-6	-9852	-10	-9741	-14	-9573	-19	-9331	-24	-9001	-29	-8573	-32	-8045+	-33	11.25
11.50	-9940	-5	-9887	-8	-9798	-12	-9661	-17	-9459	-22	-9177	-26	-8803	-30	-8332	-32	11.50
11.75	-9955+	-4	-9914	-6	-9844	-10	-9732	-14	-9505-	-19	-9236	-23	-9003	-28	-8587	-31	11.75
12.00	-9967	-3	-9935-	-5	-9880	-8	-9790	-11	-9552	-15	-9452	-21	-9176	-25	-8812	-29	12.00
12.25	-9976	-2	-9951	-4	-9908	-6	-9836	-9	-9724	-14	-9558	-18	-9323	-23	-9007	-27	12.25
12.50	-9982	-2	-9963	-3	-9930	-5	-9873	-8	-9782	-11	-9645-	-15	-9447	-20	-9176	-24	12.50
12.75	-9987	-1	-9973	-2	-9947	-4	-9902	-6	-9820	-10	-9717	-13	-9551	-17	-9320	-22	12.75
13.00	-9990	-1	-9980	-2	-9960	-3	-9925+	-5	-9867	-9	-9775+	-11	-9638	-15	-9443	-19	13.00
13.25	-9993	-1	-9985+	-1	-9970	-2	-9943	-4	-9897	-7	-9823	-9	-9710	-13	-9546	-17	13.25
13.50	-9995+	—	-9989	-1	-9978	-2	-9957	-3	-9920	-5	-9861	-8	-9709	-11	-9632	-15	13.50
13.75	-9996	—	-9992	-1	-9983	-1	-9967	-3	-9938	-6	-9892	-9	-9817	-9	-9704	-13	13.75
14.00	-9997	—	-9994	—	-9988	-1	-9975+	-2	-9953	-3	-9910	-5	-9856	-8	-9763	-11	14.00
14.25	-9998	—	-9996	—	-9991	-1	-9982	-2	-9964	-2	-9935+	-4	-9887	-6	-9811	-9	14.25
14.50	-9999-	—	-9997	—	-9993	—	-9986	-1	-9973	-2	-9950+	-3	-9912	-5	-9850+	-8	14.50
14.75	-9999+	—	-9998	—	-9995-	—	-9990	-1	-9980	-2	-9961	-3	-9931	-4	-9882	-6	14.75
15.00	-9999+	—	-9998	—	-9996	—	-9992	-1	-9985-	-1	-9971	-3	-9947	-3	-9906	-5	15.00
15.25	-9999+	—	-9999-	—	-9997	—	-9994	—	-9988	-1	-9978	-2	-9959	-3	-9928	-4	15.25
15.50	-9999+	—	-9999+	—	-9998	—	-9996	—	-9991	-1	-9984	-2	-9968	-2	-9944	-3	15.50
15.75	-9999+	—	-9999+	—	-9999-	—	-9997	—	-9994	-1	-9988	-1	-9976	-2	-9957	-3	15.75
16.00	-9999+	—	-9999+	—	-9999+	—	-9998	—	-9995-	—	-9991	-1	-9982	-1	-9967	-2	16.00
16.25	-9999+	—	-9999+	—	-9999+	—	-9998	—	-9997	—	-9993	-1	-9985+	-1	-9974	-2	16.25
16.50	-9999+	—	-9999+	—	-9999+	—	-9999-	—	-9998	—	-9995-	—	-9990	-1	-9981	-1	16.50
16.75	-9999+	—	-9999+	—	-9999+	—	-9999+	—	-9998	—	-9996	—	-9992	—	-9985+	-1	16.75
17.00	-1.0000-	—	-9999+	—	-9999+	—	-9999+	—	-9999-	—	-9997	—	-9994	—	-9989	-1	17.00
17.25			-9999+	—	-9999+	—	-9999+	—	-9999+	—	-9998	—	-9996	—	-9991	-1	17.25

P_{λ_n} Test for RandomnessTable of the P_{λ_n} Function (cont.)

$-\log_{10} \lambda_n$	n=15 (cont.)		n=16 (cont.)		n=17 (cont.)		n=18 (cont.)		n=19 (cont.)		n=20 (cont.)		n=21 (cont.)		n=22 (cont.)		$-\log_{10} \lambda_n$
	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	
17.25			.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9996	—	.9991	— 1	17.25
17.50			1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9997	—	.9994	—	17.50
17.75					.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9995 ⁺	—	17.75
18.00					.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9996	—	18.00
18.25					.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9997	—	18.25
18.50					1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9998	—	18.50
18.75					.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9998	—	18.75
19.00					.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9998	—	19.00
19.25					1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	.9998	—	.9998	—	.9998	—	19.25
19.50									1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	19.50
19.75									.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	19.75
20.00									.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	20.00
20.25									1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	20.25
20.50									.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	20.50
20.75											1.0000 ⁻	—	.9999 ^a	—	.9999 ^a	—	20.75
21.00											.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	21.00
21.25											.9999 ^a	—	.9999 ^a	—	.9999 ^a	—	21.25
21.50															1.0000 ⁻	—	21.50

$-\log_{10} \lambda_n$	n=23		n=24		n=25		n=26		n=27		n=28		n=29		n=30		$-\log_{10} \lambda_n$
	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	P_{λ_n}	$\delta^2 P_{\lambda_n}$	
3.50	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	3.50
3.75	.0000	+ 1	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	3.75
4.00	.0001	—	.0000	+ 1	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	4.00
4.25	.0002	+ 2	.0001	—	.0000	+ 1	.0000	—	.0000	—	.0000	—	.0000	—	.0000	—	4.25
4.50	.0005 ⁺	+ 2	.0002	+ 1	.0001	—	.0000	+ 1	.0000	—	.0000	—	.0000	—	.0000	—	4.50
4.75	.0010	+ 3	.0004	+ 2	.0002	+ 1	.0001	—	.0000	+ 1	.0000	—	.0000	—	.0000	—	4.75
5.00	.0018	+ 7	.0009	+ 3	.0004	+ 2	.0002	+ 1	.0001	—	.0000	+ 1	.0000	—	.0000	—	5.00
5.25	.0033	+ 9	.0016	+ 6	.0008	+ 2	.0004	—	.0002	—	.0001	—	.0000	+ 1	.0000	—	5.25
5.50	.0057	+ 12	.0029	+ 9	.0014	+ 5	.0006	+ 4	.0003	+ 2	.0001	+ 1	.0001	—	.0000	+ 1	5.50
5.75	.0093	+ 17	.0049	+ 11	.0025 ⁺	+ 7	.0012	+ 4	.0006	+ 2	.0003	+ 2	.0001	+ 1	.0001	—	5.75
6.00	.0146	+ 21	.0080	+ 15	.0043	+ 10	.0022	+ 5	.0011	+ 4	.0005 ⁺	+ 2	.0002	+ 2	.0001	+ 1	6.00
6.25	.0220	+ 26	.0126	+ 18	.0070	+ 13	.0037	+ 8	.0019	+ 5	.0010	+ 3	.0005 ⁺	+ 2	.0002	+ 1	6.25
6.50	.0321	+ 31	.0190	+ 24	.0109	+ 16	.0060	+ 11	.0032	+ 7	.0017	+ 4	.0008	+ 3	.0004	+ 2	6.50
6.75	.0453	+ 36	.0278	+ 27	.0165 ⁺	+ 20	.0094	+ 14	.0052	+ 9	.0028	+ 6	.0014	+ 4	.0008	+ 2	6.75
7.00	.0620	+ 39	.0393	+ 32	.0241	+ 25	.0142	+ 18	.0082	+ 12	.0045 ⁺	+ 8	.0024	+ 5	.0013	+ 3	7.00
7.25	.0826	+ 41	.0540	+ 36	.0341	+ 29	.0209	+ 22	.0123	+ 16	.0071	+ 11	.0039	+ 7	.0021	+ 4	7.25
7.50	.1074	+ 42	.0723	+ 38	.0471	+ 33	.0297	+ 26	.0181	+ 19	.0107	+ 14	.0061	+ 9	.0034	+ 6	7.50
7.75	.1364	+ 41	.0945 ⁻	+ 40	.0633	+ 36	.0411	+ 29	.0258	+ 23	.0157	+ 17	.0093	+ 12	.0053	+ 8	7.75
8.00	.1695 ⁺	+ 39	.1206	+ 40	.0831	+ 38	.0554	+ 33	.0358	+ 27	.0224	+ 21	.0136	+ 15	.0080	+ 11	8.00
8.25	.2066	+ 35	.1508	+ 39	.1066	+ 38	.0730	+ 35	.0485 ⁻	+ 30	.0312	+ 24	.0195 ⁻	+ 18	.0118	+ 13	8.25
8.50	.2472	+ 30	.1848	+ 36	.1340	+ 38	.0941	+ 37	.0641	+ 33	.0424	+ 27	.0272	+ 22	.0170	+ 16	8.50
8.75	.2908	+ 24	.2235 ⁻	+ 32	.1651	+ 36	.1189	+ 37	.0831	+ 34	.0563	+ 30	.0371	+ 25	.0237	+ 19	8.75
9.00	.3368	+ 17	.2633	+ 27	.1999	+ 33	.1473	+ 36	.1054	+ 35	.0733	+ 32	.0495 ⁻	+ 28	.0324	+ 22	9.00
9.25	.3844	+ 9	.3069	+ 21	.2380	+ 29	.1794	+ 34	.1313	+ 35	.0934	+ 34	.0646	+ 30	.0434	+ 25	9.25
9.50	.4340	+ 1	.3524	+ 14	.2790	+ 24	.2148	+ 30	.1607	+ 34	.1169	+ 34	.0827	+ 32	.0569	+ 28	9.50
9.75	.4817	— 5	.3994	+ 7	.3224	+ 18	.2532	+ 26	.1935 ⁻	+ 31	.1438	+ 33	.1040	+ 33	.0732	+ 30	9.75
10.00	.5299	— 12	.4470	—	.3675 ⁻	+ 11	.2942	+ 21	.2294	+ 28	.1741	+ 32	.1286	+ 32	.0925 ⁻	+ 31	10.00
10.25	.5768	— 18	.4946	— 7	.4137	+ 5	.3373	+ 15	.2680	+ 23	.2074	+ 29	.1564	+ 31	.1148	+ 31	10.25
10.50	.6220	— 23	.5415 ⁻	— 13	.4604	— 8	.3819	+ 9	.3090	+ 18	.2437	+ 25	.1873	+ 29	.1408	+ 31	10.50
10.75	.6648	— 27	.5871	— 18	.5069	— 8	.4274	+ 3	.3518	+ 12	.2825 ⁻	+ 21	.2212	+ 27	.1689	+ 30	10.75
11.00	.7051	— 29	.6309	— 22	.5525 ⁺	— 14	.4732	— 3	.3959	+ 7	.3234	+ 16	.2578	+ 23	.2005 ⁻	+ 27	11.00
11.25	.7424	— 31	.6724	— 26	.5969	— 18	.5186	— 9	.4406	+ 1	.3658	+ 10	.2966	+ 19	.2348	+ 24	11.25

ON CORRECTIONS FOR THE MOMENT COEFFICIENTS OF FREQUENCY DISTRIBUTIONS WHEN THE START OF THE FREQUENCY IS ONE OF THE CHARACTERISTICS TO BE DETERMINED.

By E. S. MARTIN, B.Sc.

1. *Introduction.* In certain statistical data, for example, in cases of disease incidence in infancy, wages, incomes and house-valuations, the exact point at which the frequency commences is not evident. Thus the number of cases of Whooping Cough under one year of age may be recorded, but it is very unlikely that any of these occurred under three months; the number of houses valued for income-tax purposes under £20 per annum is published, but no house is actually valued at nothing per annum; the number of divorces occurring in the first year of marriage has been given, but it is clear that there is a certain minimum time which the legal formalities must take. Methods hitherto devised for fitting curves to such distributions have assumed the frequency to start at "zero," but this assumption may not only give poor results for the moments, but, in the case when the first frequency is less than the second, may make the distribution appear

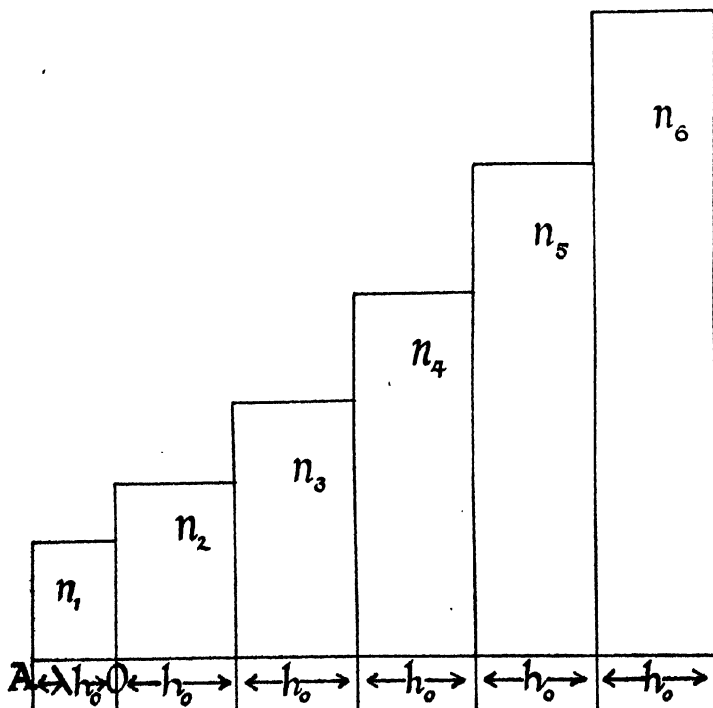


Fig. 1.

non-asymptotic when it is really asymptotic. The object of this paper is to investigate the possibility of determining the start of the frequency by means of an auxiliary curve and to examine what improvement in the evaluated moments of the distribution is thereby obtained.

2. Let the histogram of the first few sub-frequencies be as in Fig. 1, where the range of the first is λh_0 and the ranges of the others are h_0 . Let there be p sub-ranges taken from $x=0$ (at A) to $x=x_p$. As in the paper by Eleanor Pairman and Karl Pearson*, let $Z = \int_x^{x_p} y dx$, and assume that Z is given in the neighbourhood of A by the 5th order parabola

$$Z = N + c_1 \frac{x}{h_0} + c_2 \left(\frac{x}{h_0}\right)^2 + c_3 \left(\frac{x}{h_0}\right)^3 + c_4 \left(\frac{x}{h_0}\right)^4 + c_5 \left(\frac{x}{h_0}\right)^5,$$

where N is the total frequency†. The c 's are now determined in terms of λ by fitting the parabola to the first five frequencies. When

$$x = \lambda h_0, \quad Z = N - n_1,$$

$$x = (1 + \lambda) h_0, \quad Z = N - n_1 - n_2, \text{ etc.,}$$

whence

$$\left. \begin{aligned} -n_1 &= \lambda c_1 + \lambda^2 c_2 + \lambda^3 c_3 + \lambda^4 c_4 + \lambda^5 c_5 \\ -n_1 - n_2 &= (1 + \lambda) c_1 + (1 + \lambda)^2 c_2 + (1 + \lambda)^3 c_3 + (1 + \lambda)^4 c_4 + (1 + \lambda)^5 c_5 \\ -n_1 - n_2 - n_3 &= (2 + \lambda) c_1 + (2 + \lambda)^2 c_2 + (2 + \lambda)^3 c_3 + (2 + \lambda)^4 c_4 + (2 + \lambda)^5 c_5 \\ -n_1 - n_2 - n_3 - n_4 &= (3 + \lambda) c_1 + (3 + \lambda)^2 c_2 + (3 + \lambda)^3 c_3 + (3 + \lambda)^4 c_4 + (3 + \lambda)^5 c_5 \\ -n_1 - n_2 - n_3 - n_4 - n_5 &= (4 + \lambda) c_1 + (4 + \lambda)^2 c_2 + (4 + \lambda)^3 c_3 + (4 + \lambda)^4 c_4 + (4 + \lambda)^5 c_5 \end{aligned} \right\} \dots\dots(I).$$

If we put

$$\lambda = \mu - 2, \quad n_1 = m_1, \quad n_1 + n_2 = m_2, \quad n_1 + n_2 + n_3 = m_3,$$

$$n_1 + n_2 + n_3 + n_4 = m_4, \quad n_1 + n_2 + n_3 + n_4 + n_5 = m_5,$$

the solution of these equations is

$$\begin{aligned} 24c_1 &= -\frac{\mu(\mu^3-1)(\mu+2)}{\mu-2} m_1 + 4\frac{\mu(\mu+1)(\mu^3-4)}{\mu-1} m_2 \\ &\quad - 6\frac{(\mu^3-1)(\mu^3-4)}{\mu} m_3 + 4\frac{\mu(\mu-1)(\mu^3-4)}{\mu+1} m_4 - \frac{\mu(\mu^3-1)(\mu-2)}{\mu+2} m_5, \\ 24c_2 &= +\frac{(4\mu+2)(\mu^3+\mu-1)}{\mu-2} m_1 - 4\frac{4\mu^3+3\mu^2-8\mu-4}{\mu-1} m_2 \\ &\quad + 6\frac{4\mu^3-10\mu}{\mu} m_3 - 4\frac{4\mu^3-3\mu^2-8\mu+4}{\mu+1} m_4 + \frac{(4\mu-2)(\mu^3-\mu-1)}{\mu+2} m_5, \end{aligned}$$

* *Biometrika*, Vol. XII. pp. 231-258.

† In the paper by Pairman and Pearson the form

$$Z = N \left[1 + c_1 \left(\frac{x}{h_0}\right) + c_2 \left(\frac{x}{h_0}\right)^2 + c_3 \left(\frac{x}{h_0}\right)^3 + c_4 \left(\frac{x}{h_0}\right)^4 + c_5 \left(\frac{x}{h_0}\right)^5 \right]$$

was taken, but the above form is chosen here as it dispenses altogether with the use of "proportional frequencies."

$$\begin{aligned}
24c_3 &= -\frac{6\mu^2 + 6\mu - 1}{\mu - 2} m_1 + 4\frac{6\mu^2 + 3\mu - 4}{\mu - 1} m_2 \\
&\quad - 6\frac{6\mu^2 - 5}{\mu} m_3 + 4\frac{6\mu^2 - 3\mu - 4}{\mu + 1} m_4 - \frac{6\mu^2 - 6\mu - 1}{\mu + 2} m_5, \\
24c_4 &= +\frac{4\mu + 2}{\mu - 2} m_1 - 4\frac{4\mu + 1}{\mu - 1} m_2 + 6\frac{4\mu}{\mu} m_3 - 4\frac{4\mu - 1}{\mu + 1} m_4 + \frac{4\mu - 2}{\mu + 2} m_5, \\
24c_5 &= -\frac{1}{\mu - 2} m_1 + \frac{4}{\mu - 1} m_2 - \frac{6}{\mu} m_3 + \frac{4}{\mu + 1} m_4 - \frac{1}{\mu + 2} m_5.
\end{aligned}$$

The coefficients in these equations have been evaluated for values of λ between 0 and 1. On substituting back for the m 's in terms of the n 's, the c 's are obtained as linear functions of n_1, n_2, n_3, n_4, n_5 . The coefficients in these expressions are given to twelve places of decimals in Table III.

Now let $n_1\mu_s''$ be the s th moment of the first frequency about O , i.e., about $x = \lambda h_0$, as given by the auxiliary curve. Then

$$n_1\mu_s'' = \int_0^{\lambda h_0} y (x - \lambda h_0)^s dx,$$

whence the first four moments are

$$\begin{aligned}
n_1\mu_1'' &= h_0 \left(\frac{1}{2}\lambda^2 c_1 + \frac{1}{3}\lambda^3 c_2 + \frac{1}{4}\lambda^4 c_3 + \frac{1}{5}\lambda^5 c_4 + \frac{1}{6}\lambda^6 c_5 \right), \\
n_1\mu_2'' &= -h_0^2 \left(\frac{1}{3}\lambda^3 c_1 + \frac{1}{6}\lambda^4 c_2 + \frac{1}{10}\lambda^5 c_3 + \frac{1}{15}\lambda^6 c_4 + \frac{1}{24}\lambda^7 c_5 \right), \\
n_1\mu_3'' &= h_0^3 \left(\frac{1}{4}\lambda^4 c_1 + \frac{1}{10}\lambda^5 c_2 + \frac{1}{20}\lambda^6 c_3 + \frac{1}{35}\lambda^7 c_4 + \frac{1}{60}\lambda^8 c_5 \right), \\
n_1\mu_4'' &= -h_0^4 \left(\frac{1}{5}\lambda^5 c_1 + \frac{1}{15}\lambda^6 c_2 + \frac{1}{35}\lambda^7 c_3 + \frac{1}{70}\lambda^8 c_4 + \frac{1}{128}\lambda^9 c_5 \right).
\end{aligned}$$

With the help of Table III, these may be expressed as linear functions of n_1, n_2, n_3, n_4, n_5 ; the coefficients are given in Table IV.

Next, the formula

$$a_s = h_0^s \left(\frac{d^s Z}{dx^s} \right)_{x=\lambda h_0}$$

gives the abruptness coefficients at $x = \lambda h_0$ as linear functions of the true frequencies. These are

$$\begin{aligned}
a_1 &= c_1 + 2\lambda c_2 + 3\lambda^2 c_3 + 4\lambda^3 c_4 + 5\lambda^4 c_5, \\
a_2 &= 2c_2 + 6\lambda c_3 + 12\lambda^2 c_4 + 20\lambda^3 c_5, \\
a_3 &= 6c_3 + 24\lambda c_4 + 60\lambda^2 c_5, \\
a_4 &= 24c_4 + 120\lambda c_5, \\
a_5 &= 120c_5,
\end{aligned}$$

and again using Table III, the abruptness coefficients were calculated and are given in Table V.

Now suppose the raw moments of the remainder of the frequency, when n_1 is excluded, are calculated about O , using a sub-range h . Let them be $(N - n_1)\nu_1'''$,

etc., and write $p_0 = h/h_0$ and $a_s' = a_s p_0^s$. Then the corrected moments of the distribution about $x = \lambda h_0$ are as follows:

$$\left. \begin{aligned} N\mu_1' &= (N - n_1) \nu_1''' + \frac{1}{12} h (a_1' - \frac{1}{80} a_3' + \frac{1}{3840} a_5') + n_1 \mu_1'' \\ N\mu_2' &= (N - n_1) (\nu_2''' - \frac{1}{12} h^2) - \frac{1}{120} h^3 (a_2' - \frac{5}{128} a_4') + n_1 \mu_2'' \\ N\mu_3' &= (N - n_1) (\nu_3''' - \frac{1}{4} h^2 \nu_1''') - \frac{1}{40} h^3 (a_1' - \frac{5}{32} a_3' + \frac{1}{240} a_5') + n_1 \mu_3'' \\ N\mu_4' &= (N - n_1) (\nu_4''' - \frac{1}{2} h^2 \nu_2''' + \frac{7}{240} h^4) + \frac{1}{120} h^4 (a_2' - \frac{7}{80} a_4') + n_1 \mu_4'' \end{aligned} \right\} \dots (II),$$

provided the distribution be not abrupt at the other end; in that case the usual corrections for abruptness must be applied for that terminal.

If $h = h_0$, then $a_s' = a_s$, and the equations may be simplified by putting

$$K_1 = \frac{1}{12} (a_1 - \frac{1}{80} a_3 + \frac{1}{3840} a_5) + \frac{1}{h} n_1 \mu_1'',$$

$$K_2 = -\frac{1}{120} (a_2 - \frac{5}{128} a_4) + \frac{1}{h^2} n_1 \mu_2'',$$

$$K_3 = -\frac{1}{40} (a_1 - \frac{5}{32} a_3 + \frac{1}{240} a_5) + \frac{1}{h^3} n_1 \mu_3'',$$

$$K_4 = \frac{1}{120} (a_2 - \frac{7}{80} a_4) + \frac{1}{h^4} n_1 \mu_4'',$$

whence the corrected moments are

$$\left. \begin{aligned} N\mu_1' &= (N - n_1) \nu_1''' + h K_1 \\ N\mu_2' &= (N - n_1) (\nu_2''' - \frac{1}{12} h^2) + h^2 K_2 \\ N\mu_3' &= (N - n_1) (\nu_3''' - \frac{1}{4} h^2 \nu_1''') + h^3 K_3 \\ N\mu_4' &= (N - n_1) (\nu_4''' - \frac{1}{2} h^2 \nu_2''' + \frac{7}{240} h^4) + h^4 K_4 \end{aligned} \right\} \dots (III).$$

The values of the K 's are given in Table VI.

3. We now come to the most troublesome part of the problem: namely, to determine from a given set of frequencies the best value of λ . Three methods will be proposed.

1st Method. Following on the lines of Miss Pearson's paper on asymptotic distributions*, we may select λ by making our parabola fit the sixth frequency n_6 . When $x = (5 + \lambda) h_0$, $Z = N - n_1 - n_2 - n_3 - n_4 - n_5 - n_6$ and on substituting in the equation of the parabola we get

$$\begin{aligned} n_6 = -n_1 - n_2 - n_3 - n_4 - n_5 - (5 + \lambda) c_1 - (5 + \lambda)^2 c_2 \\ - (5 + \lambda)^3 c_3 - (5 + \lambda)^4 c_4 - (5 + \lambda)^5 c_5, \end{aligned}$$

giving n_6 as a linear function of n_1, n_2, n_3, n_4, n_5 , whose coefficients are given in Table II.

In order to select λ in any given case, the values of n_6 are calculated from this table for various values of λ and that value of λ is chosen which yields a result nearest to the given n_6 .

* "On Corrections for the Moment Coefficients of Frequency Distributions when there are Infinite Ordinates at one or both of the Terminals of the Range," *Biometrika*, Vol. xx^A, pp. 314—355.

This method does not always work very well, thus (a) the value of λ so found sometimes differs widely from the true value, (b) the table may not give any value of λ , i.e., there is no auxiliary parabola, for which λ is less than unity, which fits the first six frequencies, (c) the table may give two values of λ , i.e., there are two such parabolas.

2nd Method. If it is expected that the final frequency curve will rise abruptly from the x -axis, we may, instead of making our parabola fit the sixth frequency, impose the condition that its initial ordinate shall be zero. Now since $y = -\frac{dZ}{dx}$, the

ordinate at $x = 0$ is $-\frac{c_1}{h_0}$. Hence we may use Table III to find that value of λ which makes c_1 most nearly equal to zero. This test, while not always giving an accurate result, has been found in several cases to give a better result than the first test, and it has not yet in any case been found to give a double value of λ .

3rd Method. The poor results which the above tests sometimes give are no doubt due to the inadequacy of our parabola to represent various types of curves.

Accordingly, for the purpose of choosing λ a new type of auxiliary curve may be proposed.

(i) If $Z = N + Ax^q$ be taken as auxiliary curve*, then Z contains three unknowns, A , q and λ . We may hence fit this curve to the first three frequencies. We have

$$\begin{aligned} -n_1 &= A\lambda^q, \\ -n_1 - n_2 &= A(1 + \lambda)^q, \\ -n_1 - n_2 - n_3 &= A(2 + \lambda)^q. \end{aligned}$$

Writing $n_1 = m_1$, $n_1 + n_2 = m_2$, $n_1 + n_2 + n_3 = m_3$, we get, eliminating A ,

$$\frac{m_2}{m_1} = \left(\frac{1 + \lambda}{\lambda}\right)^q, \quad \frac{m_3}{m_2} = \left(\frac{2 + \lambda}{1 + \lambda}\right)^q,$$

whence
$$\frac{\log\left(\frac{m_2}{m_1}\right)}{\log\left(\frac{m_3}{m_2}\right)} = \frac{\log\left(\frac{1 + \lambda}{\lambda}\right)}{\log\left(\frac{2 + \lambda}{1 + \lambda}\right)}$$

is the equation from which to determine λ . Writing

$$\frac{\log\left(\frac{m_2}{m_1}\right)}{\log\left(\frac{m_3}{m_2}\right)} = \frac{{}_3M_1}{{}_3M_2} \quad \text{and} \quad \frac{\log\left(\frac{1 + \lambda}{\lambda}\right)}{\log\left(\frac{2 + \lambda}{1 + \lambda}\right)} = \frac{{}_3L_1}{{}_3L_2},$$

we have calculated the values of the latter ratio for various values of λ , the results being given in Table I. All the computer has to do, therefore, is to calculate the value of $\frac{{}_3M_1}{{}_3M_2}$ from the given frequencies and, referring to Table I, select that value

of λ for which $\frac{{}_3L_1}{{}_3L_2}$ is most nearly equal to the computed value of $\frac{{}_3M_1}{{}_3M_2}$.

* x stands here for x/h_0 .

If q also be required, we have $q = \frac{{}_2M_1}{{}_2L_1}$ and the appropriate value of ${}_2L_1$ can be taken from the column adjacent to $\frac{{}_2L_1}{{}_2L_2}$.

(ii) Now three frequencies will often be insufficient from which to determine λ , especially if h_0 be large. To get a better result we may fit the curve

$$Z = N + Ax^q e^{ax}$$

to the first four frequencies. Eliminating A , q and a from the resulting equations by successive divisions we get

$$\frac{\log\left(\frac{m_2^3}{m_1 m_3}\right)}{\log\left(\frac{m_3^2}{m_2 m_4}\right)} = \frac{\log\left\{\frac{(1+\lambda)^2}{\lambda(2+\lambda)}\right\}}{\log\left\{\frac{(2+\lambda)^2}{(1+\lambda)(3+\lambda)}\right\}},$$

or

$$\frac{{}_4M_1}{{}_4M_2} = \frac{{}_4L_1}{{}_4L_2},$$

and

$$q = \frac{{}_4M_1}{{}_4L_1}.$$

The values of $\frac{{}_4L_1}{{}_4L_2}$ and ${}_4L_1$ are given in Table I and the procedure is as for three frequencies.

(iii) Similarly, fitting $Z = N + Ax^q e^{ax+bx^2}$ to the first five frequencies, we have

$$\frac{\log\left(\frac{m_2^3 m_4}{m_1 m_3^3}\right)}{\log\left(\frac{m_3^3 m_5}{m_2 m_4^3}\right)} = \frac{\log\left\{\frac{(1+\lambda)^3(3+\lambda)}{\lambda(2+\lambda)^3}\right\}}{\log\left\{\frac{(2+\lambda)^3(4+\lambda)}{(1+\lambda)(3+\lambda)^3}\right\}},$$

or

$$\frac{{}_5M_1}{{}_5M_2} = \frac{{}_5L_1}{{}_5L_2},$$

and

$$q = \frac{{}_5M_1}{{}_5L_1}.$$

The values of $\frac{{}_5L_1}{{}_5L_2}$ and ${}_5L_1$ are given in Table I.

(iv) If the curve $Z = N + Ax^q e^{ax+bx^2+cx^3}$ be fitted to the first six frequencies, the equation for λ is

$$\frac{\log\left(\frac{m_2^4 m_4^4}{m_1 m_3^6 m_5}\right)}{\log\left(\frac{m_3^4 m_5^4}{m_2 m_4^6 m_6}\right)} = \frac{\log\left\{\frac{(1+\lambda)^4(3+\lambda)^4}{\lambda(2+\lambda)^6(4+\lambda)}\right\}}{\log\left\{\frac{(2+\lambda)^4(4+\lambda)^4}{(1+\lambda)(3+\lambda)^6(5+\lambda)}\right\}},$$

or

$$\frac{{}_6M_1}{{}_6M_2} = \frac{{}_6L_1}{{}_6L_2},$$

and

$$q = \frac{{}_6M_1}{{}_6L_1}.$$

The values of $\frac{{}_6L_1}{{}_6L_2}$ and ${}_6L_1$ are given in Table I.

(v) If the curve $Z = N + Ax^q e^{ax+bx^2+cx^3+dx^4}$

be fitted to the first seven frequencies, the equation for λ is

$$\frac{\log \left(\frac{m_2^5 m_4^{10} m_6}{m_1 m_3^{10} m_5^5} \right)}{\log \left(\frac{m_2^5 m_6^{10} m_7}{m_3 m_4^{10} m_5^5} \right)} = \frac{\log \left\{ \frac{(1+\lambda)^5 (3+\lambda)^{10} (5+\lambda)}{\lambda (2+\lambda)^{10} (4+\lambda)^5} \right\}}{\log \left\{ \frac{(2+\lambda)^5 (4+\lambda)^{10} (6+\lambda)}{(1+\lambda) (3+\lambda)^{10} (5+\lambda)^5} \right\}},$$

or

$$\frac{{}_7M_1}{{}_7M_2} = \frac{{}_7L_1}{{}_7L_2},$$

and

$$q = \frac{{}_7M_1}{{}_7L_2}.$$

The values of $\frac{{}_7L_1}{{}_7L_2}$ and ${}_7L_1$ are given in Table I.

This method, though apparently rather elaborate, is simple in application. If the logarithms of $m_1, m_2, m_3, m_4, m_5, m_6$ and m_7 are found, the five values of λ corresponding to 3, 4, 5, 6 and 7 frequencies are quickly determined*. In any theoretical distribution it will be found that the more frequencies are taken, the more nearly does the corresponding value of λ approach the true value. In fact, among numerous theoretical frequencies that have been tested, we have not found one in which the test fails to yield the correct value of λ (to one significant figure) when the seven initial frequencies are taken. In experimental frequencies, however, this is not the case, as the values of M_1 and M_2 become more and more affected by the sampling errors as more frequencies are taken. Thus the value of λ given by seven frequencies may be an impossible one, while that given by four or five frequencies may yield good results. For this reason the present test may not be as efficacious as the previous tests in any particular case, but it has one advantage over them; it can be used for either asymptotic or non-asymptotic frequencies. We have placed no limit on the value of q and if it is found that $0 < q < 1$ then an asymptotic frequency is indicated. It may be remarked, however, that although the auxiliary parabola of the n_6 test is incapable of having an infinite initial ordinate, yet it can have a very large one and will yield reasonable results when applied to an asymptotic frequency, provided the degree of asymptoticity is not large.

Our third method will be referred to as the " ϵ test."

As an example consider the Type VI curve

$$y = 33 \times (160)^{\frac{x-4}{x^{13}}}, \text{ starting at } x = 4.$$

Let $h_0 = .2, \lambda = .5$, then from the formula

$$\int_4^x y dx = 10^8 \left(1 - \frac{12}{(x/4)^{11}} + \frac{11}{(x/4)^{13}} \right)$$

* We note that: ${}_7M_1 = -\log m_1 + 5 \log m_2 - 10 \log m_3 + 10 \log m_4 - 5 \log m_5 + \log m_6$, and similarly for the other M 's.

we have the "frequencies

$$\begin{aligned} n_1 &= 33,377, & n_5 &= 123,600, \\ n_2 &= 168,900, & n_6 &= 91,023, \\ n_3 &= 189,434, & n_7 &= 65,653. \\ n_4 &= 160,671, \end{aligned}$$

Tables II and III give the following results:

λ	n_s	c_1
·1	966,781	- 368,500
·2	276,218	- 171,696
·3	111,069	- 90,182
·4	61,994	- 39,885
·5	51,108	- 3,641
·6	54,606	+ 24,479
·7	63,513	+ 47,148
·8	74,072	+ 65,788
·9	84,638	+ 81,248
1·0	94,507	+ 94,078

These results are shown in Figs. 2 and 3. It is seen that the n_s test gives (without interpolation) the two values ·3 and 1·0 for λ , while the c_1 test gives the correct value ·5.

Applying our third test we have

$$\begin{aligned} m_1 &= 33,377, & \log m_1 &= 4\cdot523\ 4473, \\ m_2 &= 202,277, & \log m_2 &= 5\cdot305\ 9465, \\ m_3 &= 391,711, & \log m_3 &= 5\cdot592\ 9658, \\ m_4 &= 552,382, & \log m_4 &= 5\cdot742\ 2395, \\ m_5 &= 675,982, & \log m_5 &= 5\cdot829\ 9351, \\ m_6 &= 767,005, & \log m_6 &= 5\cdot884\ 7983, \\ m_7 &= 832,658, & \log m_7 &= 5\cdot920\ 4667. \end{aligned}$$

Taking in each case the nearest value of λ (without interpolation) from Table I, we get the following results:

$$\begin{aligned} 3 \text{ frequencies.} & \quad \frac{{}_3M_1}{{}_3M_2} = \frac{\cdot782\ 4992}{\cdot287\ 0193} = 2\cdot7263, & \lambda &= \cdot3. \\ 4 \text{ frequencies.} & \quad \frac{{}_4M_1}{{}_4M_2} = \frac{\cdot495\ 4799}{\cdot137\ 7456} = 3\cdot5971, & \lambda &= \cdot4. \\ 5 \text{ frequencies.} & \quad \frac{{}_5M_1}{{}_5M_2} = \frac{\cdot357\ 7343}{\cdot076\ 1675} = 4\cdot6967, & \lambda &= \cdot5. \\ 6 \text{ frequencies.} & \quad \frac{{}_6M_1}{{}_6M_2} = \frac{\cdot281\ 5668}{\cdot047\ 4218} = 5\cdot9375, & \lambda &= \cdot5. \\ 7 \text{ frequencies.} & \quad \frac{{}_7M_1}{{}_7M_2} = \frac{\cdot234\ 1450}{\cdot032\ 3137} = 7\cdot246, & \lambda &= \cdot5. \end{aligned}$$

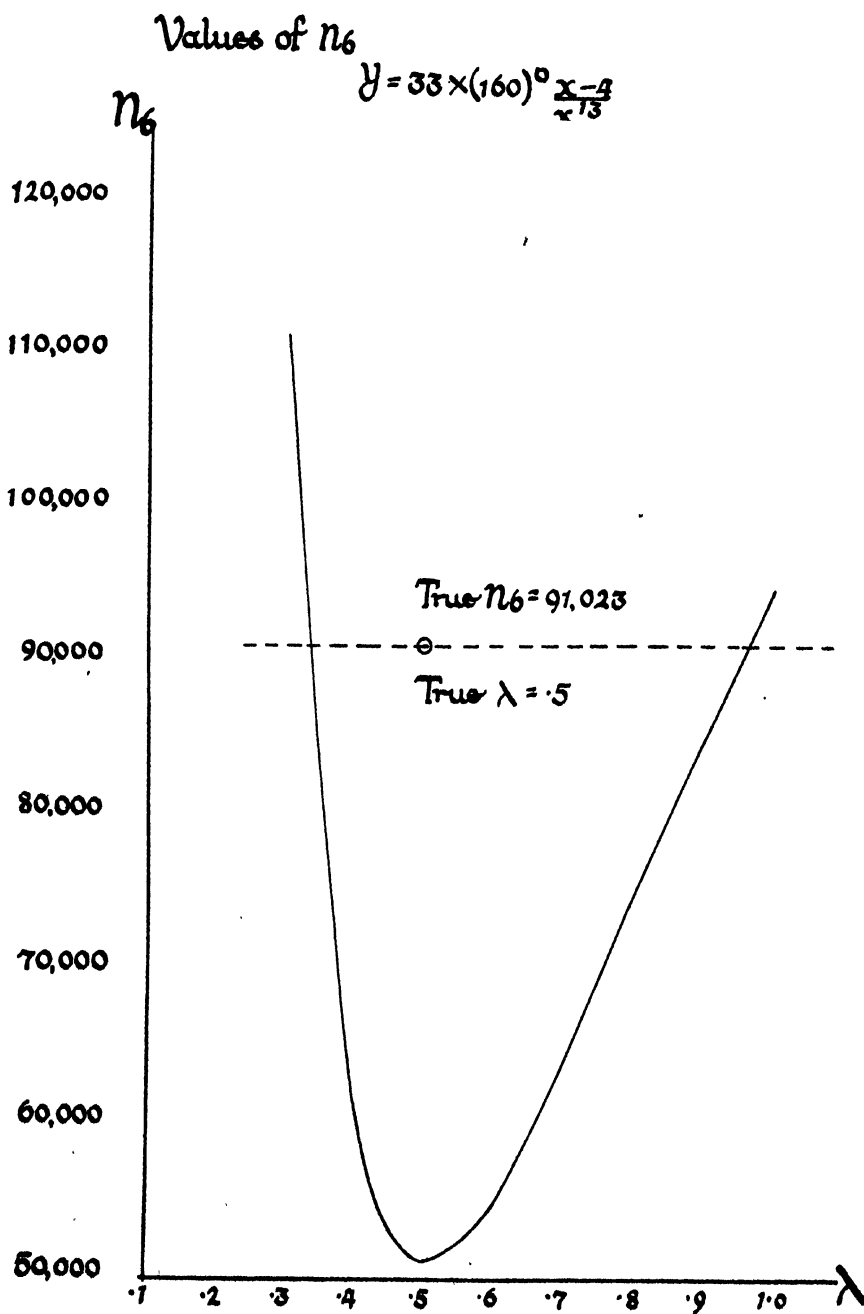


Fig. 2.

Values of C_1

$$y = 33 \times (160)^6 \frac{x-4}{x^{1/3}}$$

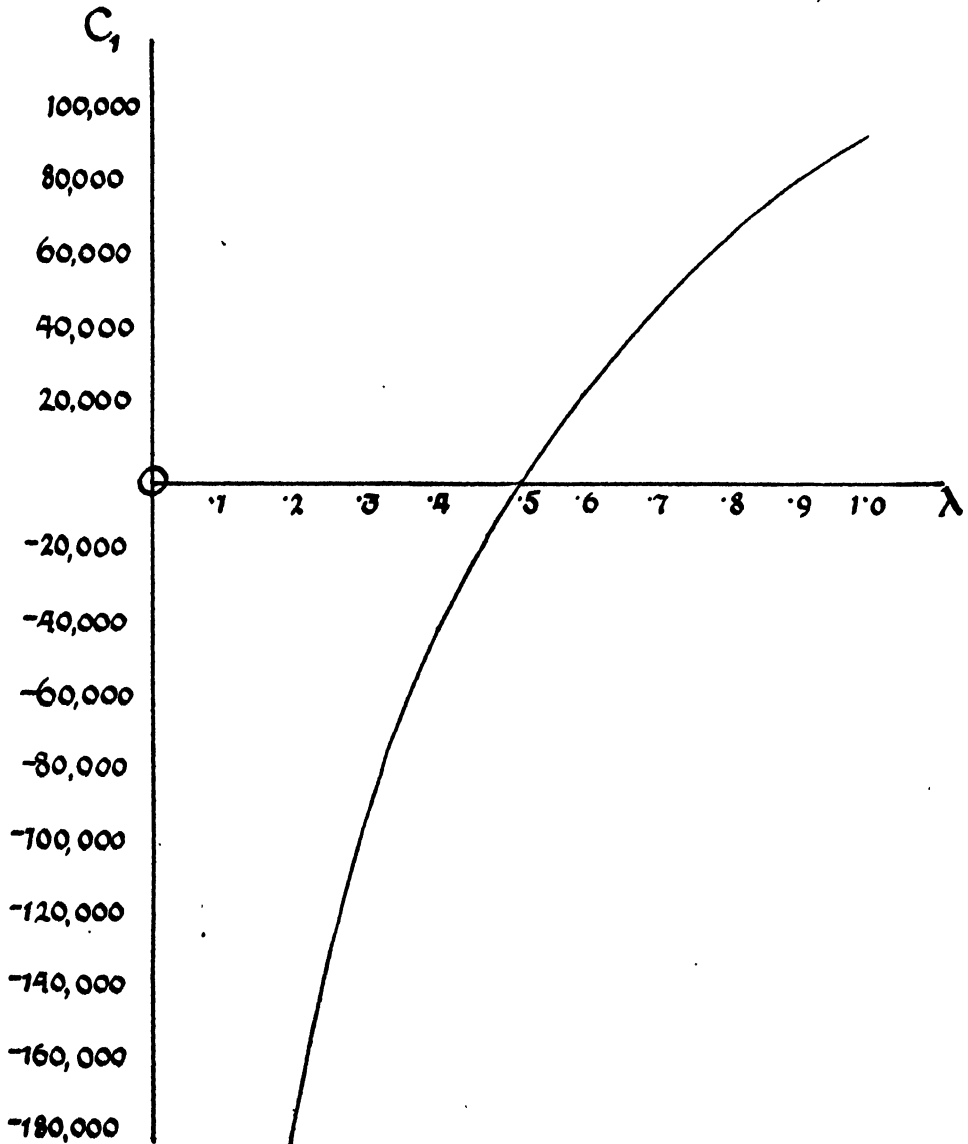


Fig. 3.

Hence the correct value of λ is obtained by the use of five frequencies.

The equation of the auxiliary parabola with the correct value .5 for λ is

$$y = 18,206 + 1,513,518 \left(\frac{x}{.2}\right) - 817,671 \left(\frac{x}{.2}\right)^2 \\ + 176,037 \left(\frac{x}{.2}\right)^3 - 14,187.94 \left(\frac{x}{.2}\right)^4.$$

The equations of the two parabolas given by the n_8 test are

$$\lambda = .3, y = 450,911 + 771,550 \left(\frac{x}{.2}\right) - 356,436 \left(\frac{x}{.2}\right)^2 \\ + 49,760.9 \left(\frac{x}{.2}\right)^3 - 1,696.00 \left(\frac{x}{.2}\right)^4,$$

$$\lambda = 1.0, y = -470,389 + 1,656,578 \left(\frac{x}{.2}\right) - 641,893 \left(\frac{x}{.2}\right)^2 \\ + 95,914.6 \left(\frac{x}{.2}\right)^3 - 5,146.46 \left(\frac{x}{.2}\right)^4.$$

These parabolas together with the original curve are shown in Fig. 4. It will be seen that the parabola for which $\lambda = 1.0$ has, besides its incorrect starting-point, another serious fault; its first frequency n_1 is composed of a negative and a positive part.

As an example of an asymptotic frequency let us take the Type III curve

$$y = \frac{10^6}{\Gamma(.9)} x^{-1} e^{-x}.$$

Take $\lambda = .8$, $h_0 = \frac{1}{2} \sqrt{.9}$, then $\lambda h_0 = .4 \sqrt{.9}$. The first seven frequencies, from the *Tables of the Incomplete Γ -Function*, are shown below:

x	u	$I(u, p)$	n	m	$\log m$
.4 $\sqrt{.9}$.4	.365 4305	365,431	365,431	5.562 8054
.9 $\sqrt{.9}$.9	.620 7737	255,343	620,774	5.792 9335
1.4 $\sqrt{.9}$	1.4	.770 3535	149,580	770,354	5.886 6903
1.9 $\sqrt{.9}$	1.9	.860 0470	89,693	860,047	5.934 5222
2.4 $\sqrt{.9}$	2.4	.914 3798	54,333	914,380	5.961 1267
2.9 $\sqrt{.9}$	2.9	.947 4827	33,103	947,483	5.976 5715
3.4 $\sqrt{.9}$	3.4	.967 7261	20,243	967,726	5.985 7524

Using the n column with Table II, we have

$$\lambda = .8, n_8 = 88,314;$$

$$\lambda = .9, n_8 = 9,583.$$

Hence, since n_8 is really 33,103, the n_8 test gives .9 as the nearest value of λ , or .87 by linear interpolation.

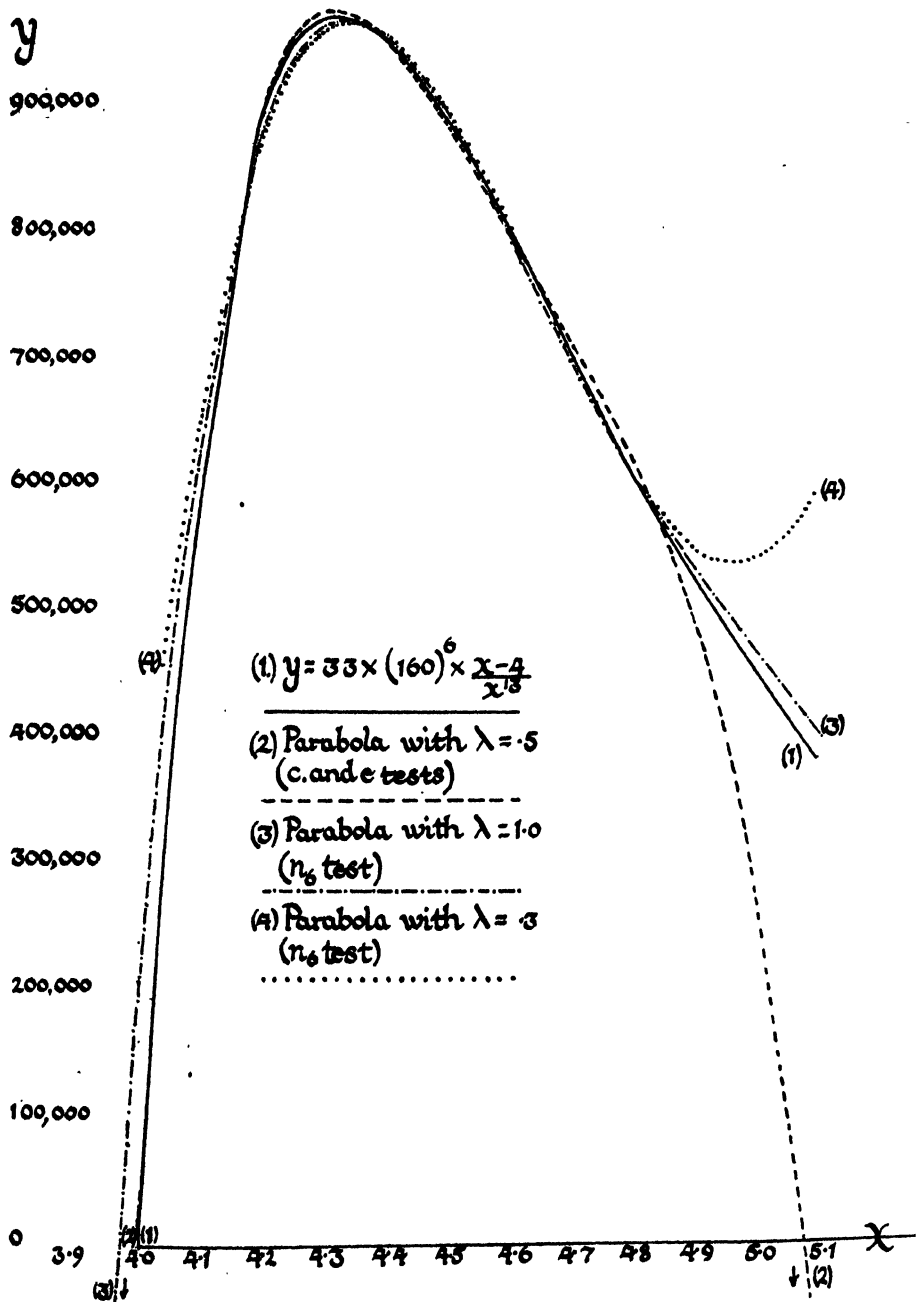


Fig. 4.

Using the log m column, we get

$$\frac{{}_3M_1}{{}_3M_2} = \frac{.230\,1281}{.093\,7568} = 2.4545, \quad \lambda = .3.$$

$$\frac{{}_4M_1}{{}_4M_2} = \frac{.136\,3713}{.045\,9249} = 2.9694, \quad \lambda = .7.$$

$$\frac{{}_5M_1}{{}_5M_2} = \frac{.090\,4464}{.024\,6975} = 3.6622, \quad \lambda = .8.$$

$$\frac{{}_6M_1}{{}_6M_2} = \frac{.065\,7489}{.014\,6298} = 4.4942, \quad \lambda = .8.$$

$$\frac{{}_7M_1}{{}_7M_2} = \frac{.051\,1191}{.009\,4579} = 5.405, \quad \lambda = .8.$$

Hence the correct value of λ is again obtained by the use of five frequencies.

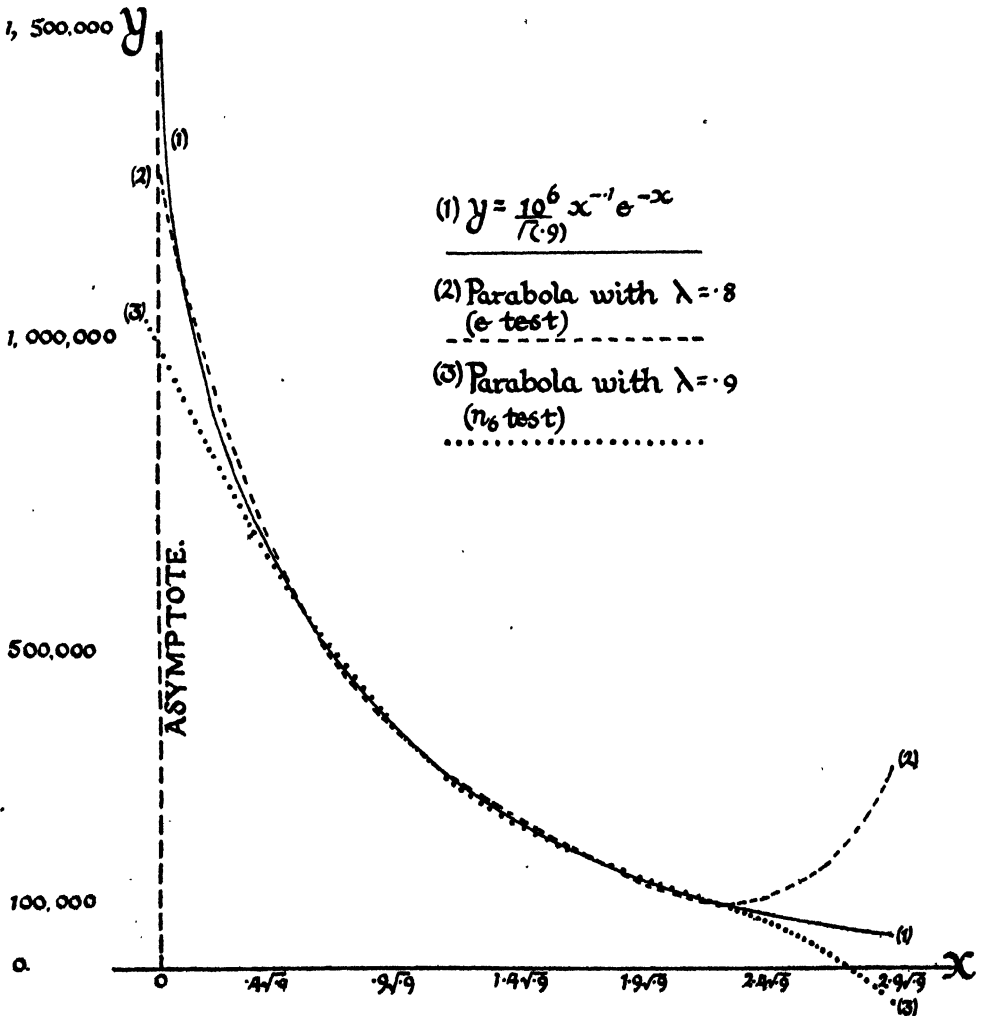


Fig. 5.

The equation of the parabola with $\lambda = \cdot 8$, referred to its start, is

$$y = 1,263,615 - 919,415 \left(\frac{x}{h_0} \right) + 356,072 \left(\frac{x}{h_0} \right)^2 - 72,594 \cdot 5 \left(\frac{x}{h_0} \right)^3 + 5,811 \cdot 89 \left(\frac{x}{h_0} \right)^4,$$

and of that with $\lambda = \cdot 9$, referred to its start, is

$$y = 1,038,998 - 438,479 \left(\frac{x}{h_0} \right) + 49,970 \cdot 5 \left(\frac{x}{h_0} \right)^2 + 5,305 \cdot 22 \left(\frac{x}{h_0} \right)^3 - 1,104 \cdot 01 \left(\frac{x}{h_0} \right)^4.$$

These, together with the original curve, are shown in Fig. 5. It is seen that there is very close agreement between the curves for frequencies n_2, n_3, n_4, n_5 and that there is a good attempt at agreement at the start, especially in the parabola with the correct value of λ .

The curves of our e -test would no doubt fit still better, and there is in fact some inconsistency in using these curves to determine the value of λ and then reverting to our original parabola. But it will be shown in the following examples that if the correct value of λ be determined, the corrections given to the moments by the parabola are usually adequate. Further, if the e -curves were used for correcting the moments, the K 's would have to be worked out *ab initio* for each example, owing to the impracticability of tabling for values of both λ and q . There is the further objection that with these curves the abruptness coefficients tend to diverge. We shall therefore be content to correct the moments by the use of the tables at the end of this paper, no matter how the value of λ may be determined.

4. We will now investigate, in some further distributions, the extent to which the moments are corrected by the method of this paper.

Example 1. Consider the curve $y = 156 \times 10^6 x(1-x)^{11}$ from 0 to 1, which has a total "frequency" 1,000,000. We will take our first sub-range as $\cdot 02$ starting from $x = 0$ and the other sub-ranges as $\cdot 06$; thus $\lambda = \frac{1}{3}$. The frequencies are as follows:

x	Frequency	x	Frequency
0— $\cdot 02$	26,951,	$\cdot 38$ — $\cdot 44$	11,965,
$\cdot 02$ — $\cdot 08$	252,423,	$\cdot 44$ — $\cdot 50$	4,264,
$\cdot 08$ — $\cdot 14$	281,978,	$\cdot 50$ — $\cdot 56$	1,302,
$\cdot 14$ — $\cdot 20$	205,002,	$\cdot 56$ — $\cdot 62$	330,
$\cdot 20$ — $\cdot 26$	122,555,	$\cdot 62$ — $\cdot 68$	66,
$\cdot 26$ — $\cdot 32$	63,781,	$\cdot 68$ — $\cdot 74$	10,
$\cdot 32$ — $\cdot 38$	29,372,	$\cdot 74$ — $\cdot 80$	1.

We have first to find λ .

(i) The n_6 test. Table II gives:

when $\lambda = \cdot 2$, $n_6 = 76,551$; when $\lambda = \cdot 3$, $n_6 = - 2,153$;

when $\lambda = \cdot 6$, $n_6 = 53,936$; when $\lambda = \cdot 7$, $n_6 = 81,889$.

Since n_6 is really 63,781, this test gives the two values $\cdot 2$ and $\cdot 6$ for λ .

(ii) The c_1 test. Table III gives:

when $\lambda = \cdot 3$, $c_1 = -22,007$; when $\lambda = \cdot 4$, $c_1 = 30,348$.

Hence c_1 is most nearly equal to zero when $\lambda = \cdot 3$, or $\cdot 34$ by linear interpolation. This is a good result.

(iii) The e test. We have

$m_1 = 26,951,$	$\log m_1 = 4.430\ 5749,$
$m_2 = 279,374,$	$\log m_2 = 5.446\ 1860,$
$m_3 = 561,352,$	$\log m_3 = 5.749\ 2352,$
$m_4 = 766,354,$	$\log m_4 = 5.884\ 4295,$
$m_5 = 888,909,$	$\log m_5 = 5.948\ 8573,$
$m_6 = 952,690,$	$\log m_6 = 5.978\ 9516,$
$m_7 = 982,062,$	$\log m_7 = 5.992\ 1389.$

Whence, using Table I we have

$$\frac{{}_3M_1}{{}_3M_2} = 3.35131, \quad \lambda = \cdot 15-,$$

$$\frac{{}_4M_1}{{}_4M_2} = 4.24511, \quad \lambda = \cdot 31,$$

$$\frac{{}_5M_1}{{}_5M_2} = 5.6104, \quad \lambda = \cdot 35+,$$

$$\frac{{}_6M_1}{{}_6M_2} = 7.3797, \quad \lambda = \cdot 34,$$

$$\frac{{}_7M_1}{{}_7M_2} = 9.291, \quad \lambda = \cdot 333$$

The values of λ are obtained by linear interpolation. It is seen that 4 frequencies give a value correct to 1 significant figure while 7 frequencies give a very exact value.

Let us take $\lambda = \cdot 3$ and test Table IV to find the accuracy with which our auxiliary curve gives the moments of the first frequency about $x = \cdot 02$. We get the following results:

	From Table	True value
$n_1\mu_1''$	-185.36	-186.44
$n_1\mu_2''$	+ 1.868	+ 1.906
$n_1\mu_3''$	- .022 25	- .023 21
$n_1\mu_4''$	+ .000 2914	+ .000 3126

It is seen that the first moment is in error in the units figure, but since the total first moment of the whole curve about $x = \cdot 02$ is 122,857 $\frac{1}{2}$, this error will not seriously affect the results. Similarly the total second moment about $x = \cdot 02$ is 23,257 $\frac{1}{2}$, and hence the error in the second moment will not affect the sixth significant figure. Also since the total third and fourth moments are respectively 5592 and 1593.572, we should again get at least six figures correct on adding the

corresponding moments of the first frequency. We can therefore say that the moments of the first frequency as given by Table IV are sufficiently accurate.

We will now find the corrected moments of the total frequency about $x = .02$, taking $h = 1$ as our working sub-range. With the notation of Equations III (p. 15) above,

$$\begin{aligned}(N - n_1) \nu_1''' &= 2,063,961.5, & (N - n_1) \nu_3''' &= 26,400,593.4, \\ (N - n_1) \nu_2''' &= 6,538,164.25, & (N - n_1) \nu_4''' &= 126,209,278.6.\end{aligned}$$

The values of the K 's, obtained by using $\lambda = .3$ in Table VI, are

$$\begin{aligned}K_1 &= -16,220.0, & K_3 &= +4605.9, \\ K_2 &= +3257.5, & K_4 &= -1955.0.\end{aligned}$$

The corrected moments are therefore

$$\begin{aligned}N\mu_1' &= 2,063,961.5 - 16,220.0 = 2,047,742, \\ N\mu_2' &= 6,538,164.25 - 81,087.42 + 3257.5 = 6,460,334, \\ N\mu_3' &= 26,400,593.4 - 515,990.4 + 4605.9 = 25,889,209, \\ N\mu_4' &= 126,209,278.6 - 3,269,082.1 + 28,380.6 - 1955.0 = 122,966,622,\end{aligned}$$

whence
$$\begin{aligned}\mu_1' &= 2.047\ 742, & \mu_3' &= 25.889\ 209, \\ \mu_2' &= 6.460\ 334, & \mu_4' &= 122.966\ 622.\end{aligned}$$

Transferring to the mean as here given, viz., 2.047 742 from the start of the second frequency, we obtain

$$\begin{aligned}\mu_2 &= 2.267\ 087, \\ \mu_3 &= 3.375\ 294, \\ \mu_4 &= 20.697\ 620.\end{aligned}$$

Now the true moments about the mean are easily found from the equation of the curve, and can be brought to the working scale by dividing μ_s by $(.06)^s$. They are shown in the following comparative table:

	By present method	True value	% error
Mean from start of } second frequency }	2.047 742	2.047 619	+ .006
μ_2	2.267 087	2.267 574	-.021
μ_3	3.375 294	3.374 366	+ .028
μ_4	20.697 620	20.699 890	-.011

It is seen that the percentage errors in all four moments are very small.

It is of interest to compare these results with those which would be obtained by supposing the first frequency group on the same base as the others and finding the moments about its supposed start. We shall use the abruptness coefficients as given in the paper by Fairman and Pearson cited above. The raw moment-coefficients are

$$\begin{aligned}\nu_1' &= 3.050\ 486, & \nu_3' &= 53.183\ 388, \\ \nu_2' &= 11.645\ 874, & \nu_4' &= 280.271\ 217.\end{aligned}$$

These, if taken without correction, give the following results:

Mean from start of second frequency = 2·050 486, error = +·140%,

$$\nu_2 = 2\cdot340\ 409, \text{ error} = +3\cdot212\%,$$

$$\nu_3 = 3\cdot379\ 081, \text{ error} = +\cdot140\%,$$

$$\nu_4 = 21\cdot776\ 860, \text{ error} = +5\cdot203\%.$$

Applying Sheppard's corrections we get (ν_1' and ν_3 being uncorrected)

$$\nu_2' - \frac{1}{12} = 11\cdot562\ 541,$$

$$\nu_3' - \frac{1}{4} \nu_1' = 52\cdot420\ 767,$$

$$\nu_4' - \frac{1}{2} \nu_2' + \frac{7}{240} = 274\cdot477\ 447.$$

With these corrections alone, we get

$$\mu_2 = 2\cdot257\ 076, \text{ error} = -\cdot463\%,$$

$$\mu_4 = 20\cdot635\ 988, \text{ error} = -\cdot309\%,$$

which are a good improvement on the raw moment results.

Now the usual abruptness corrections* work out to be

$$H_1 = -\cdot013\ 828, \quad H_3 = -\cdot003\ 662,$$

$$H_2 = -\cdot004\ 091, \quad H_4 = -\cdot003\ 872,$$

whence the fully corrected moment-coefficients are

$$\mu_1' = \nu_1' - H_1 = 3\cdot064\ 314,$$

$$\mu_2' = \nu_2' - \frac{1}{12} - H_2 = 11\cdot566\ 632,$$

$$\mu_3' = \nu_3' - \frac{1}{4} \nu_1' + H_3 = 52\cdot417\ 105,$$

$$\mu_4' = \nu_4' - \frac{1}{2} \nu_2' + \frac{7}{240} + H_4 = 274\cdot473\ 575.$$

Hence H_1 acts the wrong way and gives a worse value for the mean than the raw first moment:

Mean from start of second frequency = 2·064 314, error = +·815%.

Referred to this mean,

$$\mu_2 = 2\cdot176\ 612, \text{ error} = -4\cdot011\%,$$

$$\mu_3 = 3\cdot633\ 669, \text{ error} = +7\cdot685\%,$$

$$\mu_4 = 19\cdot131\ 702, \text{ error} = -7\cdot576\%.$$

These are obviously much worse than the raw moments alone.

Thus we have the remarkable result that Sheppard's corrections, which are obviously inapplicable since there is not high contact at the start, give very good results, while the usual abruptness corrections give very bad results. Now it is shown in the paper by Pairman and Pearson that the reverse is often the case, and

* F. Sandon, *Biometrika*, Vol. xvi. pp. 193—195 (with an error in the value for μ_4'), and *Tables for Statisticians and Biometricians*, Part II. p. xciii, Introduction (error corrected).

hence one must conclude that the present result is due to the incorrect value given to the base of the first frequency. This in fact has cancelled those errors in the moments as given by Sheppard's corrections and in the raw first moment, which are normally corrected by the abruptness corrections. To prove this, one may omit the first frequency and examine the moments of the remainder about its start. The results are as follows:

Moments of the remainder of the frequency about its start.

	True value	Raw moment	With Sheppard	With abruptness correction
$(N - n_1) \mu_1'''$	2 050 726	2 063 961	2 063 961	2 049 890
$(N - n_1) \mu_2'''$	6 459 788	6 538 164	6 457 077	6 459 401
$(N - n_1) \mu_3'''$	25 888 986	26 400 593	25 884 603	25 889 420
$(N - n_1) \mu_4'''$	122 960 760	126 209 279	122 968 578	122 966 457

Thus we see that when the first frequency is excluded from consideration, the raw moments are improved by Sheppard's corrections, but the abruptness corrections make a still further improvement in all four moments. Hence the previous unusual result was due entirely to the effect of the first frequency.

But obviously Sheppard's corrections alone cannot always be relied on to give good results where their use is not justifiable, and since using the abruptness corrections for full first range may actually be worse than the raw moments, and give errors of more than 7%, the need for our present method is clearly demonstrated.

Lastly we may inquire whether the value .3 taken for λ can be improved. Using the four moments obtained by the present method and fitting a Type I curve in the usual way, we obtain the following equation, reduced to the original scale:

$$y = 154.956 \times 10^6 \left(\frac{x}{.9995} \right)^{.9975} \left(1 - \frac{x}{.9995} \right)^{10.9685}.$$

Distance from start of curve to mean = $2.3793 \times .06$, distance from start of second frequency to mean = $2.0477 \times .06$, hence $\lambda = .3316$ and the base of the first frequency is .0199.

Hence in fitting a curve it is best not to fix the start at the value given by our tables, but to fit with four moments.

Example 2. The curve in the last example was one in which our tables could reasonably be expected to give a good result, as its equation could be expanded in the form of the auxiliary curve. We will now take a curve whose equation cannot be so expanded,

$$y = y_0 x^{0.1} (1 - x)^5.$$

If $N = 10^6$ as before, $y_0 = 7,611,605.925$. Take $h = .08$, $\lambda = .5$, then $\lambda h = .04$.

The "frequencies" are as follows:

x	n_x
0—·04	180,698
·04—·12	311,260
·12—·20	212,991
·20—·28	134,844
·28—·36	79,796
·36—·44	43,790
·44—·52	21,913
·52—·60	9,726
·60—·68	3,663
·68—·76	1,082
·76—·84	216
·84—·92	21
·92—1·0	0
Total	1,000,000

(i) The n_s test. Table II gives:

when $\lambda = \cdot 4$, $n_s = 198,469$, when $\lambda = \cdot 5$, $n_s = -19,300$.

Hence, since n_s is really 43,790, $\lambda = \cdot 5$ is the nearest value and this is the correct value.

(ii) The c_1 test. From Table III we have:

when $\lambda = \cdot 8$, $c_1 = -66,710$; when $\lambda = \cdot 9$, $c_1 = +6,469$.

Hence this test gives $\lambda = \cdot 9$, a bad result.

(iii) The e test. The results are as follows:

Using 3 frequencies, $\lambda = \cdot 24$,

" 4 " $\lambda = \cdot 47$,

" 5 " $\lambda = \cdot 52$,

" 6 " $\lambda = \cdot 50$,

" 7 " $\lambda = \cdot 50$.

Hence again four frequencies are sufficient to give the value of λ correct to one figure.

Taking $\lambda = \cdot 5$ and carrying out the work as before, with $h = 1$ as working unit, we get

$$(N - n_1) \nu_1''' = 1,511,250, \quad (N - n_1) \nu_3''' = 19,012,399\cdot 5,$$

$$(N - n_1) \nu_2''' = 4,643,887\cdot 5, \quad (N - n_1) \nu_4''' = 92,962,029.$$

$$K_1 = -74,940\cdot 3, \quad K_3 = +3318\cdot 0,$$

$$K_2 = +14,650\cdot 5, \quad K_4 = +2832\cdot 7.$$

$$N\mu_1' = 1,511,250 - 74,940 = 1,436,310,$$

$$N\mu_2' = 4,643,887.5 - 68,275.2 + 14,650.5 = 4,590,263,$$

$$N\mu_3' = 19,012,399.5 - 377,812.5 + 3318.0 = 18,637,905,$$

$$N\mu_4' = 92,962,029 - 2,321,944 + 23,896 + 2833 = 90,666,814.$$

$$\mu_1' = 1.436\ 310,$$

$$\mu_2' = 18.637\ 905,$$

$$\mu_3' = 4.590\ 263,$$

$$\mu_4' = 90.666\ 814.$$

The last three moments are now referred to the mean and give the following comparison with the true values:

	True value	By present method	Error %
Mean from start of second frequency }	1.436 620	1.436 310	- .022
μ_2	2.525 587	2.527 277	+ .067
μ_3	4.788 490	4.784 959	- .074
μ_4	27.616 548	27.637 739	+ .077

Let us again compare these results with those obtained when the first frequency is placed on the same base as the other frequencies, taking the origin at the supposed start of the first frequency. We get

$$\nu_1' = 2.420\ 901,$$

$$\nu_2' = 38.319\ 701,$$

$$\nu_3' = 8.530\ 864,$$

$$\nu_4' = 203.750\ 548.$$

These if taken without correction give:

$$\text{Mean from start of second frequency} = 1.420\ 901, \text{ error} = -1.094\%,$$

$$\nu_2 = 2.670\ 102, \text{ error} = +5.722\%,$$

$$\nu_3 = 4.739\ 218, \text{ error} = -1.029\%,$$

$$\nu_4 = 29.616\ 321, \text{ error} = +7.241\%.$$

Applying Sheppard's corrections,

$$\nu_1' = 2.420\ 901,$$

$$\nu_2' - \frac{1}{12} = 8.447\ 531,$$

$$\nu_3' - \frac{1}{4}\nu_1' = 37.714\ 476,$$

$$\nu_4' - \frac{1}{2}\nu_2' + \frac{7}{240} = 199.514\ 283.$$

Taking these as true moment-coefficients and transferring to the mean, we get

$$\mu_2 = 2.586\ 769, \text{ error} = +2.422\%,$$

$$\mu_3 = \nu_3 = 4.739\ 218, \text{ error} = -1.029\%,$$

$$\mu_4 = 28.310\ 447, \text{ error} = +2.513\%.$$

Hence Sheppard's corrections make a considerable improvement in the second and fourth moments, but the errors are still large; the good results which they gave in the first example were due to a good luck which is not repeated here.

The Pairman-Pearson abruptness corrections are

$$H_1 = -\cdot004\ 598, \quad H_3 = +\cdot000\ 216,$$

$$H_2 = -\cdot006\ 360, \quad H_4 = +\cdot005\ 776,$$

whence the fully corrected moment-coefficients are

$$\mu_1' = 2\cdot425\ 499, \quad \mu_3' = 37\cdot714\ 692,$$

$$\mu_2' = 8\cdot453\ 891, \quad \mu_4' = 199\cdot520\ 059.$$

The mean is thus 1·425 499 from the start of the second frequency, the error being $-\cdot774\%$. We get also

$$\mu_1 = 2\cdot570\ 846, \quad \text{error} = +1\cdot792\%,$$

$$\mu_2 = 4\cdot738\ 621, \quad \text{error} = -1\cdot041\%,$$

$$\mu_3 = 28\cdot189\ 347, \quad \text{error} = +2\cdot074\%.$$

These results are an improvement on the results with Sheppard's correction only, but are considerably worse than those obtained by the method of this paper.

Example 3. We will now try our corrections on an asymptotic frequency distribution.

For this purpose let us take the Type III curve

$$y = \frac{10^6}{\Gamma(\cdot8)} x^{-2} e^{-x}.$$

Taking $h = \sqrt{\cdot8}$, $\lambda = \cdot5$, we get, from the *Tables of the Incomplete Γ -Function*:

x	u	$I(u, p)$	Frequencies
$\frac{1}{2}\sqrt{\cdot8}$	$\cdot5$	$\cdot466\ 3930$	466,393
$1\cdot5\sqrt{\cdot8}$	1·5	$\cdot807\ 4006$	341,008
$2\cdot5\sqrt{\cdot8}$	2·5	$\cdot926\ 7582$	119,358
$3\cdot5\sqrt{\cdot8}$	3·5	$\cdot971\ 5767$	44,818
$4\cdot5\sqrt{\cdot8}$	4·5	$\cdot988\ 8457$	17,269
$5\cdot5\sqrt{\cdot8}$	5·5	$\cdot995\ 5913$	6,746
$6\cdot5\sqrt{\cdot8}$	6·5	$\cdot998\ 2487$	2,657
$7\cdot5\sqrt{\cdot8}$	7·5	$\cdot999\ 3018$	1,053
$8\cdot5\sqrt{\cdot8}$	8·5	$\cdot999\ 7208$	419
$9\cdot5\sqrt{\cdot8}$	9·5	$\cdot999\ 8881$	167
$10\cdot5\sqrt{\cdot8}$	10·5	$\cdot999\ 9551$	67
$11\cdot5\sqrt{\cdot8}$	11·5	$\cdot999\ 9820$	27
$12\cdot5\sqrt{\cdot8}$	12·5	$\cdot999\ 9927$	11
$13\cdot5\sqrt{\cdot8}$	13·5	$\cdot999\ 9971$	4
$14\cdot5\sqrt{\cdot8}$	14·5	$\cdot999\ 9988$	2
$15\cdot5\sqrt{\cdot8}$	15·5	$\cdot999\ 9995$	1
		Total	1,000,000

(i) The n_6 test. Table II gives:

when $\lambda = \cdot6$, $n_6 = 101,244$; when $\lambda = \cdot7$, $n_6 = -133,346$.

Hence, since n_6 is really 6,746, this test gives $\lambda = \cdot6$.

(ii) The ϵ test. This test gives :

Using 3 frequencies, $\lambda = \cdot 1$,

„ 4 „ $\lambda = \cdot 3$,

„ 5 „ $\lambda = \cdot 4$,

„ 6 „ $\lambda = \cdot 5$,

„ 7 „ $\lambda = \cdot 5$.

Hence six frequencies give the correct value of λ .

We find, with $h = 1$ as working unit,

$$(N - n_1) \nu_1''' = 579,542\cdot 5, \quad (N - n_1) \nu_3''' = 3,632,652\cdot 625,$$

$$(N - n_1) \nu_2''' = 1,154,247\cdot 75, \quad (N - n_1) \nu_4''' = 15,517,049\cdot 4375.$$

With $\lambda = \cdot 5$, Table VI gives

$$K_1 = -186,671\cdot 9, \quad K_3 = -4,253\cdot 5,$$

$$K_2 = +38,430\cdot 9, \quad K_4 = +13,794\cdot 1.$$

Applying the full corrections we get

$$\mu_1' = \cdot 397\ 8706, \quad \mu_3' = 3\cdot 483\ 5135,$$

$$\mu_2' = 1\cdot 148\ 2114, \quad \mu_4' = 14\cdot 969\ 2831,$$

whence the moment-coefficients about the mean are

$$\mu_2 = \cdot 989\ 910,$$

$$\mu_3 = 2\cdot 238\ 961,$$

$$\mu_4 = 10\cdot 440\ 733.$$

Now h has been taken as 1, whereas it should have been $\sqrt{\cdot 8}$. Hence we multiply μ_s by $(\sqrt{\cdot 8})^s$ and get the following results :

$$\mu_1' = \cdot 355\ 8663, \quad \mu_3 = 1\cdot 602\ 070,$$

$$\mu_2 = \cdot 791\ 928, \quad \mu_4 = 6\cdot 682\ 069.$$

Now the true mean is $\cdot 8$ from the start of the curve, or $\cdot 8 - \lambda h = \cdot 352\ 7864$ from the start of the second frequency. Thus the error in the mean as here obtained is $+0\cdot 87\%$.

The true values of μ_2 , μ_3 and μ_4 are $\cdot 8$, $1\cdot 6$ and $6\cdot 72$ respectively. Hence the errors in our values are $-1\cdot 01\%$, $+0\cdot 13\%$ and $-0\cdot 56\%$ respectively.

These results, while not as good as those obtained for non-asymptotic frequencies, are still very reasonable, especially when it is remembered that nearly half the total frequency is contained in the first group.

We will again compare these results with those obtained by assuming the first sub-range to be equal to the others.

On trying Miss Pearson's method we find no value of q , that is, the method fails to indicate an asymptotic distribution.

The raw moment-coefficients about the supposed start of the first frequency are

$$\begin{aligned}\nu_1' &= 1.346\ 346, & \nu_3' &= 9.425\ 9295, \\ \nu_2' &= 2.963\ 538, & \nu_4' &= 39.854\ 073.\end{aligned}$$

The uncorrected moments about the mean work out to be

$$\nu_2 = 1.150\ 890, \quad \nu_3 = 2.336\ 990, \quad \nu_4 = 11.465\ 859.$$

Reducing to the base $h = \sqrt{8}$, we get:

Mean from start of second frequency $= .346\ 346 \times \sqrt{8} = .309\ 781$; error $= -12.2\%$,

$$\begin{aligned}\nu_2 &= 1.150\ 890 \times .8 = .920\ 7120; & \text{error} &= +15.1\%, \\ \nu_3 &= 2.336\ 990 \times (\sqrt{8})^3 = 1.672\ 214; & \text{error} &= +4.51\%, \\ \nu_4 &= 11.465\ 859 \times .64 = 7.338\ 150; & \text{error} &= +9.20\%.\end{aligned}$$

The percentage errors given are those which would be made if these results were taken instead of the true moment-coefficients.

Now applying Sheppard's corrections, we have

$$\begin{aligned}\mu_1' &= 1.346\ 346, \\ \mu_2' &= 2.880\ 205, \\ \mu_3' &= 9.089\ 343, \\ \mu_4' &= 38.375\ 221,\end{aligned}$$

giving, with base $\sqrt{8}$,

$$\begin{aligned}\mu_2 &= 1.067\ 557 \times .8 = .854\ 0456; & \text{error} &= +6.76\%, \\ \mu_3 &= 2.336\ 988 \times (\sqrt{8})^3 = 1.672\ 214; & \text{error} &= +4.51\%, \\ \mu_4 &= 10.893\ 331 \times .64 = 6.971\ 732; & \text{error} &= +3.75\%.\end{aligned}$$

The mean is as above, with an error of -12.2% .

Hence Sheppard's corrections make a good improvement in the second and fourth moments.

Applying now the abruptness corrections on the assumption that $h_0 = h$, we obtain

$$\begin{aligned}H_1 &= +.032\ 0855, & H_3 &= +.011\ 2586, \\ H_2 &= -.003\ 6942, & H_4 &= -.003\ 1652.\end{aligned}$$

Hence $\mu_1' = \nu_1' - H_1 = 1.314\ 2605$, and H_1 has again acted the wrong way. The mean is now $.314\ 2605 \times \sqrt{8} = .281\ 0832$ from the start of the second frequency and is in error to the extent of 20.3% . We have also

$$\mu_2' = 2.883\ 899, \quad \mu_3' = 9.100\ 602, \quad \mu_4' = 38.372\ 055,$$

whence, reduced to base $\sqrt{8}$, the moment-coefficients about the mean are

$$\begin{aligned}\mu_2 &= 1.156\ 618 \times .8 = .925\ 294; & \text{error} &= +15.7\%, \\ \mu_3 &= 2.270\ 212 \times (\sqrt{8})^3 = 1.624\ 431; & \text{error} &= +1.53\%, \\ \mu_4 &= 11.467\ 132 \times .64 = 7.338\ 964; & \text{error} &= +9.21\%.\end{aligned}$$

Thus three of these moments are actually worse than the raw moments. The results of this example are shown in the following table.

$$\text{Moments of Asymptotic Curve } y = \frac{10^6}{\Gamma(8)} x^{-2} e^{-x}.$$

	Mean, from $x = \frac{1}{2} \sqrt{8}$	μ_2	μ_3	μ_4
True value ...	352 786	8	1.6	6.72
Present method ...	355 866	791 928	1.602 070	6.682 069
Raw moments ...	309 781	920 712	1.672 214	7.338 150
With Sheppard ...	309 781	854 046	1.672 214	6.971 732
Assumption, $h = h_0$	281 083	925 294	1.624 431	7.338 964

Example 4. Age incidence of Whooping Cough cases in Metropolitan Boroughs in 1912:

Under 1 year	212	} 1463.
1—2 years	427	
2—3 „	344	
3—4 „	270	
4—5 „	210	
5—10 „	253	
10—15 „	14	
Total	1730	

One case between 40 and 45 years has been omitted.

Now here we have the difficulty, common in such statistics, that after the first five years the frequencies are given only for 5-yearly groupings. To get λ from the n_6 test a knowledge of the frequency in the sixth year is required; further, in finding the raw moments omitting the first frequency, the sub-ranges must be equal, and this is impossible as the frequencies stand at present. We have therefore to make an estimate of the frequency in the sixth year, and either arrange the remaining frequency in groups 6—11 years, 11—16 years, etc., or split it entirely into yearly groupings. In the present case, as there is not a long tail, the second method is feasible, and may be performed as follows:

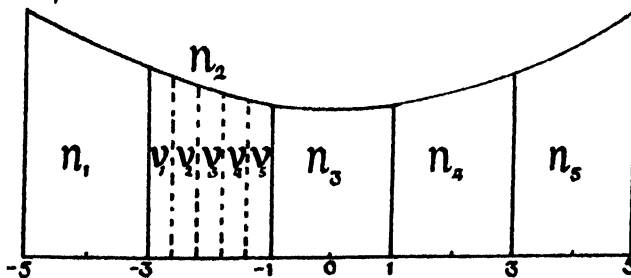


Fig. 6.

Let us take five frequencies n_1, n_2, n_3, n_4, n_5 on equal ranges of 2 units, and fit the parabola

$$\int_0^x y dx = ax + bx^2 + cx^3 + dx^4 + ex^5$$

to them. We get, taking the origin as in the figure,

$$\begin{aligned} n_1 &= \int_{-5}^{-3} y dx = 2a - 16b + 98c - 544d + 2882e, \\ n_2 &= 2a - 8b + 26c - 80d + 242e, \\ n_3 &= 2a + 2c + 2e, \\ n_4 &= 2a + 8b + 26c + 80d + 242e, \\ n_5 &= 2a + 16b + 98c + 544d + 2882e, \end{aligned}$$

whence

$$\begin{aligned} 3840a &= 9n_1 - 116n_2 + 2134n_3 - 116n_4 + 9n_5, \\ 384b &= 5n_1 - 34n_2 + 34n_4 - 5n_5, \\ 384c &= -n_1 + 12n_2 - 22n_3 + 12n_4 - n_5, \\ 768d &= -n_1 + 2n_2 - 2n_4 + n_5, \\ 3840e &= n_1 - 4n_2 + 6n_3 - 4n_4 + n_5. \end{aligned}$$

We now divide n_2 into five frequencies $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ on equal bases, getting

$$\begin{aligned} \nu_1 &= \int_{-3}^{-13/5} y dx = \frac{1}{15625} (399n_1 + 4054n_2 - 1931n_3 + 729n_4 - 126n_5), \\ \nu_2 &= \int_{-13/5}^{-11/5} y dx = \frac{1}{15625} (69n_1 + 3849n_2 - 1086n_3 + 349n_4 - 56n_5), \\ \nu_3 &= \int_{-11/5}^{-9/5} y dx = \frac{1}{15625} (-111n_1 + 3319n_2 - 41n_3 - 56n_4 + 14n_5), \\ \nu_4 &= \int_{-9/5}^{-7/5} y dx = \frac{1}{15625} (-181n_1 + 2599n_2 + 1039n_3 - 401n_4 + 69n_5), \\ \nu_5 &= \int_{-7/5}^{-1} y dx = \frac{1}{15625} (-176n_1 + 1804n_2 + 2019n_3 - 621n_4 + 99n_5). \end{aligned}$$

To apply these formulae to the Whooping Cough statistics, we suppose for this purpose that the frequency starts at 0 years, and we have the five frequencies 1463, 253, 14, 0, 0 on equal bases. Then the frequency 253 divides into 101, 68, 43, 26, 15.

If we take the five frequencies 253, 14, 0, 0, 0 and apply the above formulae to split up the 14 frequency, unsatisfactory results are obtained, some of the resulting frequencies being negative. We accordingly take the same five frequencies as before and divide n_2 into $\nu_1', \nu_2', \nu_3', \nu_4', \nu_5'$ by the formulae

$$\begin{aligned} \nu_1' &= \int_{-1}^{-3/5} y dx = \frac{1}{15625} (-126n_1 + 1029n_2 + 2794n_3 - 671n_4 + 99n_5), \\ \nu_2' &= \int_{-3/5}^{-1/5} y dx = \frac{1}{15625} (-56n_1 + 349n_2 + 3289n_3 - 526n_4 + 69n_5), \end{aligned}$$

$$\nu_3' = \int_{-1/5}^{+1/5} y dx = \frac{1}{15625} (14n_1 - 181n_2 + 3459n_3 - 181n_4 + 14n_5),$$

$$\nu_4' = \int_{1/5}^{3/5} y dx = \frac{1}{15625} (69n_1 - 526n_2 + 3289n_3 + 349n_4 - 56n_5),$$

$$\nu_5' = \int_{3/5}^1 y dx = \frac{1}{15625} (99n_1 - 671n_2 + 2794n_3 + 1029n_4 - 126n_5).$$

These divide the frequency of 14 roughly into 7, 3, 2, 1, 1.

We thus have the revised data:

Under 1 year	212
1—2 years	427
2—3 "	344
3—4 "	270
4—5 "	210
5—6 "	101
6—7 "	68
7—8 "	43
8—9 "	26
9—10 "	15
10—11 "	7
11—12 "	3
12—13 "	2
13—14 "	1
14—15 "	1
Total	1730

(i) The n_6 test. Table II gives:

when $\lambda = \cdot 4$, $n_6 = 269\cdot 840$; when $\lambda = \cdot 5$, $n_6 = 30\cdot 555$.

Hence $\cdot 5$ is the nearest value for λ .

(ii) The c_1 test. Table III gives:

when $\lambda = \cdot 8$, $c_1 = -33\cdot 903$; when $\lambda = \cdot 9$, $c_1 = +53\cdot 638$.

Hence this gives $\lambda = \cdot 8$.

(iii) The e test gives the following results:

Using 3 frequencies, $\lambda = \cdot 3$,		
" 4	"	$\lambda = \cdot 4$,
" 5	"	$\lambda = \cdot 5$,
" 6	"	$\lambda > 1\cdot 0$,
" 7	"	$\lambda > 1\cdot 5$.

We have to choose from these the best value of λ , i.e. the value which gives the best fit to the total frequency. We may reject, perhaps, in this case the values of λ greater than unity, but there is no guide as to which of the other values is best.

Accordingly, curves have been fitted for the values .5 and .8, using four moments. In each case a Type I curve is indicated by the β 's and the value .8 gives a better fit than the value .5.

The fitting for $\lambda = .8$ is, with the usual notation, as follows:

About the start of the second frequency,

$$\begin{aligned}(N - n_1)\nu_1''' &= 3717, & (N - n_1)\nu_3''' &= 83873.25, \\(N - n_1)\nu_2''' &= 15171.5, & (N - n_1)\nu_4''' &= 570882.875. \\K_1 &= -99.9286, & K_3 &= -3.7007, \\K_2 &= +30.0474, & K_4 &= +7.5620,\end{aligned}$$

whence, using the full corrections,

$$\begin{aligned}\mu_1' &= 2.090\,793, & \mu_2' &= 8.713\,900, \\ \mu_3' &= 47.942\,369, & \mu_4' &= 325.635\,238.\end{aligned}$$

These give

$$\begin{aligned}\mu_2 &= 4.342\,485, \\ \mu_3 &= 11.564\,934, \\ \mu_4 &= 95.909\,587. \\ \beta_1 &= 1.633\,320, & \beta_2 &= 5.086\,106.\end{aligned}$$

These values indicate a Type I curve. Fitting with four moments, we have

$$\begin{aligned}r &= \frac{6\beta_2 - 6\beta_1 - 6}{3\beta_1 - 2\beta_2 + 6} = 20.22227, \\ \epsilon &= \frac{r^2}{4 + \frac{(r+2)^2}{4(r+1)}\beta_1} = 30.28830, \\ b &= r \sqrt{\frac{\mu_2(r+1)}{\epsilon}} = 35.27423.\end{aligned}$$

$m_1 + 1$, $m_2 + 1$ are the roots of the quadratic $m'^2 - rm' + \epsilon = 0$, whence

$$\begin{aligned}m_1 &= 0.62899, & m_2 &= 17.59328. \\ a_1 &= \frac{bm_1}{m_1 + m_2} = 1.217584, & a_2 &= \frac{bm_2}{m_1 + m_2} = 34.05665, \\ y_0 &= \frac{N}{b} \frac{m_1^{m_1} m_2^{m_2}}{(m_1 + m_2)^{m_1 + m_2}} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1) \Gamma(m_2 + 1)} = 425.880.\end{aligned}$$

$$\text{Mean to mode} = \frac{1}{2} \frac{r+2}{r-2} \frac{\mu_2}{\mu_1} = 1.623\,905.$$

The equation, referred to the mode, is

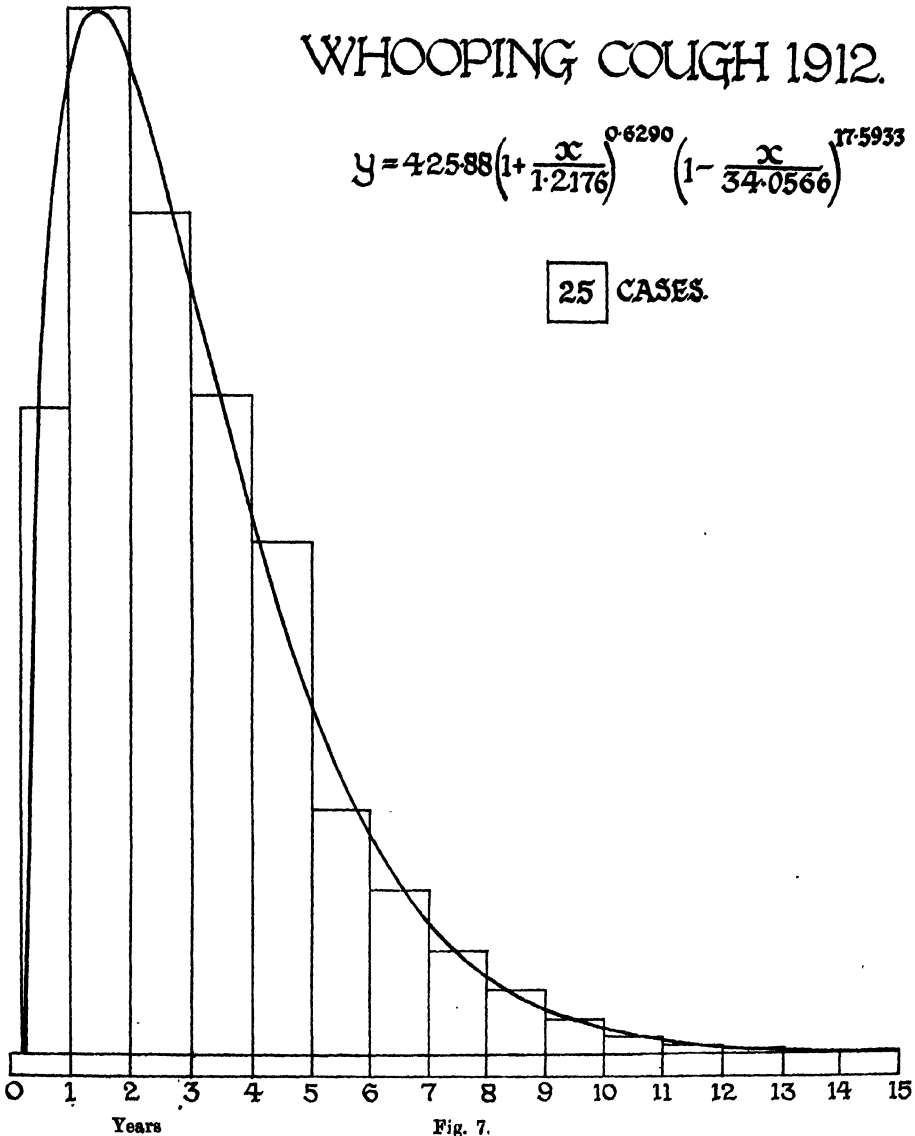
$$y = 425.880 \left(1 + \frac{x}{1.217\,584}\right)^{0.62899} \left(1 - \frac{x}{34.05665}\right)^{17.59328}.$$

This curve is drawn in the diagram (Fig. 7).

WHOOPING COUGH 1912.

$$y = 425.88 \left(1 + \frac{x}{1.2176}\right)^{0.6290} \left(1 - \frac{x}{34.0566}\right)^{17.5933}$$

25 CASES.



To find the theoretical frequencies, the mid-ordinates of the groups (except the first) were calculated, and the frequencies were calculated from the formula

$$n_r = \frac{h}{5760} \{5178y_r + 308(y_{r-1} + y_{r+1}) - 17(y_{r-2} + y_{r+2})\}.$$

The first two frequencies were calculated from the formula

$$\int_0^x y dx' = y_0 \left(\frac{b}{a_2}\right)^{m_2} \left(\frac{x'}{a_1}\right)^{m_2+1} a_1 \left[\frac{1}{m_1+1} - \frac{m_2}{m_1+2} \left(\frac{x'}{b}\right) + \frac{m_2(m_2-1)}{2!(m_1+3)} \left(\frac{x'}{b}\right)^2 - \dots \right]$$

obtained by expanding the equation of the curve referred to its start as origin, and integrating. The results are as follows:

<i>n</i> observed	<i>n</i> theoretical	Contribution to χ^2
212	213.6	.01
427	418.2	.19
344	363.5	1.05
270	268.0	.01
210	182.1	4.27
101	117.1	2.21
68	72.1	.23
43	42.8	.00
26	24.5	.09
15	13.6	.14
7	7.3	14.4
3	3.9	
2	1.9	
1	0.9	
1	0.4	.01
		$\chi^2 = 8.21$

For goodness of fit, this gives $P = .61$. The goodness of fit to the original seven groups is given by $P = .36$.

Now $d = \text{mean} - \text{mode} = 1.623\ 905$,
and AO (Fig. 1) = base of 1st frequency
 $= a_1 - \mu_1' + d = .750\ 696$.

Now this indicates that the value .7507 for λ would be a better one to take; if this is done, and the K 's evaluated by linear interpolation in Table VI, and a curve again fitted by four moments, a new value of λ is found.

The question naturally arises whether the series of values of λ found by successive fittings tend to a limit; further, is this limit eventually attained if we start off with a bad value of λ , e.g. the value .5 obtained from the n_6 test? This point has been investigated, with the following results:

$c_1 = 0$ gives $\lambda_1 = .8$ (nearest).

1st fitting with $\lambda_1 = .8$ gives $\lambda_2 = .7507$,
2nd fitting with $\lambda_2 = .7507$ gives $\lambda_3 = .7421$,
3rd fitting with $\lambda_3 = .7421$ gives $\lambda_4 = .7406$,
4th fitting with $\lambda_4 = .7406$ gives $\lambda_5 = .7403$,
5th fitting with $\lambda_5 = .7403$ gives $\lambda_6 = .7403$.

The n_6 test gives $\lambda_1 = .5$ (nearest).

1st fitting with $\lambda_1 = .5$ gives $\lambda_2 = .6586$,
2nd fitting with $\lambda_2 = .6586$ gives $\lambda_3 = .7198$,
3rd fitting with $\lambda_3 = .7198$ gives $\lambda_4 = .7367$,
4th fitting with $\lambda_4 = .7367$ gives $\lambda_5 = .7396$,
5th fitting with $\lambda_5 = .7396$ gives $\lambda_6 = .7402$.

Hence we may conclude that the values of λ obtained by successive fittings tend to a limit, here .7403, and this is the case even if a bad value is originally selected.

These data therefore indicate that Whooping Cough starts at .2597 year of age.

The curve fitted with the value .7403 for λ should thus give the best fit possible; the details of this fitting are:

$$\begin{aligned}
 K_1 &= -98.7699, & K_2 &= +28.5936, \\
 K_3 &= -2.4606, & K_4 &= +6.6076. \\
 \mu_1' &= 2.091462, & \mu_2' &= 8.713060, \\
 \mu_3' &= 47.943086, & \mu_4' &= 325.634686, \\
 \mu_2 &= 4.338847, \\
 \mu_3 &= 11.570984, \\
 \mu_4 &= 95.825542. \\
 \beta_1 &= 1.639145, & \beta_2 &= 5.090175, \\
 r &= 19.95181, \\
 \epsilon &= 29.65197, \\
 m_1 &= .617275, \\
 m_2 &= 17.334535, \\
 b &= 34.93440, \\
 a_1 &= 1.201223, \\
 a_2 &= 33.73318, \\
 y_0 &= 427.027, \\
 d &= 1.630527, \\
 AO \text{ (Fig. 1)} &= .740288.
 \end{aligned}$$

The equation, referred to the mode, is

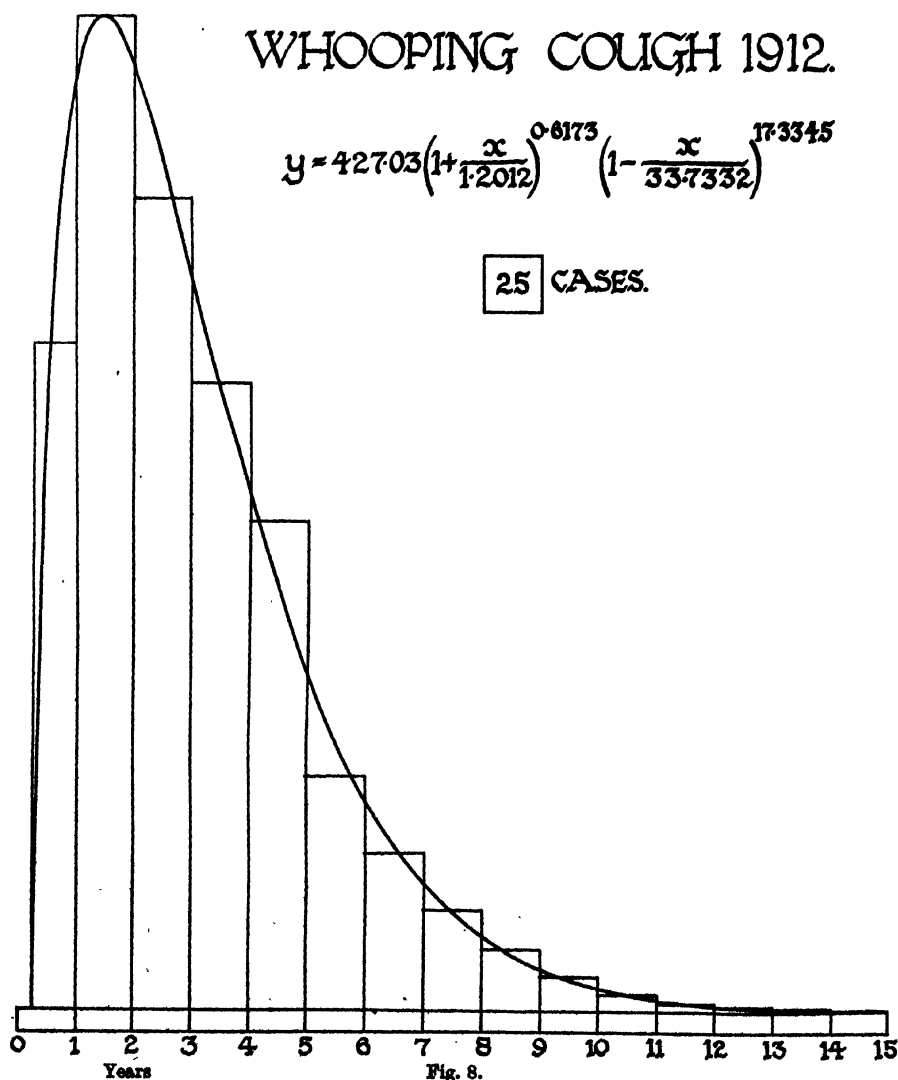
$$y = 427.027 \left(1 + \frac{x}{1.201223}\right)^{0.617275} \left(1 - \frac{x}{33.73318}\right)^{17.3345}.$$

The frequencies are calculated as before, with the following results:

n observed	n theoretical	Contribution to χ^2
212	212.5	.00
427	419.2	.15
344	363.7	1.07
270	267.9	.02
210	182.0	4.31
101	117.1	2.21
68	72.1	.23
43	42.8	.00
26	24.6	.08
15	13.6	.14
7	7.3	.01
3	3.8	
2	1.9	
1	0.9	
1	0.4	
		$\chi^2 = 8.22, n' = 11$

This gives $P = .61$. For the original seven frequencies P is still .36. Hence the result of our successive fittings has been to improve very slightly the fit at the start of the curve, but without any sensible increase of P . The conclusion we may draw is that if the start of the frequency obtained by fitting with four moments does not differ greatly from the value selected from the tables without interpolation, it is not worth while to make a second fitting. If this is not the case, then successive fittings steadily improve the goodness of fit; thus the above curve is much better than the one which would be obtained by using the value $\lambda = .5$ obtained from the n_4 test.

The curve with $\lambda = .7403$ is shown in Fig. 8.



Example 5. Dwelling houses in England and Wales first assessed for Income Tax in 1929—30.

Valuation	No. of Houses
Under £20	40,272
20—40	104,243
40—60	27,589
Over 60	12,226

The information here is meagre; no indication is given of the minimum valuation taken, and all houses valued at more than £60 are grouped together. The first point, namely the start of the frequency, must be discovered by our present method. The second difficulty can be met by a hypothetical spreading out of the frequency 12,226 into a suitable "tail."

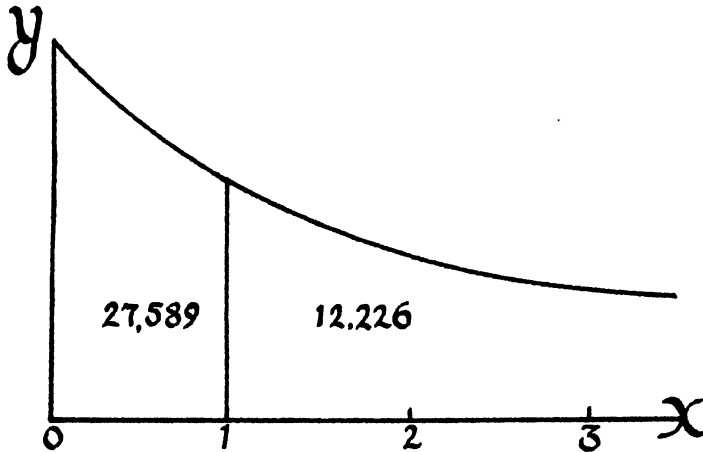


Fig. 9.

Let us assume the form $y = \frac{N'}{\sigma} e^{-x/\sigma}$ for the tail, the origin being as in the figure.

Then $\int_0^x y dx = N' (1 - e^{-x/\sigma})$, and we have

$$\int_0^1 y dx = 27,589 = N' (1 - e^{-1/\sigma}),$$

$$\int_0^\infty y dx = 39,815 = N',$$

whence

$$\frac{1}{\sigma} = 1.180679,$$

and

$$\int_0^\infty y dx = 39,815 (1 - e^{-1.180679x}).$$

Putting $x = 2, 3, 4$, etc. we get the corresponding frequencies which are shown in the following distribution :

Valuation	No. of Houses
Under £20	40,272
20— 40	104,243
40— 60	27,589
60— 80	8,472
80—100	2,602
100—120	799
120—140	245
140—160	75
160—180	23
180—200	7
200—220	2
220—240	1
Total	184,330

(i) The n_0 test. From Table II,

when $\lambda = \cdot 1$, $n_0 = 868,753$; when $\lambda = \cdot 2$, $n_0 = 247$.

Hence this test gives $\lambda = \cdot 2$.

(ii) The e test.

Using 3 frequencies, $\lambda < \cdot 05$, $q < \cdot 4$,

„ 4 „ $\lambda = \cdot 05$, $q = \cdot 5$,

„ 5 „ $\lambda = \cdot 1$, $q = \cdot 7$,

„ 6 „ $\lambda = \cdot 1$, $q = \cdot 7$,

„ 7 „ $\lambda = \cdot 1$, $q = \cdot 7$.

The values of q indicate an asymptotic curve, therefore the $c_1 = 0$ test was not tried.

We find that the value $\lambda = \cdot 2$ gives the best results. The e test is remarkably constant for 5, 6 and 7 frequencies, but this is due to the fact that all the frequencies except the first two are already smoothed.

With $\lambda = \cdot 2$ we have

$$K_1 = -18,898\cdot 41,$$

$$K_3 = +4,045\cdot 02,$$

$$K_2 = -1,092\cdot 24,$$

$$K_4 = +1,550\cdot 88.$$

With the usual notation,

$$(N - n_1) \nu_1''' = 129,484,$$

$$(N - n_1) \nu_3''' = 501,121,$$

$$(N - n_1) \nu_2''' = 201,810\cdot 5,$$

$$(N - n_1) \nu_4''' = 1,691,053\cdot 625.$$

Applying the full corrections we get the following moment-coefficients about the start of the second frequency :

$$\mu_1' = \cdot 599\,9327,$$

$$\mu_3' = 2\cdot 564\,9380,$$

$$\mu_2' = 1\cdot 023\,7803,$$

$$\mu_4' = 8\cdot 657\,8470,$$

leading to

$$\begin{aligned}\mu_2 &= .663\ 8611, \\ \mu_3 &= 1.154\ 1947, \\ \mu_4 &= 4.324\ 9300, \\ \beta_1 &= 4.553\ 304, & \beta_2 &= 9.813\ 515.\end{aligned}$$

Now $2\beta_2 - 3\beta_1 - 6 = -.032\ 882$, which is sufficiently near to zero to allow a Type III curve to be taken.

If the equation referred to the start of the curve is

$$y = y_0 \left(\frac{x}{a}\right)^p e^{-(p+1)\frac{x}{a}},$$

we have

$$p = \frac{4}{\beta_1} - 1 = -.121\ 5170,$$

$$a = 2\sqrt{\frac{\mu_2}{\beta_1}} = .763\ 6693,$$

$$y_0 = \frac{N(1+p)^{p+1}}{a\Gamma(p+1)} = 198,236.8.$$

Start of curve to mean $= a = .763\ 6693$.

Start of second frequency to mean $= \mu_1' = .599\ 9327$, whence base of first frequency $= .163\ 7366$, and the curve starts at $.836\ 2634 \times £20$, or about £16.7 per annum.

To get the theoretical frequencies, we have

$$\int_0^x y dx = N \frac{\Gamma^{(p+1)\frac{x}{a}}(p+1)}{\Gamma(p+1)} = NI(u, p)$$

in the notation of the *Tables of the Incomplete Γ -Function*, where

$$u = \frac{(p+1)\frac{x}{a}}{\sqrt{p+1}} = \sqrt{p+1} \frac{x}{a} = .937\ 2742 \frac{x}{a}.$$

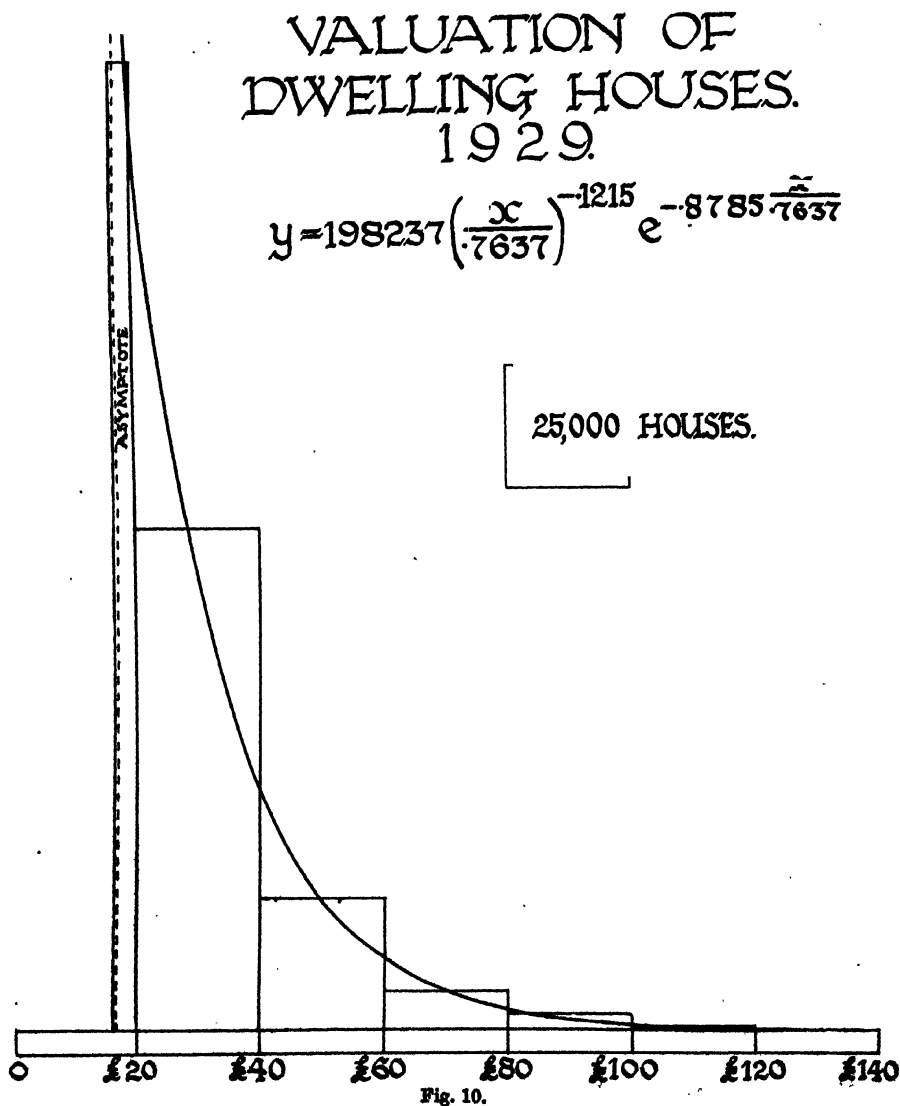
The tables mentioned give the following theoretical frequencies which are placed beside the observed frequencies for comparison.

Observed	Theoretical
40,272	40,858
104,243	102,972
27,589	28,352
8,472	8,441
2,602	2,566
799	788
245	243.5
75	75.5
23	23.5
7	7
2	2
1	1
$\left. \begin{array}{l} 40,272 \\ 104,243 \\ 27,589 \\ 8,472 \\ 2,602 \\ 799 \\ 245 \\ 75 \\ 23 \\ 7 \\ 2 \\ 1 \end{array} \right\} 12,226$	
$\left. \begin{array}{l} 40,858 \\ 102,972 \\ 28,352 \\ 8,441 \\ 2,566 \\ 788 \\ 243.5 \\ 75.5 \\ 23.5 \\ 7 \\ 2 \\ 1 \end{array} \right\} 12,147.5$	

These results are in fairly good accord but the probability P is zero to five places of decimals. This is explained by the fact that the given frequencies are so large that they would have to be reproduced with extreme fidelity in order to give a reasonable value of P .

It may be remarked that an attempt to fit a curve by four moments, assuming the base of the first frequency to be the same as the others and using the ordinary abruptness corrections, fails owing to the appropriate Type I curve omitting the first frequency.

The curve obtained above is shown in Fig. 10.



Example 6. Cricket Scores, 1931. In order to make the scores as homogeneous as possible, only First Class County Championship matches have been considered and all "not outs" and uncompleted innings ("declared" or abandoned) have been excluded. The distribution is as follows:

Score (Runs)	Number of Innings	Score (Runs)	Number of Innings
0	701	81—90	35
1, 2	627	91—100	22
3, 4	470	101—110	21
5, 6	374	111—120	15
7, 8	335	121—130	10
9, 10	303	131—140	8
11—20	895	141—150	3
21—30	524	151—160	6
31—40	330	161—170	1
41—50	170	171—180	0
51—60	105	181—190	1
61—70	83	191—200	0
71—80	56	201—210	1
		Total 5096	

(i) The n_s test gives, for $\lambda = \cdot 8$, $n_s = 419$;

$$\lambda = \cdot 9, n_s = 284.$$

Hence, since $n_s = 303$, $\lambda = \cdot 9$ is the nearest value.

(ii) The e test gives the following results:

Using 3 frequencies, $\lambda = \cdot 5$ ($q = \cdot 6$),

„ 4 „ $\lambda = \cdot 7$ ($q = \cdot 8$),

„ 5 „ $\lambda > 1$ ($q > 1$).

Now we can see from the data that λ should not be greater than $\cdot 5$, hence we take the value $\cdot 5$ given by three frequencies in the e test*. The value of q indicates that the distribution is asymptotic.

* This choice is supported by results obtained if we do not use the subdivision of the group 1—10 runs, i.e., if we take as the first seven frequencies 701, 2109, 895, 524, 330, 170, 105. With these the e test gives:

Using 3 frequencies, $\lambda = \cdot 05$;

„ 4 „ $\lambda = \cdot 1$;

„ 5 „ $\lambda = \cdot 1$;

„ 6 „ $\lambda = \cdot 1$;

„ 7 „ $\lambda = \cdot 2$.

Hence, since our sub-ranges are now five times as great, the value $\lambda = \cdot 1$ given by 4, 5 and 6 frequencies corresponds to the value $\lambda = \cdot 5$ taken above. The e test here gives more constant results as the frequencies are more regular than those in the subdivisions.

Table VI cannot be used in this example, as h_0 is not equal to h . We get, from Table IV, taking $h = 1$, $h_0 = \cdot 2$, $\lambda = \cdot 5$,

$$\frac{1}{h_0} n_1 \mu_1'' = -195\cdot 362, \quad \therefore n_1 \mu_1'' = -39\cdot 072.$$

$$\frac{1}{h_0^3} n_1 \mu_2'' = + 68\cdot 727, \quad \therefore n_1 \mu_2'' = + 2\cdot 749.$$

$$\frac{1}{h_0^5} n_1 \mu_3'' = - 26\cdot 624, \quad \therefore n_1 \mu_3'' = - 0\cdot 213.$$

$$\frac{1}{h_0^4} n_1 \mu_4'' = + 10\cdot 884, \quad \therefore n_1 \mu_4'' = + 0\cdot 017.$$

Table V gives

$$a_1 = - 977\cdot 525+, \quad \therefore a_1' = 5a_1 = -4,887\cdot 63.$$

$$a_2 = + 1,266\cdot 217, \quad \therefore a_2' = 25a_2 = + 31,655\cdot 4.$$

$$a_3 = - 2,260\cdot 555, \quad \therefore a_3' = 125a_3 = - 282,569.$$

$$a_4 = + 2,510\cdot 921, \quad \therefore a_4' = 625a_4 = + 1,569,326.$$

$$a_5 = - 1,253\cdot 460, \quad \therefore a_5' = 3125a_5 = - 3,917,063.$$

The abruptness functions are

$$\frac{a_1'}{12} - \frac{a_2'}{720} + \frac{a_5'}{30240} = - 144\cdot 378.$$

$$- \frac{a_3'}{120} + \frac{a_4'}{3024} = + 255\cdot 162.$$

$$- \frac{a_1'}{40} + \frac{a_2'}{504} - \frac{a_5'}{9600} = - 30\cdot 435.$$

$$\frac{a_2'}{126} - \frac{a_4'}{1440} = - 838\cdot 577.$$

Now to calculate the moments of the remainder about its start, when the first frequency is omitted, we must group together the next five frequencies with a total 2109 in order to have all the ranges equal, as required by the theory. We get, with $h = 1$,

$$(N - n_1) \nu_1''' = 8,488\cdot 5, \quad (N - n_1) \nu_3''' = 282,622\cdot 125,$$

$$(N - n_1) \nu_2''' = 38,074\cdot 75, \quad (N - n_1) \nu_4''' = 2,801,954\cdot 6875,$$

whence, using Equations II on p. 15,

$$\mu_1' = 1\cdot 629\ 719, \quad \mu_3' = 55\cdot 037\ 157,$$

$$\mu_2' = 7\cdot 450\ 238, \quad \mu_4' = 545\cdot 958\ 976.$$

Transferring to the mean, we obtain

$$\mu_1 = 4\cdot 794\ 254,$$

$$\mu_2 = 27\cdot 268\ 791,$$

$$\mu_4 = 284\cdot 742\ 098,$$

whence $\beta_1 = 6.747\ 900$, $\beta_2 = 12.388\ 240$.

These values indicate a Type I curve. When we endeavour, however, to fit a curve by four moments, we find that the curve just omits the first frequency. We therefore fix the start at the point given by $\lambda = .5$ and fit with three moments. We have:

$$\begin{aligned} c &= \text{distance of mean from start} = .1 + 1.629\ 719 \\ &= 1.729\ 719, \\ r &= \frac{2(\mu_2 c^2 + \mu_3 c - \mu_2^2)}{2\mu_2^2 - \mu_3 c} = -64.339\ 415. \end{aligned}$$

The negative value of r indicates that the type has changed to Type VI as a result of fixing the start. The working for a Type VI curve with fixed start is as follows:

$$\gamma_2 = \frac{\mu_2}{c^2} = 1.602\ 39628,$$

$$\gamma_3 = \frac{\mu_3}{2\mu_2 c} = 1.644\ 14188.$$

If the equation is
$$y = y_0 \frac{(x-a)^{q_2}}{x^{q_1}},$$

then

$$q_2 - q_1 = \frac{1 - 3\gamma_2 + 4\gamma_3}{\gamma_2 - \gamma_3} = -66.339\ 415,$$

$$q_2 + q_1 = \frac{\gamma_3(1 + \gamma_2)(\gamma_2 - 1 - 2\gamma_3)}{(2\gamma_2 - \gamma_3 + \gamma_2\gamma_3)(\gamma_2 - \gamma_3)} = 65.619\ 867.$$

$$\therefore q_1 = 65.979\ 641,$$

$$q_2 = -.359\ 774,$$

$$a = -\frac{cr}{q_2 + 1} = 173.8278,$$

$$y_0 = \frac{N}{a^{r-1}} \frac{\Gamma(q_1)}{\Gamma(1-r) \Gamma(q_2+1)} = 1.228\ 559 \times 10^{151}.$$

Hence the equation is

$$y = 1.228\ 559 \times 10^{151} \frac{(x - 173.8278)^{-.359\ 774}}{x^{65.979\ 641}}.$$

The first seven frequencies can be calculated from the formula

$$\int_a^x y dx = y_0 a^{r-1} z^{q_2+1} \left[\frac{1}{q_2+1} + \frac{r}{q_2+2} z + \frac{r(r+1)}{2!(q_2+3)} z^2 + \dots \right],$$

where

$$z = 1 - a/x, \quad y_0 a^{r-1} = 52,640.23.$$

The remaining frequencies can be calculated from the formula

$$n_r = \frac{h}{5760} [5178y_r + 308(y_{r-1} + y_{r+1}) - 17(y_{r-2} + y_{r+2})].$$

The results are as follows:

Score	Observed Frequency	Theoretical Frequency	χ^2
0	701	682.6	0.496
1, 2	627	657.1	1.379
3, 4	470	466.0	0.034
5, 6	374	372.1	0.010
7, 8	335	310.7	1.901
9, 10	303	265.7	5.236
11—20	895	913.0	0.355
21—30	524	522.6	0.004
31—40	330	319.3	0.359
41—50	170	201.5	4.924
51—60	105	129.7	4.704
61—70	83	84.7	0.034
71—80	56	55.9	0.000
81—90	35	37.2	0.130
91—100	22	24.9	0.338
101—110	21	16.8	1.050
111—120	15	11.4	1.137
121—130	10	7.7	24.8
131—140	8	5.3	
141—150	3	3.6	
151—160	6	2.5	
161—170	1	1.7	
171—180	0	1.2	
181—190	1	0.8	
191—200	0	0.6	
201—210	1	0.4	1.090
over 210	0	1.0	
Totals	5096	5096	23.181

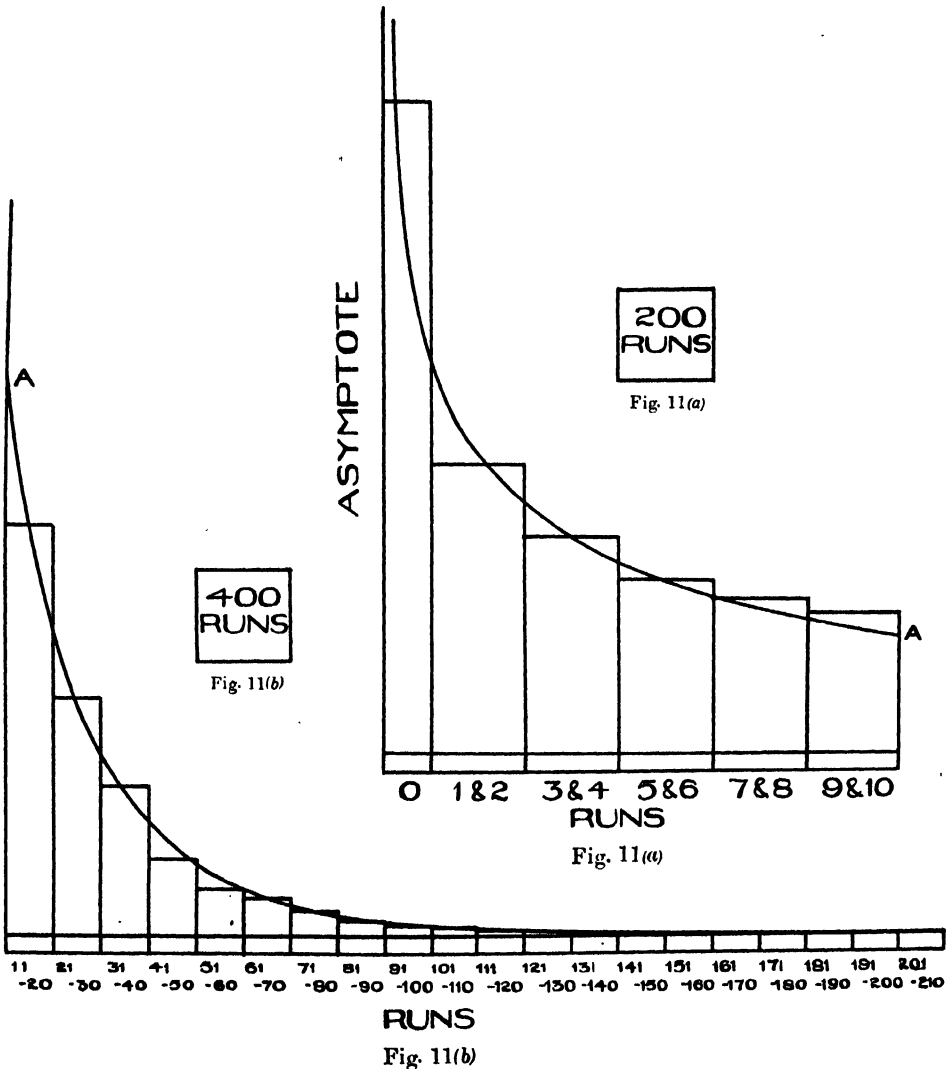
Hence, since $n' = 18$, $P = .144$ from Palin Elderton's Table.

The curve has been drawn in Figs. 11(a) and 11(b). Owing to the large frequencies on small ranges at the start, the distribution cannot be adequately represented in a single diagram. The first figure therefore gives the first six groups and the second figure gives the remaining groups on a reduced scale. The points *A* in the two diagrams correspond. It is seen that the observed frequencies are most irregular and therefore the curve fits worst in the neighbourhood of 10 and of 50 runs. This can probably be explained on psychological grounds.

5. *Conclusion.* To the question "Where does a given frequency distribution start?" we can give only a qualified answer. Three tests for λ are suggested in this paper and they may give conflicting results. If, however, the distribution be a theoretical one, i.e., if there are no sampling or experimental errors, then the ϵ test should always be used, as it can be relied on to give the start correctly. In other cases the choice of the best value may be guided from some *a priori* knowledge of the distribution or it may have to be made by fitting curves for each value of λ and selecting that which fits best. In all the examples we have chosen, at least one of the tests has given a suitable value of λ . When the best value of λ has been found, the tables at the end of this paper provide adequate corrections for the

CRICKET SCORES 1931.

$$Y = 1.22856 \times 10^{151} \frac{(X-173.828)^{-358774}}{X^{65.9796}}$$



moments. The best results will usually be obtained by fitting with four moments, but if this fails, the start of the curve should be fixed at the point given by the chosen value of λ .

I have to thank Professor Karl Pearson, F.R.S., under whose guidance the work was carried out, for his continual help and stimulating advice.

Moment Corrections with unknown Curve-Start

TABLE I. (Exponential Test.)

3 frequencies		
λ	$\frac{3L_1}{3L_2}$	$3L_1$
·05	4·550 518	1·322 219
·1	3·708 312	1·041 393
·2	2·956 036	·778 151
·3	2·570 064	·636 822
·4	2·324 251	·544 068
·5	2·150 660	·477 121
·6	2·020 213	·425 969
·7	1·917 981	·385 351
·8	1·835 378	·352 183
·9	1·767 062	·324 511
1·0	1·709 511	·301 030
1·5	1·518 181	·221 849

4 frequencies		
λ	$\frac{4L_1}{4L_2}$	$4L_1$
·05	8·741 460	1·031 655 ⁻
·1	6·809 970	·760 566
·2	5·122 760	·514 910
·3	4·275 215 ⁻	·389 038
·4	3·743 076	·309 985 ⁻
·5	3·371 237	·255 273
·6	3·094 107	·215 115 ⁺
·7	2·878 355 ⁺	·184 436
·8	2·704 980	·160 297
·9	2·562 247	·140 867
1·0	2·442 475 ⁻	·124 939
1·5	2·047 415 ⁺	·075 721

5 frequencies		
λ	$\frac{5L_1}{5L_2}$	$5L_1$
·05	13·312 918	·913 636
·1	10·144 490	·648 882
·2	7·414 998	·414 396
·3	6·061 800	·298 039
·4	5·219 689	·227 169
·5	4·635 132	·179 552
·6	4·201 718	·145 591
·7	3·865 712	·120 359
·8	3·596 643	·101 037
·9	3·375 784	·085 889
1·0	3·190 921	·073 786
1·5	2·584 325 ⁻	·038 737

6 frequencies		
λ	$\frac{6L_1}{6L_2}$	$6L_1$
·05	18·150 433	·845 008
·1	13·640 942	·584 918
·2	9·794 426	·358 510
·3	7·905 153	·248 872
·4	6·736 800	·183 648
·5	5·929 568	·140 815 ⁻
·6	5·333 254	·110 941
·7	4·872 341	·089 224
·8	4·504 169	·072 945 ⁻
·9	4·202 601	·060 446
1·0	3·950 642	·050 662
1·5	3·126 988	·023 748

7 frequencies		
λ	$\frac{7L_1}{7L_2}$	$7L_1$
·05	23·190 383	·798 453
·1	17·259 963	·542 038
·2	12·239 659	·321 906
·3	9·791 384	·217 390
·4	8·284 568	·156 387
·5	7·247 196	·117 067
·6	6·483 021	·090 139
·7	5·893 711	·070 912
·8	5·423 872	·056 750 ⁻
·9	5·039 654	·046 063
1·0	4·719 088	·037 839
1·5	3·674 188	·016 153

TABLE II. n_6 .

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	11·416 667	13·583 333	11·416 667	5·250 000
·1	40·871 795 ⁺	10·128 205 ⁻	13·053 614	11·232 101	5·219 512
·2	16·910 173	9·089 827	12·576 840	11·059 524	5·190 476
·3	9·427 714	8·238 952	12·145 663	10·897 815 ⁺	5·162 791
·4	5·968 296	7·531 704	11·754 011	10·745 989	5·136 364
·5	4·063 492	6·936 508	11·396 825 ⁺	10·603 175 ⁻	5·111 111
·6	2·903 196	6·430 137	11·069 863	10·468 599	5·086 957
·7	2·147 687	5·995 170	10·769 535 ⁺	10·341 576	5·063 830
·8	1·631 683	5·618 317	10·492 794	10·221 491	5·041 667
·9	1·266 271	5·289 285 ⁺	10·237 031	10·107 797	5·020 400
1·0	1·000 000	5·000 000	10·000 000	10·000 000	5·000 000

TABLE III. c_1 .

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	∞	0	0	0	0
·1	11·951 764 469 467	·281 610 530 533	·122 798 560 376	·043 641 915 814	·007 277 439 024
·2	6·838 474 025 974	·553 525 974 026	·267 807 359 307	·098 738 095 238	·016 761 904 762
·3	5·072 935 153 825	·819 856 512 842	·435 412 717 928	·166 119 890 768	·028 683 139 535
·4	4·152 342 755 284	1·083 657 244 716	·626 057 040 998	·246 609 625 668	·043 272 727 273
·5	3·574 603 174 603	1·347 271 825 397	·840 228 174 603	·341 021 825 397	·060 763 888 889
·6	3·171 451 133 408	1·612 548 866 592	1·078 451 133 408	·450 164 251 208	·081 391 304 348
·7	2·870 213 320 777	1·880 983 107 795	1·341 281 598 087	·574 838 772 283	·105 390 957 447
·8	2·634 189 640 769	2·153 810 359 231	1·629 300 751 880	·715 842 105 263	·133 000 000 000
·9	2·442 746 375 855	2·432 073 068 589	1·943 111 141 937	·873 966 444 270	·164 456 632 653
1·0	2·283 333 333 333	2·716 666 666 667	2·283 333 333 333	1·050 000 000 000	·200 000 000 000

 C_2 .

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	2·881 944 444 444	1·118 055 555 556	·381 944 444 444	·062 500 000 000
·1	20·813 961 805 307	3·062 704 861 360	1·403 052 714 398	·506 947 285 602	·085 203 252 033
·2	10·354 662 698 413	3·235 337 301 587	1·696 884 920 635	·643 115 079 365	·110 634 920 635
·3	6·848 448 708 914	3·402 662 402 197	1·999 645 290 110	·790 354 709 890	·138 837 209 302
·4	5·084 988 540 871	3·566 678 125 796	2·311 417 112 299	·948 582 887 701	·169 848 484 548
·5	4·021 164 021 164	3·728 835 978 836	2·632 275 132 275	1·117 724 867 725	·203 703 703 704
·6	3·308 675 523 349	3·890 213 365 539	2·962 286 634 461	1·297 713 365 539	·240 434 782 609
·7	2·797 898 918 049	4·051 624 891 475	3·301 512 363 427	1·488 487 636 573	·280 070 921 986
·8	2·413 803 606 238	4·213 696 393 762	3·650 007 309 942	1·689 992 690 058	·322 638 888 889
·9	2·114 566 024 340	4·376 915 457 141	4·007 821 384 964	1·902 178 615 036	·368 163 265 306
1·0	1·875 000 000 000	4·541 666 666 667	4·375 000 000 000	2·125 000 000 000	·416 666 666 667

TABLE III (continued).

 C_3 .

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	∞	2.621 527 777 778	1.711 805 555 556	.663 194 444 444	.114 583 333 333
.1	13.317 393 333 129	2.540 940 000 204	1.816 635 757 371	.738 126 147 391	.130 691 056 911
.2	6.115 845 959 596	2.475 820 707 071	1.918 623 737 374	.813 194 444 444	.147 222 222 222
.3	3.763 229 243 512	2.422 881 867 599	2.018 143 773 427	.888 377 965 704	.164 147 286 822
.4	2.616 103 259 486	2.379 730 073 848	2.115 508 021 390	.963 659 645 276	.181 439 393 939
.5	1.947 089 947 090	2.344 576 719 577	2.210 978 835 979	1.039 021 164 021	.199 074 074 074
.6	1.514 500 495 479	2.318 055 060 077	2.304 778 273 257	1.114 452 495 974	.217 028 985 507
.7	1.215 232 754 657	2.293 100 578 676	2.397 095 499 755	1.189 941 537 282	.235 283 687 943
.8	.998 046 331 106	2.274 870 335 561	2.488 092 627 402	1.265 478 801 170	.253 819 444 444
.9	.834 683 230 328	2.260 687 140 043	2.577 909 351 185	1.341 056 166 056	.272 619 047 619
1.0	.708 333 333 333	2.250 000 000 000	2.666 666 666 667	1.416 666 666 667	.291 666 666 667

 C_4 .

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	.826 388 888 889	.673 611 111 111	.326 388 888 889	.062 500 000 000
.1	3.576 282 097 132	.757 051 236 202	.667 191 188 041	.332 808 811 959	.065 046 350 407
.2	1.550 099 206 349	.699 900 793 651	.661 210 317 460	.338 789 682 540	.067 460 317 460
.3	.903 489 275 582	.652 066 279 973	.655 626 027 719	.344 373 972 281	.069 767 441 860
.4	.596 829 640 947	.611 503 692 386	.650 401 069 519	.349 598 930 481	.071 968 696 970
.5	.423 280 423 280	.576 719 576 720	.645 502 645 503	.354 497 354 497	.074 074 074 074
.6	.314 512 882 448	.546 598 228 663	.640 901 771 337	.359 098 228 663	.076 076 956 522
.7	.241 614 759 762	.520 290 002 142	.636 572 742 956	.363 427 257 044	.078 074 184 397
.8	.190 363 060 429	.497 136 939 571	.632 492 690 058	.367 507 309 942	.079 461 111 111
.9	.153 007 671 805	.476 621 957 825	.628 641 200 070	.371 358 799 930	.081 632 653 061
1.0	.125 000 000 000	.458 333 333 333	.625 000 000 000	.375 000 000 000	.083 333 333 333

 C_5 .

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	∞	.086 805 555 556	.079 861 111 111	.045 138 888 889	.010 479 666 667
.1	.340 598 294 965	.076 068 371 702	.075 446 779 813	.043 600 839 234	.010 162 601 626
.2	.140 918 109 668	.067 415 223 665	.071 473 665 224	.042 162 698 413	.009 920 634 921
.3	.078 564 284 833	.060 324 604 056	.067 880 524 149	.040 815 128 024	.009 689 922 481
.4	.049 736 803 412	.054 430 863 254	.064 616 755 793	.039 549 910 873	.009 469 696 970
.5	.033 862 433 862	.049 470 899 471	.061 640 211 640	.038 359 788 360	.009 259 259 259
.6	.024 183 298 650	.045 251 145 795	.058 915 520 872	.037 238 325 282	.009 057 971 014
.7	.017 897 389 612	.041 626 419 912	.056 412 795 774	.036 179 796 818	.008 865 248 227
.8	.013 597 361 459	.038 485 871 874	.054 106 620 718	.035 179 093 567	.008 680 555 556
.9	.010 552 253 228	.035 744 043 068	.051 975 255 177	.034 231 641 375	.008 503 401 361
1.0	.008 333 333 333	.033 333 333 333	.050 000 000 000	.033 333 333 333	.008 333 333 333

TABLE IV. $\frac{1}{h_0} n_1 \mu_1''$.

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	0	0	0	0	0
·1	·053 146 674	·000 449 173	·000 190 403	·000 067 022	·000 011 125
·2	·111 505 682	·003 389 207	·001 557 015 ⁺	·000 563 833	·000 094 889
·3	·173 837 033	·010 866 342	·005 373 119	·001 998 761	·000 340 875
·4	·239 262 372	·024 618 517	·013 025 673	·004 970 771	000 858 667
·5	·307 142 857	·046 199 157	·026 023 065 ⁺	·010 174 851	·001 779 514
·6	·377 003 205 ⁺	·077 054 795 ⁻	·046 002 205 ⁺	·018 407 333	·003 258 000
·7	·448 482 797	·118 575 634	·074 735 131	·030 571 163	·005 473 708
·8	·521 303 258	·172 129 631	·114 135 258	·047 681 123	·008 632 889
·9	·595 246 405 ⁺	·239 086 220	·166 263 359	·070 869 029	·012 970 125
1·0	·670 138 889	·320 833 333	·233 333 333	·101 388 889	·018 750 000

 $\frac{1}{h_0^2} n_1 \mu_2''$.

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	0	0	0	0	0
·1	·003 650 103	·000 045 316	·000 019 321	·000 006 814	·000 001 132
·2	·015 663 867	·000 689 594	·000 320 269	·000 116 407	·000 019 625 ⁺
·3	·037 282 384	·003 342 812	·001 678 446	·000 627 664	·000 107 316
·4	·069 407 157	·010 174 304	·005 487 556	·002 108 127	·000 365 345 ⁻
·5	·112 711 010	·024 038 742	·013 849 817	·005 458 219	·000 958 306
·6	·167 707 836	·048 444 736	·029 670 121	·011 979 901	·002 129 814
·7	·234 798 023	·087 549 158	·056 754 081	·023 449 694	·004 219 488
·8	·314 298 975 ⁻	·146 167 819	·099 910 037	·042 196 086	·007 681 354
·9	·406 466 153	·229 797 043	·165 055 118	·071 181 282	·013 103 669
1·0	·511 507 937	·344 642 857	·259 325 397	·114 087 302	·021 230 159

 $\frac{1}{h_0^3} n_1 \mu_3''$.

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	0	0	0	0	0
·1	·000 278 636	·000 004 102	·000 001 756	·000 000 620	·000 000 103
·2	·002 423 051	·000 125 549	·000 058 724	·000 021 396	·000 003 611
·3	·008 740 137	·000 917 673	·000 465 341	·000 174 614	·000 029 904
·4	·021 876 387	·003 742 508	·002 043 440	·000 788 408	·000 136 921
·5	·044 716 081	·011 104 929	·006 490 247	·002 570 858	·000 452 500 ⁻
·6	·080 315 486	·026 975 571	·016 788 729	·006 818 053	·001 215 604
·7	·131 858 723	·057 118 020	·037 682 934	·015 669 565 ⁻	·002 828 545 ⁺
·8	·202 637 203	·109 429 483	·076 221 175 ⁺	·032 414 776	·005 921 321
·9	·295 977 878	·194 300 820	·142 370 050 ⁺	·061 853 498	·011 429 177
1·0	·415 327 381	·325 000 000	·249 702 381	·110 714 286	·020 684 524

TABLE IV (continued).

$$\frac{1}{h_0^4} n_1 \mu_4''.$$

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	0	0	0	0	0
•1	•000 022 553	•000 000 366	•000 000 157	•000 000 056	•000 000 009
•2	•000 395 663	•000 022 502	•000 010 577	•000 003 860	•000 000 652
•3	•002 155 292	•000 247 619	•000 126 434	•000 047 558	•000 008 154
•4	•007 232 436	•001 351 176	•000 744 069	•000 287 950	•000 050 082
•5	•018 564 080	•005 028 172	•002 967 929	•001 179 826	•000 208 029
•6	•040 168 896	•014 703 127	•009 252 508	•003 772 720	•000 674 000
•7	•077 200 853	•036 429 862	•024 325 338	•010 160 223	•001 838 150-
•8	•135 988 366	•079 992 630	•056 439 374	•024 117 893	•004 416 456
•9	•224 063 645-	•160 224 084	•119 005 540	•051 968 585+	•009 627 849
1•0	•350 185 185+	•298 558 201	•232 658 730	•103 716 931	•019 431 217

TABLE V. a_1 .

λ	$-n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	0	0	0	0
•1	8•174 359 079	•257 692 412	•105 943 951	•036 913 192	•006 097 561
•2	3•382 034 632	•465 367 965+	•201 298 701	•071 428 571	•011 904 762
•3	1•885 542 836	•635 542 836	•287 534 087	•103 770 261	•017 441 860
•4	1•193 659 282	•776 992 615+	•365 864 528	•134 135 472	•022 727 273
•5	•812 698 413	•896 031 746	•437 301 587	•162 698 413	•027 777 778
•6	•580 639 168	•997 305 834	•502 694 166	•189 613 527	•032 698 696
•7	•429 537 351	1•084 299 255+	•562 759 568	•215 018 210	•037 234 043
•8	•326 336 675+	1•159 670 008	•618 107 769	•239 035 088	•041 666 667
•9	•253 254 077	1•225 476 300	•669 260 542	•261 773 940	•045 918 367
1•0	•200 000 000	1•283 333 333	•716 666 667	•283 333 333	•050 000 000

 a_2 .

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	5•763 888 889	+2•236 111 111	-•763 888 889	+•125 000 000
•1	34•059 829 496	4•690 170 504	+1•794 677 981	-•610 083 923	+•099 593 496
•2	14•091 810 967	3•824 855 700	+1•397 366 522	-•466 269 841	+•075 396 825+
•3	7•856 428 483	3•115 793 739	+1•038 052 415-	-•331 512 802	+•052 325 581
•4	4•973 580 341	2•526 419 659	+•711 675 579	-•204 991 087	+•030 303 030
•5	3•386 243 386	2•030 423 280	+•414 021 164	-•085 978 836	+•009 259 259
•6	2•419 329 865-	1•608 447 913	+•141 552 087	+•026 167 472	-•010 869 565+
•7	1•799 738 961	1•245 975 325-	-•108 720 423	+•132 020 318	-•030 141 844
•8	1•359 736 146	•931 930 521	-•339 337 928	+•232 090 643	-•048 611 111
•9	1•055 225 323	•657 737 640	-•552 474 482	+•326 835 863	-•066 326 531
1•0	•833 333 333	•416 666 667	-•750 000 000	+•416 666 667	-•083 333 333

TABLE V (continued).

 a_3 .

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	∞	15·729 166 667	10·270 833 333	3·979 166 667	·687 500 000
·1	71·525 641 943	13·474 358 057	9·343 823 761	3·656 176 239	·634 146 341
·2	29·592 803 030	11·657 196 970	8·509 469 697	3·354 166 667	·583 333 333
·3	16·498 499 815 ⁻	10·168 166 852	7·754 910 071	3·071 176 885 ⁺	·534 883 721
·4	10·444 518 717	8·930 481 283	7·069 518 717	2·805 481 283	·488 636 364
·5	7·111 111 111	7·888 888 889	6·444 444 444	2·555 555 556	·444 444 444
·6	5·080 592 716	7·002 740 617	5·872 259 383	2·320 048 309	·402 173 913
·7	3·758 451 819	6·241 548 181	5·346 687 113	2·097 757 332	·361 702 128
·8	2·855 445 906	5·582 054 094	4·862 390 351	1·887 609 649	·322 916 667
·9	2·215 973 178	5·006 249 044	4·414 803 587	1·688 644 689	·285 714 286
1·0	1·750 000 000	4·500 000 000	4·000 000 000	1·500 000 000	·250 000 000

 a_4 .

λ	$+n_1$	$-n_2$	$+n_3$	$-n_4$	$+n_5$
0	∞	19·833 333 333	16·166 666 667	7·833 333 333	1·500 000 000
·1	81·743 590 792	17·256 409 208	15·107 227 155 ⁺	7·464 201 416	1·439 024 390
·2	33·820 346 320	15·179 653 680	14·153 679 654	7·119 047 619	1·380 952 381
·3	18·855 428 360	13·477 904 973	13·291 325 796	6·795 630 726	1·325 581 395 ⁺
·4	11·936 592 819	12·063 407 181	12·508 021 390	6·491 978 610	1·272 727 273
·5	8·126 984 127	10·873 015 873	11·793 650 794	6·206 349 206	1·222 222 222
·6	5·806 391 676	9·860 274 991	11·139 725 009	5·937 198 068	1·173 913 043
·7	4·295 373 507	8·990 340 779	10·539 070 986	5·683 151 236	1·127 659 574
·8	3·263 366 750 ⁺	8·236 633 250 ⁻	9·985 588 972	5·444 982 456	1·083 333 333
·9	2·532 540 775 ⁻	7·578 570 336	9·474 061 243	5·215 593 930	1·040 816 327
1·0	2·000 000 000	7·000 000 000	9·000 000 000	5·000 000 000	1·000 000 000

 a_5 .

λ	$-n_1$	$+n_2$	$-n_3$	$+n_4$	$-n_5$
0	∞	10·416 666 667	9·583 333 333	5·416 666 667	1·250 000 000
·1	40·871 795 396	9·128 204 604	9·053 613 578	5·232 100 708	1·219 512 195 ⁺
·2	16·910 173 160	8·089 826 840	8·576 839 827	5·059 523 810	1·190 476 190
·3	9·427 714 180	7·238 952 487	8·145 662 898	4·897 815 363	1·162 790 698
·4	5·968 296 409	6·531 703 591	7·754 010 695 ⁺	4·745 989 305 ⁻	1·136 363 636
·5	4·063 492 063	5·936 507 937	7·396 825 397	4·603 174 603	1·111 111 111
·6	2·903 195 838	5·430 137 495 ⁺	7·069 862 505 ⁻	4·468 599 034	1·086 956 522
·7	2·147 686 753	4·995 170 389	6·769 535 493	4·341 575 618	1·063 829 787
·8	1·631 683 375 ⁺	4·618 316 625 ⁻	6·492 794 486	4·221 491 228	1·041 666 667
·9	1·266 270 387	4·289 285 168	6·237 030 621	4·107 796 965 ⁻	1·020 408 163
1·0	1·000 000 000	4·000 000 000	6·000 000 000	4·000 000 000	1·000 000 000

TABLE VI. K_1 .

λ	n_1	n_2	n_3	n_4	n_5
0	$-\infty$	-.021 501 598	+.013 948 137	-.005 347 498	+.000 913 525 ⁺
.1	-.636 353 675 ⁺	-.039 437 722	+.021 316 401	-.007 914 081	+.001 337 436
.2	-.352 799 985 ⁻	-.051 314 487	+.026 752 959	-.009 879 800	+.001 667 992
.3	-.308 362 782	-.055 978 631	+.029 089 396	-.010 752 319	+.001 817 055 ⁺
.4	-.324 425 067	-.052 318 318	+.027 025 398	-.009 946 742	+.001 676 356
.5	-.365 125 556	-.039 230 633	+.019 124 747	-.006 780 511	+.001 115 842
.6	-.418 429 429	-.015 600 485 ⁺	+.003 811 099	-.000 468 312	-.000 017 978
.7	-.479 128 525 ⁺	+.019 713 730	-.020 636 406	+.009 882 997	-.001 903 697
.8	-.544 586 042	+.067 890 333	-.056 087 666	+.025 279 452	-.004 746 618
.9	-.613 315 045 ⁺	+.130 151 913	-.104 566 227	+.046 845 034	-.008 780 513
1.0	-.684 408 069	+.207 771 164	-.168 253 968	+.075 826 720	-.014 269 180

 K_2 .

λ	n_1	n_2	n_3	n_4	n_5
0	$-\infty$	+.041 473 765 ⁺	-.013 288 139	+.003 775 353	-.000 545 635 ⁻
.1	-.253 150 199	+.033 332 954	-.009 940 553	+.002 608 898	-.000 352 946
.2	-.090 583 915 ⁻	+.026 164 477	-.006 644 003	+.001 414 993	-.000 152 018
.3	-.021 952 593	+.018 165 156	-.002 576 711	-.000 112 290	+.000 109 623
.4	+.031 907 940	+.006 889 972	+.003 693 177	-.002 546 686	+.000 533 695 ⁺
.5	+.087 179 810	-.010 714 122	+.014 299 657	-.006 794 093	+.001 285 319
.6	+.149 466 856 ⁽⁵⁾	-.038 301 676	+.032 174 292	-.014 161 322	+.002 608 592
.7	+.221 303 859	-.080 139 026	+.061 145 227	-.026 429 212	+.004 843 573
.8	+.304 046 996	-.141 125 485 ⁺	+.106 039 965 ⁽⁵⁾	-.045 930 103	+.008 444 692
.9	+.398 510 089	-.226 822 037	+.172 792 029	-.075 629 647	+.014 000 575 ⁺
1.0	+.505 224 868	-.343 485 450 ⁻	+.268 551 587	-.119 212 963	+.022 255 291

 K_3 .

λ	n_1	n_2	n_3	n_4	n_5
0	$+\infty$	+.030 123 595 ⁻	-.019 380 374	+.007 330 936	-.001 233 879
.1	+.066 421 864	+.032 230 396	-.020 246 603	+.007 632 757	-.001 283 736
.2	+.025 173 412	+.034 046 417	-.021 081 639	+.007 935 169	-.001 334 630
.3	+.006 645 309	+.036 227 120	-.022 191 913	+.008 352 286	-.001 406 104
.4	-.012 136 459	+.040 206 147	-.024 409 167	+.009 213 852	-.001 556 248
.5	-.038 084 688	+.048 539 894	-.029 438 880	+.011 229 368	-.001 913 038
.6	-.075 577 632	+.065 236 904	-.040 270 947	+.015 696 183	-.002 715 561
.7	-.128 353 817	+.096 089 196	-.061 655 270	+.024 754 989	-.004 366 244
.8	-.199 964 386	+.149 015 662 ⁽⁵⁾	-.100 645 137	+.041 696 171	-.007 495 188
.9	-.293 911 395 ⁺	+.234 423 961	-.167 211 404	+.071 320 436	-.013 037 737
1.0	-.413 695 437	+.365 595 238	-.274 930 556	+.120 357 143	-.022 326 389

ON ASYMPTOTIC FORMULAE FOR THE HYPERGEOMETRIC SERIES

II. HYPERGEOMETRIC SERIES IN WHICH THE FOURTH ELEMENT, x , IS NOT NECESSARILY UNITY

BY O. L. DAVIES, PH.D.

THE hypergeometric series was first introduced by Euler in 1778 when attempting to obtain solutions of the second order differential equation

$$x^n(a + bx^n) \frac{d^2 y}{dx^2} + x(c + ex^n) \frac{dy}{dx} + (f + gx^n)y = 0.$$

A particular case of this equation is

$$x(1-x) \frac{d^2 y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{dy}{dx} - \alpha\beta y = 0,$$

which has the hypergeometric function

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1!\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} x^2 + \dots$$

as one of its solutions.

The chief problem in the study of series is to determine the partial sums and in particular, the total sum. In this paper we will obtain approximations to the sum of a number of terms of the hypergeometric series by fitting certain systems of frequency curves to the series and then expressing, approximately, the required sum by means of an integral.

The hypergeometric series will be looked upon as a discrete frequency distribution; we restrict our attention, therefore, to series with positive terms only.

In § 1 the system of frequency curves which arises from the differential equation

$$\frac{1}{y} \frac{dy}{dz} = \frac{y_{r+1} - y_r}{\frac{c}{2}(y_{r+1} + y_r)},$$

where y_r is the r th term of the series, is discussed. This equation was originally employed by Professor Karl Pearson to obtain curves to fit the series $F(\alpha, \beta, \gamma, 1)$ in which the last element is unity. The curves which arise from its integration are known as the Pearson System of Frequency Curves. These will be denoted in the subsequent work by $P(z)$. When the last element of F is not unity, the resulting system of frequency curves has the more general form $e^{-\rho z} P(z)$. The goodness of fit of this new set of curves to the general series is tested in a variety of examples.

In § 2 an alternative method of curve fitting is given.

§§ 3—6 are devoted to the evaluation of the integral $\int_{z_1}^{z_2} e^{-xz} P(z) dz$.

§§ 7—9 give further and more accurate methods for summing the tail of the series, while § 10 is mainly a summary of the results and conclusions arrived at in the previous paragraphs.

§ 1. *On the Fitting of a System of Frequency Curves to the Hypergeometric Series $F(\alpha, \beta, \gamma, x)$.*

Let y_r denote the r th term of the hypergeometric series

$$1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \dots,$$

$$y_r = \frac{\alpha(\alpha+1)\dots(\alpha+r-2)\beta(\beta+1)\dots(\beta+r-2)}{(r-1)!\gamma(\gamma+1)\dots(\gamma+r-2)}x^{r-1}.$$

Form the ratio

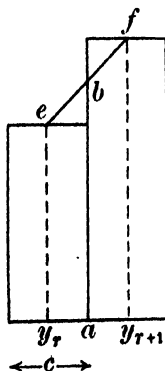
$$\frac{y_{r+1} - y_r}{y_{r+1} + y_r} = \frac{\frac{(\alpha+r-1)(\beta+r-1)}{r(\gamma+r-1)}x - 1}{\frac{(\alpha+r-1)(\beta+r-1)}{r(\gamma+r-1)}x + 1}$$

$$= \frac{r^2(x-1) + r(\alpha+\beta-2x-\gamma-1) + (\alpha-1)(\beta-1)x}{r^2(x+1) + r(\alpha+\beta-2x+\gamma-1) + (\alpha-1)(\beta-1)x}.$$

Let each term of the hypergeometric series be represented proportionally by a rectangular block of width c and assume the weight of each block to be concentrated at the mid-vertical. The ratio

$$\frac{\Delta y_r}{\frac{c}{2}(y_r + y_{r+1})}$$

is equal to the slope of the line ef divided by ab , ab being the ordinate of ef at the point a .

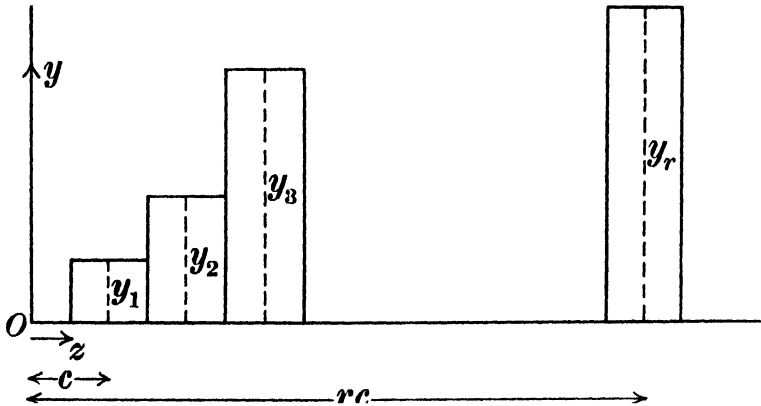


We wish to test the goodness of fit of the curves for which $\frac{1}{y} \frac{dy}{dz}$ is equal to this ratio at all points a midway between the mid-verticals of the rectangular blocks composing the histogram of the series*. Evidently, the equation of the required curve is

$$\frac{1}{y} \frac{dy}{dz} = \frac{\Delta y_r}{\frac{c}{2} (y_r + y_{r+1})} \dots \dots \dots (1),$$

where $rc = z$, measured from a suitable origin.

The distance of the r th mid-ordinate from a point O , distance c before the first mid-ordinate, is rc .



The relation (1) is true for all points at distances $(r + \frac{1}{2})c$, ($r = 0, 1, 2, \dots$) from O . Hence the equation of the curve which fits the series is

$$\frac{1}{y} \frac{dy}{dz} = \int \frac{\Delta y_r}{c \left(\frac{y_r + y_{r+1}}{2} \right)} \Big|_{(r + \frac{1}{2})c = z}$$

z being measured from the origin O .

It will simplify matters considerably if the origin be moved to the start of the histogram. This necessitates replacing rc of (1) by z . The equation of the required curve referred to the new origin is then

$$\frac{1}{y} \frac{dy}{dz} = \frac{2}{c} \frac{z^2(x-1) + zc(\alpha + \beta - 2x - \gamma - 1) + (\alpha - 1)(\beta - 1)xc^2}{z^2(x+1) + zc(\alpha + \beta - 2x + \gamma - 1) + (\alpha - 1)(\beta - 1)xc^2} \dots \dots (2).$$

The mid-verticals of the rectangular blocks are now at the points

$$z = (r + \frac{1}{2})c, \quad (r = 0, 1, 2, 3, \dots).$$

* The reader must bear in mind that we are *not* proceeding to a limit; c is finite.

The terms of the hypergeometric series will continue to increase as long as $y_{r+1}/y_r > 1$. It is reasonable, therefore, to determine the position of the modes from the values of r which satisfy $y_{r+1}/y_r = 1$, or what is equivalent,

$$\Delta y_r = 0 \dots\dots\dots(3).$$

If one value of r which satisfies (3) is positive, the modal term is the largest integer contained in $(r + 1)$.

Imagine for the moment the terms of the hypergeometric series to be ordinates of a continuous curve. If r_0 be a value of r satisfying (3), then the ordinates at r_0 and $r_0 + 1$ are equal. The mode (or antinode) of the curve if bell- (or U-) shaped lies somewhere between r_0 and $r_0 + 1$. When the standard deviation of the curve is fairly large, a good approximation to the position of the mode is $r_0 + \frac{1}{2}$. We shall take this to be our definition of the modal position of the hypergeometric series. The modal term is the nearest integer to $r_0 + \frac{1}{2}$. This is measured from a point distance c before the first mid-ordinate, hence, referred to start of histogram, the distance of the mode is r_0 . Of course, a more accurate determination of the position of the mode may be obtained by fitting a high order parabola

$$y = a_0 + a_1 z + a_2 z^2 + \dots$$

to the mid-ordinates of the series around the mode and solving $(dy/dz) = 0$.

Let us return now to the fundamental equation (2) which we will proceed to solve. It may be put in the form

$$\frac{1}{y} \frac{dy}{dz} = \frac{2}{c} \left(\frac{x-1}{x+1} \right) \frac{z^2 + zc \frac{(\alpha + \beta - 2)x - (\gamma - 1)}{x-1} + xc^2 \frac{(\alpha - 1)(\beta - 1)}{x-1}}{z^2 + zc \frac{(\alpha + \beta - 2)x + (\gamma - 1)}{x+1} + xc^2 \frac{(\alpha - 1)(\beta - 1)}{x+1}}.$$

$$\begin{aligned} \text{Let } \delta &\equiv c^2 \left[\left\{ \frac{(\alpha + \beta - 2)x + (\gamma - 1)}{x+1} \right\}^2 - 4 \frac{(\alpha - 1)(\beta - 1)}{x+1} x \right] \\ &= \frac{c^2}{(x+1)^2} \Delta, \end{aligned}$$

where, on simplification,

$$\Delta = [x^2(\alpha - \beta)^2 + 2x\{(\alpha + \beta)(\gamma + 1) - 2(\alpha\beta + \gamma)\} + (\gamma - 1)^2].$$

Three different cases arise according as Δ is positive, negative or zero. These will be considered separately.

I. Δ positive.

The denominator of the fundamental differential equation will split up into the two real factors

$$\begin{aligned} &\left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1) + \sqrt{\Delta}\} \right] \\ &\left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1) - \sqrt{\Delta}\} \right] \equiv (z + F_1)(z + F_2). \end{aligned}$$

Then
$$\frac{1}{y} \frac{dy}{dz} = \frac{2}{c} \left[\frac{(x-1)}{(x+1)} + \frac{2xc}{(x+1)^2} \left\{ \frac{z(\alpha + \beta - \gamma - 1) + (\alpha - 1)(\beta - 1)c}{(z + F_1)(z + F_2)} \right\} \right]$$

$$+ \frac{2}{c} \frac{(x-1)}{(x+1)} + \frac{4x}{(x+1)^2} \left\{ \frac{A}{z + F_1} + \frac{B}{z + F_2} \right\},$$

where $2A = -(\epsilon + 2) - \frac{1}{\sqrt{\Delta}} \{2(x+1)(\alpha\beta - \gamma) + (\epsilon + 2)(\alpha + \beta \cdot x + \gamma + 1)\},$

$$2B = -(\epsilon + 2) + \frac{1}{\sqrt{\Delta}} \{2(x+1)(\alpha\beta - \gamma) + (\epsilon + 2)(\alpha + \beta \cdot x + \gamma + 1)\}$$

and $\epsilon = \gamma - \alpha - \beta - 1.$

Integrating the original differential equation, we find

$$y = y_0 e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} [z + F_1]^{\frac{4Ax}{(x+1)^2}} [z + F_2]^{\frac{4Bx}{(x+1)^2}}.$$

This is a product of an exponential term into a Pearson Type I or VI curve, the former when F_1 and F_2 are of opposite sign and the latter when F_1 and F_2 are of the same sign.

II. Δ negative.

The denominator of the differential equation can now be expressed as a sum of two squares, namely,

$$\left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right]^2 + \frac{c^2 D}{4(x+1)^2},$$

where $D = |\Delta|$. Then

$$\frac{1}{y} \frac{dy}{dz} = -\frac{2}{c} \left(\frac{1-x}{1+x} \right) + \frac{4x}{(x+1)^2} \cdot \frac{z(\alpha + \beta - \gamma - 1) + (\alpha - 1)(\beta - 1)c}{R^2 + \frac{c^2 D}{4(x+1)^2}},$$

where $R = \left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right].$

Let $\zeta = R \frac{2(x+1)}{c\sqrt{D}}$, then

$$\frac{1}{y} \frac{dy}{d\zeta} = -\frac{\sqrt{D}(1-x)}{(1+x)^2} - \frac{2x(\epsilon + 2)}{(1+x)^2} \cdot \frac{2\zeta}{1 + \zeta^2} + \frac{\nu}{1 + \zeta^2},$$

where $\nu = \frac{8x}{\sqrt{D}(x+1)} \left[(\alpha - 1)(\beta - 1) + \frac{\epsilon + 2}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right].$

Hence, on integration,

$$y = y_0 e^{-\sqrt{D} \left\{ \frac{1-x}{(1+x)^2} \right\} \zeta} (1 + \zeta^2)^{-2x} \cdot \frac{\epsilon + 2}{(x+1)^2} e^{\nu \arctan \zeta}$$

$$= y_0' e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} e^{\frac{\nu \arctan \zeta}{c\sqrt{D}}} \left[R^2 + \frac{c^2 D}{4(x+1)^2} \right]^{\frac{2x(\epsilon + 2)}{(x+1)^2}}.$$

This is a product of an exponential term into a Pearson Type IV curve.

III. $\Delta = 0$.

For this case

$$\frac{1}{y} \frac{dy}{dz} = \frac{2(x-1)}{c(x+1)} + \frac{4x}{(x+1)^2} \cdot \left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right]^2$$

and we readily find

$$y = y_0 e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} \left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right]^{-\frac{4x(\epsilon+2)}{(x+1)^2}} \\ \times e^{-\frac{4xc\nu'}{(x+1)^2} \left[z + \frac{c}{2(x+1)} \{(\alpha + \beta - 2)x + (\gamma - 1)\} \right]},$$

where $\nu' = [(\alpha - 1)(\beta - 1) + \{(\alpha + \beta - 2)x + (\gamma - 1)\}(\epsilon + 2)/2(x + 1)]$.

This is a product of an exponential term into a Pearson Type V curve.

IV. Type III Curves.

When either α or β is unity, the differential equation reduces to the fairly simple form

$$\frac{1}{y} \frac{dy}{dz} = \frac{2z(x-1) + c(\alpha + \beta - 2x - \gamma - 1)}{c z(x+1) + c(\alpha + \beta - 2x + \gamma - 1)} \\ = -\frac{2}{c} \left(\frac{1-x}{1+x} \right) - \frac{4x(\epsilon+2)}{(x+1)^2} \cdot \frac{1}{z + \frac{c}{x+1}(\alpha + \beta - 2x + \gamma - 1)} \dots (i),$$

which gives the solution

$$y = y_0 e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} \left[z + \frac{c}{x+1}(\alpha + \beta - 2x + \gamma - 1) \right]^{-\frac{4x}{(x+1)^2} \left(\frac{\epsilon+2}{2} \right)} \dots (ii).$$

Since the hypergeometric series is symmetrical with respect to α and β , we may take $\beta = 1$. (ii) then reduces to the final form:

$$y = y_0 e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} \left[z + \frac{c}{x+1} \{(\alpha - 1)x + (\gamma - 1)\} \right]^{-\frac{4x(\gamma-\alpha)}{(x+1)^2}}, \quad 0 \leq z \leq \infty.$$

This is a Type III curve when the quantity $\psi = c \frac{(\alpha - 1)x + (\gamma - 1)}{x + 1}$ is positive.

When negative, the solution of (i) is

$$y = y_0' e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} [\psi' - z]^{-\frac{4x(\gamma-\alpha)}{(x+1)^2}}, \quad 0 \leq z \leq \psi',$$

where $\psi' = |\psi|$. This curve is not of Type III because its range is limited. It may be looked upon as a product of an exponential term into a Pearson Type VIII or IX curve, which we may symbolise as G_{VIII} or G_{IX} .

When the range of the series extends beyond ψ' , as would be the case were α and γ positive, fractional and less than unity, the equation of the curve corresponding to terms of the series beyond ψ' is

$$y = y_0'' e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} [z - \psi']^{-\frac{4x(\gamma-\alpha)}{(x+1)^2}}, \quad \psi' \leq z \leq \infty.$$

Hence, the complete solution of (i) when ψ is negative is the following:

$$y = y_0' e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} [\psi' - z]^{-4x \frac{(\gamma - \alpha)}{(x+1)^2}}, \quad 0 \leq z \leq \psi',$$

$$= y_0'' e^{-\frac{2}{c} \left(\frac{1-x}{1+x} \right) z} [z - \psi']^{-4x \frac{(\gamma - \alpha)}{(x+1)^2}}, \quad z > \psi'.$$

When $z = \psi'$, y becomes zero. We cannot, therefore, expect a good fit near this point.

When α and γ are negative and fairly large numbers, the terms of the series beyond the point ψ' are small and generally negligible compared with the modal terms. For such cases the second part of the solution of the differential equation will not be required for computing purposes.

Sufficient conditions for a Type III curve are, therefore,

- (a) α or β or both unity,
- (b) $[(\alpha - 1)x + (\gamma - 1)]/(x + 1)$ positive (for $\beta = 1$).

A necessary condition is that the numerator and denominator of the fundamental differential equation have a factor in common.

The curves which arise out of the solution of the differential equation

$$\frac{1}{y} \frac{dy}{dz} = \frac{az^2 + 2bz + d}{Az^2 + 2Bz + D}$$

have the form

$$y = e^{-\rho z} P(z),$$

where $P(z)$ is a Pearson type curve. These may, therefore, be divided into three main types, namely,

$$G_I: \quad y = y_0 e^{-\rho z} z^r (a - z)^s, \quad 0 \leq z \leq a,$$

$$G_{VI}: \quad y = y_0 e^{-\rho z} (z - a)^p z^{-q}, \quad a \leq z \leq \infty,$$

$$G_{IV}: \quad y = y_0 e^{-\rho z} \frac{e^{-\rho \arctan z/a}}{(1 + z^2/a^2)^r}, \quad -\infty \leq z \leq +\infty.$$

Integrating throughout the range for curves of type G_I , we have

$$N = y_0 \int_0^a e^{-\rho z} z^r (a - z)^s dz$$

$$= y_0 \sum_{t=0}^{\infty} \frac{(-)^t}{t!} \rho^t \int_0^a z^{r+t} (a - z)^s dz$$

$$= y_0 a^{r+s+1} \frac{\Gamma(s+1) \Gamma(r+1)}{\Gamma(r+s+2)} \left[1 - \frac{r+1}{1!(r+s+2)} \rho a \right. \\ \left. + \frac{(r+1)(r+2)}{2!(r+s+2)(r+s+3)} \rho^2 a^2 - \dots \right].$$

The higher moments are given by the partial differentials of N with respect to ρ , i.e.

$$N\mu_s' = (-)^s \frac{\partial^s N}{\partial \rho^s}.$$

The moments of the curves G_{VI} and G_{IV} may similarly be expressed by infinite series. I have attempted to use these series in a number of numerical examples but have found that they converge far too slowly to be of practical use.

Examples of the Above Method of Curve Fitting.

I. Series $F(30, 30, 111, .75)$ with $c = 1$.

The differential equation of the curve for which $d \log y/dz$ is equal to

$$\Delta y_r / \frac{1}{2}(y_{r+1} + y_r), \quad r = 0, 1, 2, \dots,$$

y_r being the r th term of the series, is the following:

$$\frac{1}{y} \frac{dy}{dz} = -\frac{2}{7} + \frac{13 \cdot 203,4565}{z + 4 \cdot 322,090} - \frac{64 \cdot 142,2321}{z + 83 \cdot 392,196}.$$

Its solution is

$$y = y_0 e^{-\frac{2}{7}z} (z + 4 \cdot 322,090)^{13 \cdot 203,457} (z + 83 \cdot 392,196)^{-64 \cdot 142,232}.$$

This curve belongs to type G_{IV} .

The equation giving the position of the modes of the curve is

$$z^2(x-1) + z\{(\alpha + \beta - 2)x - (\gamma - 1)\} + (\alpha - 1)(\beta - 1)x = 0.$$

This equation also gives the position of the modes of the series. It has two roots, namely, 9.168,914; -275.16891 . The latter is well outside the range of the series and must accordingly be rejected. Hence, the true position of the mode is 9.168,914 (i.e. the tenth term) *measured from the start of the histogram*.

The ordinates of the curve corresponding to the mid-points of the rectangular blocks representing the terms of the series are obtained by putting

$$z = r + \frac{1}{2} \quad (r = 0, 1, 2, 3, \dots)$$

in y .

The constant y_0 may be found by equating the value of one of the mid-ordinates of the curve to the corresponding term of the series. The best results will be obtained by selecting the mid-ordinate nearest a point of inflexion of the curve because here the area under the curve corresponds most closely to the numerical value of the mid-ordinate. This method has the objection of throwing too much weight on a single ordinate. This objection will shortly be removed when other methods of curve fitting are discussed (§ 2).

When we require merely the probability integral of the series (or the curve) and not the actual value of the partial sums (or partial areas), the question as to the choice of y_0 does not arise, because a probability integral measures the

proportional values of the frequencies and not their absolute values. In most of the subsequent examples the constant y_0 is calculated by equating the whole area of the curve (found by quadrature) to the total sum of the series (found mechanically). This is done to compare the *Probability Integral* of the series with the *Probability Integral* of the generalised Pearson curve which fits it. We do not know the actual sum of the general hypergeometric series and no one has yet succeeded in finding a simple and accurate expression for it. One method of obtaining an approximation to this sum is to calculate y_0 by equating the ordinate of the curve at a selected point to the corresponding term of the series. How good this approximation is may be judged from the subsequent examples.

In this first example, y_0 is calculated by equating the ordinates at a selected point.

Terms of Series	Mid-ordinates of Curve	Areas under Curve	Terms of Series	Mid-ordinates of Curve	Areas under Curve
1.000	1.702	1.866	58.705	59.024	59.176
6.081	7.202	7.539	43.152	43.496	43.638
19.567	20.703	21.166	31.030	31.373	31.496
44.397	45.067	45.503	21.877	22.198	22.299
79.328	79.595	79.826	15.152	15.438	15.517
119.717	119.496	119.421	10.327	10.572	10.632
158.031	157.618	157.252	6.937	7.139	7.183
187.553	187.124	186.573	4.599	4.760	4.937
203.994	203.639	203.042	3.013	3.138	3.160
206.280	206.015	205.490	1.952	2.047	2.062
196.095	195.916	195.535	1.252	1.324	1.334
176.795	176.702	176.484	.796	.847	.896
152.250	152.250	152.178	.502	.539	.544
125.971	126.068	126.107	.314	.340	.343
100.628	100.818	100.928	.195	.213	.215
77.926	78.198	78.342	.121	.134	.135
		

II. Series $F(-30, -50, 100, .5); c = 1.$

The equation of the curve which fits this series is

$$y = y_0 e^{-\frac{1}{2}z} \frac{e^{\frac{121928}{9\sqrt{(1379)}} \arctan \frac{3R}{\sqrt{(1379)}}}}{\left(R^2 + \frac{1379}{9}\right)^{\frac{9}{2}}},$$

where

$$R = z + \frac{1.8}{3}.$$

$$R_{(\text{mode})} = 24.870, 2702.$$

This curve belongs to G_{IV} .

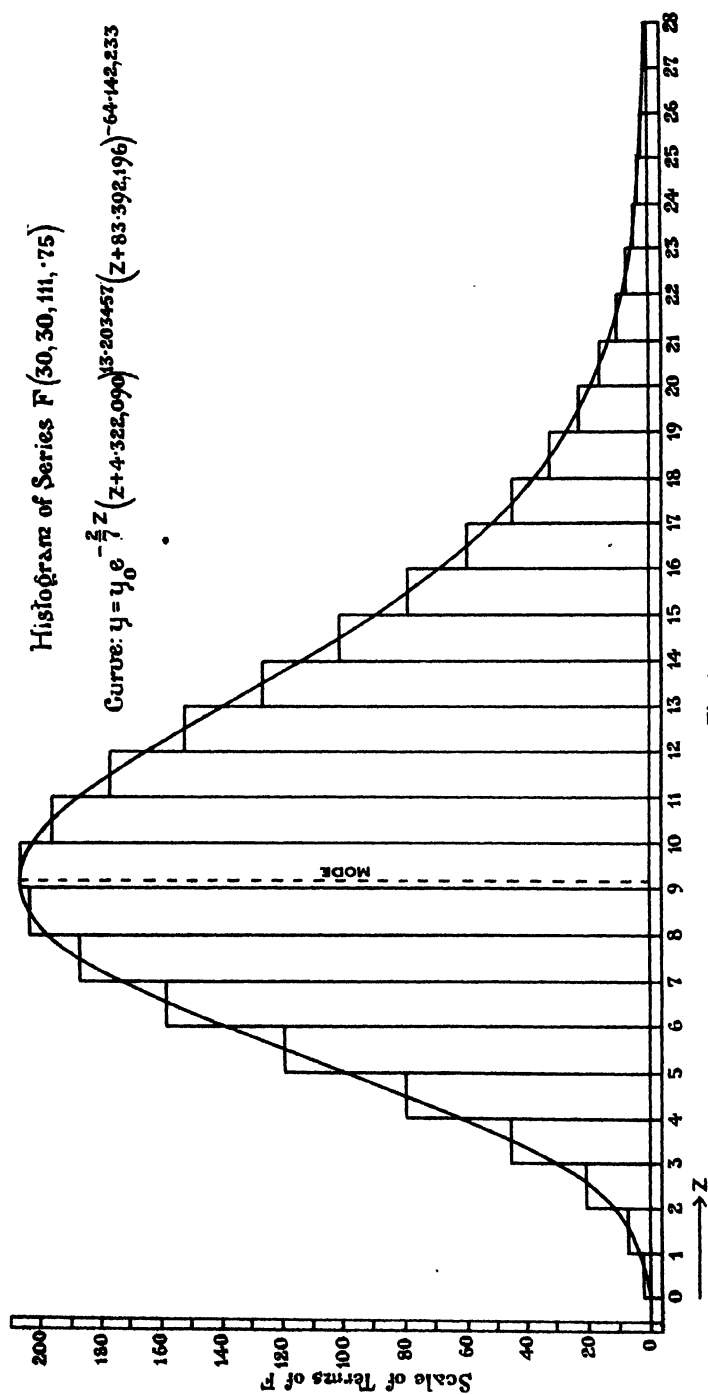
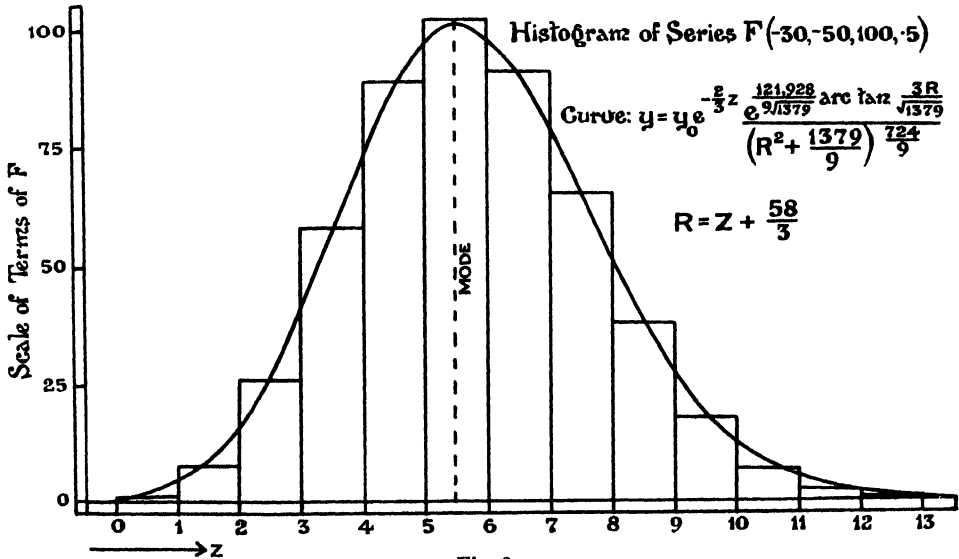


Fig. 1.

In this example, and in all subsequent ones, the constant y_0 is determined by equating the area of the curve, found by quadrature, to the sum of the series.

Terms of Series	Mid-ordinates of Curve	Areas under Curve	Terms of Series	Mid-ordinates of Curve	Areas under Curve
1.000	1.911	2.124	37.658	37.766	38.098
7.500	8.840	9.324	17.899	18.558	18.921
26.380	27.000	27.538	7.069	7.800	8.043
57.932	57.173	57.201	2.337	2.854	2.976
89.219	87.608	86.861	.650	.923	.974
102.602	100.864	99.994	.153	.268	.285
91.609	90.179	89.536	.030	.071	.076
65.188	64.446	64.423	.005	.016	.018
		



III. Series $F(-30, 60, -81, \frac{1}{3})$; $c = 1$.

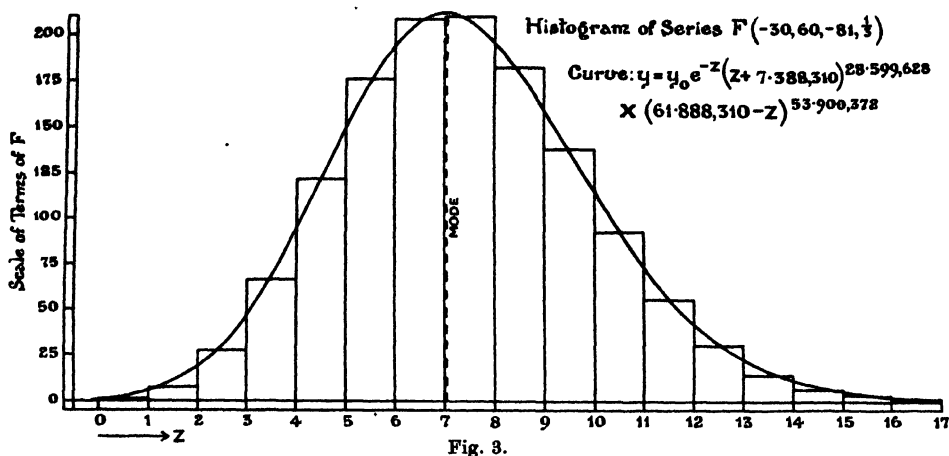
The equation of the curve which fits this series is

$$y = y_0 e^{-z} (z + 7.388,310)^{28.599,628} (61.888,310 - z)^{53.900,372}$$

which belongs to type G_1 .

$$z_{(\text{mode})} = 7.036,596.$$

Terms of Series	Mid-ordinates of Curve	Areas under Curve	Terms of Series	Mid-ordinates of Curve	Areas under Curve
1.000	1.993	2.318	92.107	92.054	92.405
7.407	11.184	11.512	55.036	55.637	56.115
27.299	29.002	29.901	29.462	30.381	30.803
66.655	67.169	67.817	14.189	15.094	15.388
121.133	120.184	120.174	6.165	6.863	7.040
174.515	172.667	172.412	2.421	2.870	2.963
207.300	205.218	203.876	.860	1.108	1.152
208.485	206.550	205.333	.276	.397	.416
180.898	179.374	178.696	.080	.131	.138
137.303	136.452	136.405



IV.

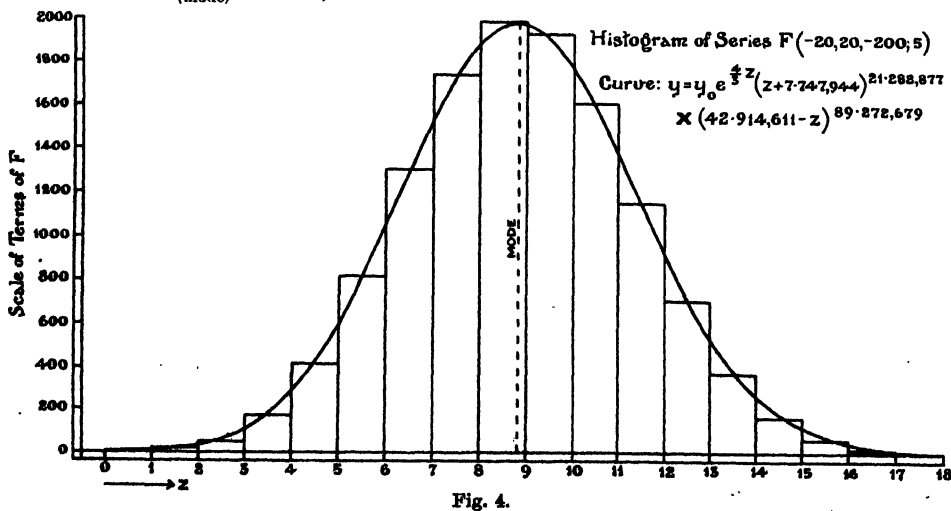
Series $F(-20, 20, -200, 5); c = 1.$

Terms of Series	Mid-ordinates of Curve	Areas under Curve	Terms of Series	Mid-ordinates of Curve	Areas under Curve
1.00	3.14	3.53	1609.69	1595.46	1589.87
10.00	16.185	17.48	1155.28	1148.56	1149.18
50.13	61.59	64.67	710.59	713.07	717.61
167.09	181.39	186.50	372.16	381.60	386.96
414.53	424.38	430.63	164.19	175.69	179.89
812.15	812.09	816.12	60.03	69.39	71.88
1301.52	1291.04	1289.33	17.74	23.38	24.56
1744.30	1727.35	1718.82	4.08	6.68	7.17
1982.67	1963.62	1951.16	.69	1.61	1.76
1927.60	1909.41	1898.31			

The curve which fits this series is

$$y = y_0 e^{\frac{1}{2}z} (z + 7.747,944)^{21.282,877} (42.914,611 - z)^{89.272,679},$$

$$z_{(\text{mode})} = 8.816,981.$$



V. *Series F*(1, -60, -65, 9); $c = 1$.

Equation of the curve which fits the above series is

$$y = y_0 e^{-105.963z} (63.631,579 - z)^{4.966,150}.$$

This curve belongs to type G_{IX} (see p. 64).

Terms of Series	Mid-ordinates of Curve	Areas under Curve	Terms of Series	Mid-ordinates of Curve	Areas under Curve
1.000,000	.993,952	.995,309	.003,534	.003,586	.003,595
.830,769	.826,180	.827,323	.002,762	.002,806	.002,813
.689,279	.685,840	.686,802	.002,150	.002,187	.002,193
.571,117	.568,580	.569,389	.001,666	.001,697	.001,702
.472,553	.470,719	.471,397	.001,285	.001,311	.001,314
.390,437	.389,146	.389,715	.000,987	.001,008	.001,011
.322,111	.321,235	.321,712	.000,753	.000,771	.000,773
.265,332	.264,769	.265,169	.000,572	.000,586	.000,589
.218,213	.217,884	.218,219	.000,432	.000,444	.000,445
.179,164	.179,008	.179,287	.000,324	.000,333	.000,334
.146,850	.146,819	.147,051	.000,241	.000,249	.000,250
.120,150	.120,205	.120,398	.000,178	.000,185	.000,186
.098,123	.098,234	.098,395	.000,131	.000,134	.000,135
.079,979	.080,126	.080,260	.000,095	.000,098	.000,098
.065,060	.065,227	.065,338	.000,068	.000,071	.000,071
.052,814	.052,989	.053,081	.000,050	.000,051	.000,051
.042,779	.042,954	.043,030	.000,034	.000,037	.000,037
.034,572	.034,742	.034,805	.000,024	.000,025	.000,025
.027,874	.028,034	.028,086	.000,016	.000,018	.000,018
.022,418	.022,566	.022,608	.000,011	.000,012	.000,012
.017,983	.018,118	.018,153	.000,007	.000,008	.000,008
.014,386	.014,507	.014,536	.000,005	.000,006	.000,006
.011,476	.011,584	.011,608	.000,003	.000,003	.000,003
.009,128	.009,222	.009,241	.000,002	.000,002	.000,002
.007,237	.007,319	.007,335	.000,001	.000,001	.000,001
.005,719	.005,790	.005,802	.000,000	.000,000	.000,000
.004,504	.004,565	.004,575			

VI. *Series F*(-100, -100, 1, $\frac{1}{2}$); $c = 1$.

The curve which fits the series is

$$y = y_0 e^{-\frac{2R}{3} e^{\frac{808}{9\sqrt{2}} \arctan \frac{3\sqrt{2}R}{202}}} / \left(R^2 + \frac{20402}{9} \right)^{\frac{808}{9}},$$

where $R = z - 101/3$. $z_{(\text{mode})} = 41.835,570$.

Significant Terms of Series $\times 10^{-45}$	Corresponding Mid-ordinates of Curve $\times 10^{-45}$	Areas under Curve $\times 10^{-45}$	Significant Terms of Series $\times 10^{-45}$	Corresponding Mid-ordinates of Curve $\times 10^{-45}$	Areas under Curve $\times 10^{-45}$
.000,002	.000,004	.000,004	.165,172	.164,379	.163,910
.000,007	.000,014	.000,015	.138,596	.137,167	.137,060
.000,027	.000,046	.000,048	.107,318	.107,009	.106,997
.000,093	.000,137	.000,142	.076,710	.076,730	.076,956
.000,287	.000,379	.000,393	.050,631	.051,513	.051,719
.000,803	.000,977	.001,008	.030,864	.031,366	.031,656
.002,048	.002,341	.002,401	.017,380	.017,935	.018,145
.004,762	.005,192	.005,300	.009,041	.009,542	.009,690
.010,110	.010,658	.010,828	.004,345	.004,731	.004,825
.019,630	.020,215	.020,451	.001,929	.002,190	.002,243
.034,900	.035,391	.035,669	.000,791	.000,949	.000,976
.056,888	.057,142	.057,404	.000,300	.000,386	.000,399
.085,104	.085,212	.085,183	.000,105	.000,147	.000,153
.116,959	.116,539	.116,503	.000,034	.000,053	.000,055
.147,794	.147,110	.146,830	.000,010	.000,018	.000,019
.171,857	.171,012	.170,664	.000,003	.000,006	.000,006
.184,023	.183,113	.182,497	.000,001	.000,002	.000,002
.181,571	.180,676	.180,092			

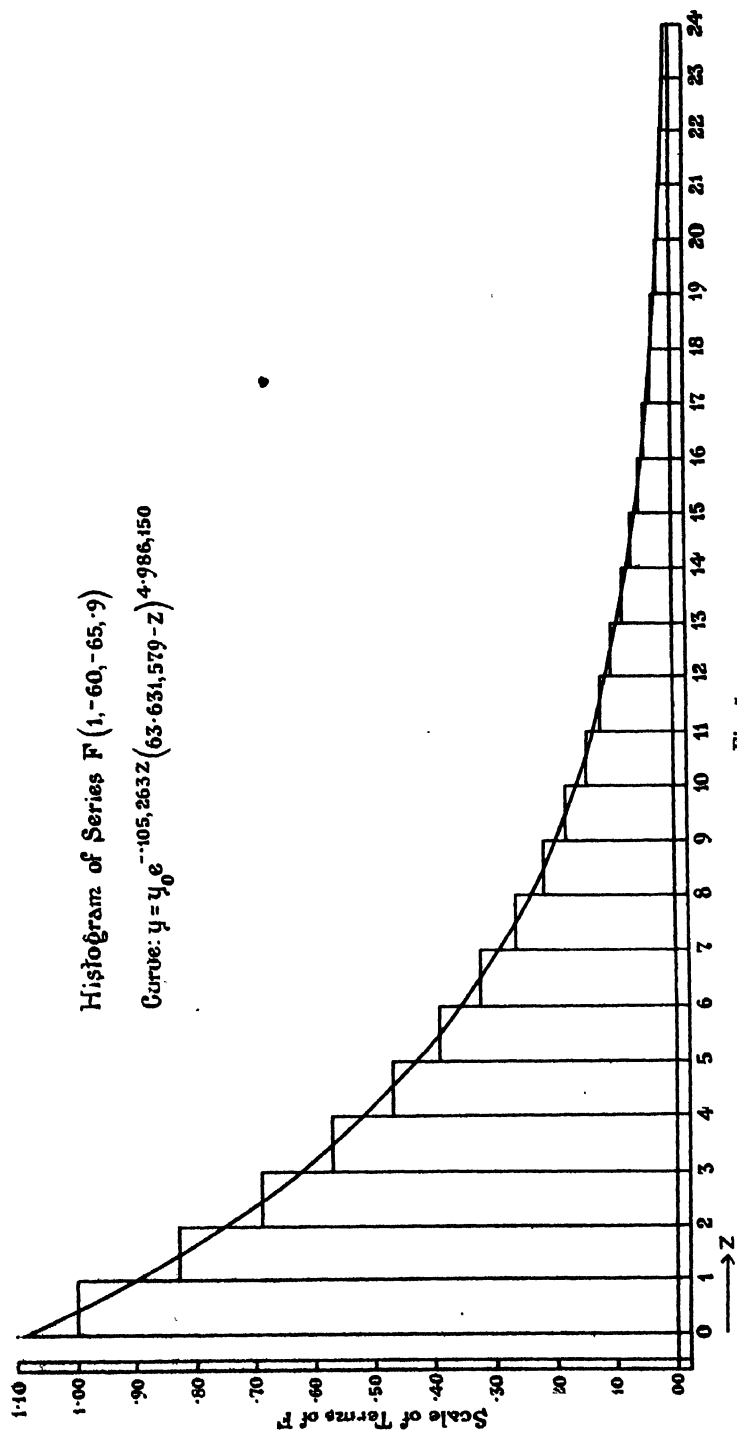


Fig. 5.

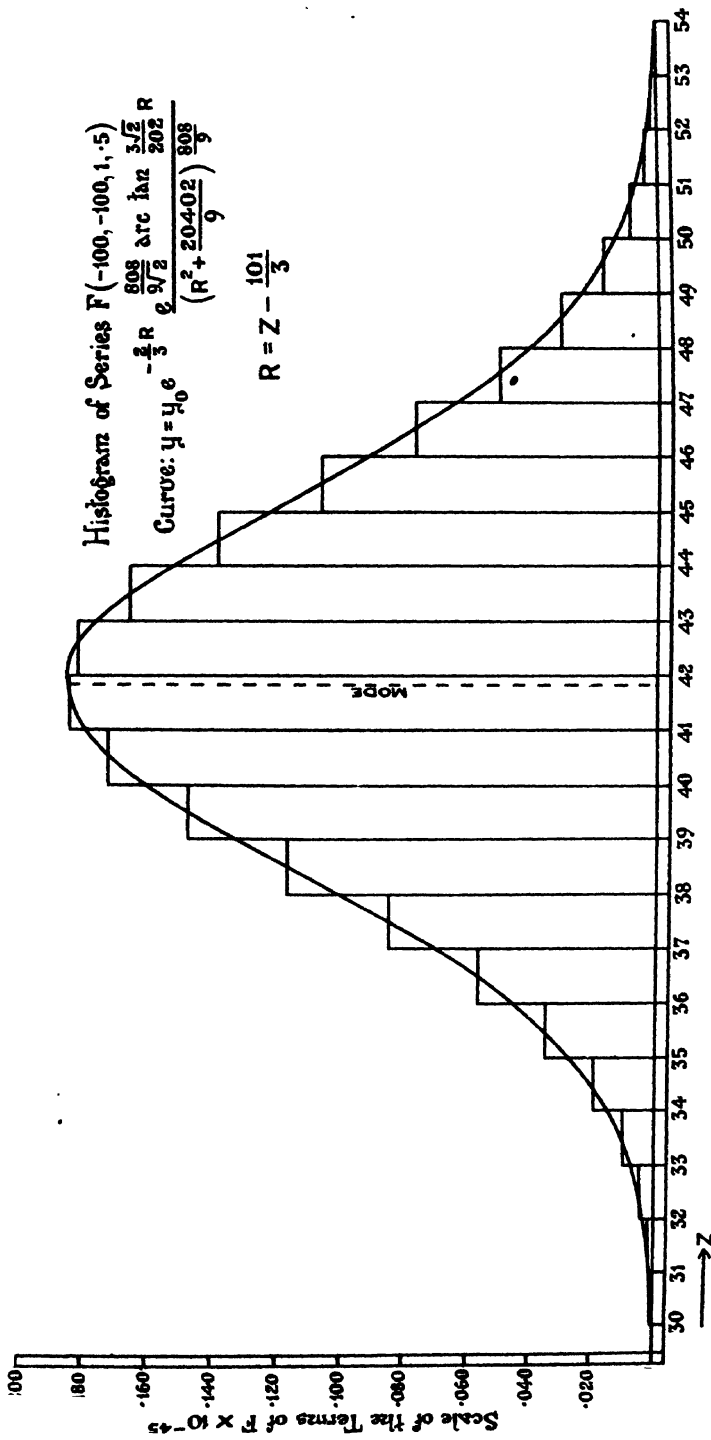


Fig. 6.

The condition $x \leq 1$ must be satisfied by all convergent infinite hypergeometric series; but for finite series (i.e., when α or β or both are negative integers) x may have any value whatsoever. A finite series with $x > 1$ may be reversed into one in which the fourth element is less than unity. Thus, if the series be

$$F(-\alpha', \beta, \gamma, x) = 1 - \frac{\alpha' \beta}{1! \gamma} + \frac{\alpha' (\alpha' - 1) \beta (\beta + 1)}{2! \gamma (\gamma + 1)} x^2 - \dots,$$

where α' is a positive integer, the last term is

$$(-)^{\alpha'} \frac{\beta (\beta + 1) \dots (\beta + \alpha' - 1)}{\gamma (\gamma + 1) \dots (\gamma + \alpha' - 1)} x^{\alpha'} = \chi.$$

Reversing the series, we have the equivalent one

$$\chi \left[1 - \frac{\alpha' \gamma + \alpha' - 1}{1! \beta + \alpha' - 1} \frac{1}{x} + \frac{\alpha' (\alpha' - 1) (\gamma + \alpha' - 1) (\gamma + \alpha' - 2)}{2! (\beta + \alpha' - 1) (\beta + \alpha' - 2)} \frac{1}{x^2} - \dots \right],$$

i.e.,

$$F(-\alpha', \beta, \gamma, x)$$

$$= (-)^{\alpha'} \frac{\beta (\beta + 1) \dots (\beta + \alpha' - 1)}{\gamma (\gamma + 1) \dots (\gamma + \alpha' - 1)} F \left\{ -\alpha', -(\gamma + \alpha' - 1), -(\beta + \alpha' - 1), \frac{1}{x} \right\},$$

i.e.

$$F(\alpha, \beta, \gamma, x) = \text{const.} \times F \left(\alpha, -\gamma + \alpha + 1, -\beta + \alpha + 1, \frac{1}{x} \right),$$

where α is a negative integer. We notice that $\epsilon = \gamma - \alpha - \beta - 1$ is the same for both series. Replacing z of the fundamental differential equation ((2), p. 61) by $-z - (\alpha - 1)/c$ results in the corresponding equation for the series

$$F \left(\alpha, -\gamma + \alpha + 1, -\beta + \alpha + 1, \frac{1}{x} \right).$$

Hence, nothing is gained by considering separately finite series in which x is greater than unity. All results which apply for finite series with $x < 1$ apply also, with a slight modification, to finite series with $x > 1$.

In the above examples the curve found by equating $\frac{1}{y} \frac{dy}{dz}$ to $\Delta y_{r/2} c (y_{r+1} + y_r)$ of the series, where y_r is the r th term, fits the significant part of the series tolerably well. There is, however, a slight deficiency about the mode and excess towards the tails. The goodness of fit of this system of curves improves rapidly when the number of significant terms of the series increases.

§ 2. *An Alternative Method for the Fitting of Generalised Pearson Curves to $F(\alpha, \beta, \gamma, 1)$.*

Consider firstly the series

$$1 + \frac{\alpha \beta}{1! \gamma} + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{2! \gamma (\gamma + 1)} + \dots \dots \dots (i),$$

in which x is unity. When its standard deviation is not small, a Pearson curve $P(z)$

may be fitted to it with sufficient accuracy for most statistical purposes*. Let, then, $P_1(z)$ be the Pearson curve having the same first four moments as (i). Let z_1 be the value of z which corresponds to the mid-ordinate of the first block of the histogram of series (i). The abscissa of the mid-vertical of the r th block is, therefore, $P_1(z_1 + r)$, the grouping unit c being taken as unity. $P_1(z + r)$ also gives quite closely the numerical value of the r th term of the series (i). Hence, the expression $x^{r-1}P_1(z_1 + r)$ gives quite closely the numerical value of the r th term of the general series

$$1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \dots \dots\dots(ii).$$

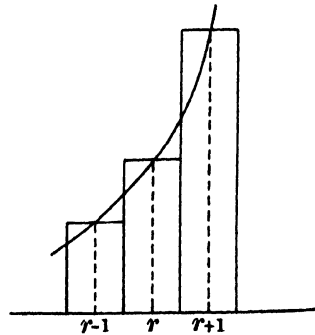
Accordingly, the generalised Pearson curve

$$x^{z-z_1-1}P_1(z) = \text{const. } x^z P_1(z)$$

will give quite closely the *ordinates* of the general series.

We know that the areas under the Pearson curve $P(z)$ accord more closely to the terms of the series (i) than the ordinates*, but owing to the presence of the factor x^z , or what is equivalent, $e^{\rho z}$ where $\rho = \log_e x$, this property no longer holds for the generalised Pearson curve. We can, however, apply a small correction so that the *areas* under $e^{\rho z} P_1(z)$ will accord to the terms of the series (ii) at least as closely as do the areas under $P_1(z)$ accord to the terms of (i). This correction is obtained as follows:

Let Y_r denote the ordinate of $x^z P_1(z)$ which corresponds to the mid-point of the $(r+1)$ st block of the histogram of the general series, and let y_r be the corresponding ordinate of $P_1(z)$. Then $Y_r = x^r y_r$.



Let N_r denote the $(r+1)$ st term of the series (ii) and n_r the $(r+1)$ st term of (i). Then, $N_r = x^r n_r$.

When the standard deviation of the series is large, the following formula expresses with sufficient accuracy, n_r in terms of y_r :

$$n_r = \frac{1}{24} (22y_r + y_{r-1} + y_{r+1}) \dots\dots\dots(iii).$$

* See O. L. Davies, *Biometrika*, Vol. xxv. pp. 295—322.

The area under the generalised Pearson curve corresponding to the $(r+1)$ st term of the series (ii) is

$$\frac{1}{24} (22Y_r + Y_{r-1} + Y_{r+1}) \dots\dots\dots(\text{iv}).$$

Now
$$N_r = x^r n_r = \frac{x^r}{24} (22y_r + y_{r-1} + y_{r+1})$$
$$= \frac{1}{24} (22Y_r + Y_{r-1}x + Y_{r+1}/x) \dots\dots\dots(\text{v}).$$

Hence, the correction to be applied to the $(r+1)$ st sectional area (iv) is

$$c_r = -\frac{1}{24} [Y_{r-1}(1-x) + Y_{r+1}(1-1/x)]$$
$$= -\frac{(1-x)}{24x} [xY_{r-1} - Y_{r+1}] \dots\dots\dots(\text{vi}).$$

If, therefore, Na_t denotes the whole area of the curve $P_1(z)x^z$ up to and including the $(t+1)$ st term $\left(= \int_0^{t+1} Y_z dz \right)$, the correction for Na_t is

$$\frac{1-x}{24x} \sum_{r=0}^t (Y_{r+1} - xY_{r-1})$$
$$= \frac{1-x}{24x} \left[\sum_{r=0}^t Y_r(1-x) + (Y_{t+1} + xY_t) - (Y_0 + xY_{-1}) \right] \dots\dots(\text{vib}).$$

If there is high contact at the start of the curve, y_{-1} and y_0 are very nearly zero and the correction simplifies to

$$\frac{1-x}{24x} \left[\sum_{r=0}^t Y_r(1-x) + (xY_t + Y_{t+1}) \right].$$

Now $\sum_{r=0}^t Y_r = Na_t + E$, where E is a small correction found by applying the Euler-Maclaurin theorem.

The correction to be applied to Na_t so that it may give a closer approximation to the sum of the first $(t+1)$ terms of the series (ii) is, therefore,

$$C_r = \left(\frac{1-x}{24x} \right) [(Na_t + E)(1-x) + (xY_t + Y_{t+1})] \dots\dots\dots(\text{vii}).$$

E is of the second degree of smallness compared with C_r , and when x is not small it may be neglected altogether. C_r may then be written in the form

$$\frac{(1-x)^2}{24x} Na_t + (xY_t + Y_{t+1}) \frac{1-x}{24x} \dots\dots\dots(\text{viii}),$$

which may be calculated quite readily.

If there be high contact at both ends, the correction for the whole area of the curve reduces to $\frac{(1-x)^2}{24x} N$. A close approximation to the sum of the whole series is, therefore,

$$N \left[1 + \frac{(1-x)^2}{24x} \right],$$

where N is the total area of the curve $Y = x^z P_1(z)$, provided x is not small. When x approaches unity, the correction tends to zero.

If the simple formula (iii) does not give sufficient accuracy, we may apply the higher quadrature formula

$$n_r = \frac{1}{5760} [5178y_r + 308(y_{r+1} + y_{r-1}) - 17(y_{r+2} + y_{r-2})].$$

The correction to be applied to the $(r+1)$ st sectional area is

$$\frac{(1-x)}{5760x} \left[308(Y_{r+1} - xY_{r-1}) - 17 \frac{1+x}{x} (Y_{r+2} - x^2Y_{r-2}) \right].$$

Accordingly, the correction to be applied to the area $N\alpha_t$ is

$$\begin{aligned} & \frac{(1-x)^2}{5760x} N\alpha_t \left[308 - 17 \frac{(1+x)^2}{x} \right] \\ & + \frac{(1-x)}{5760x} \left[308(Y_{t+1} + xY_t) - 17 \left(\frac{1+x}{x} \right) \{ (Y_{t+1} + Y_{t+2}) - x^2(Y_{t-1} + Y_t) \} \right] \\ & - \frac{(1-x)}{5760x} \left[308(Y_0 + xY_{-1}) + 17 \left(\frac{1+x}{x} \right) \{ (Y_{-1} + Y_{-2})x^2 + (Y_1 + Y_0) \} \right]. \end{aligned}$$

When there is high contact at the start of the curve, the last term approximates to zero. Further, when there is high contact at both ends of the curve, the correction for the whole area is

$$\frac{(1-x)^2}{5760x} N \left\{ 308 - 17 \frac{(1+x)^2}{x} \right\},$$

and accordingly, an approximation to the sum of the general series (ii) is

$$N \left[1 + \frac{(1-x)^2}{5760x} \left\{ 308 - 17 \frac{(1+x)^2}{x} \right\} \right],$$

where N is the total area of the curve $x^2 P_1(z)$.

The following results will give an indication of the magnitude of the correction C for the whole area when there is high contact at both ends.

$$\begin{aligned} x = .5, & \quad C = \left(\frac{4639}{3304} \right) \% \cong 2\%, \\ x = .75, & \quad C \cong .34\%, \\ x = .9, & \quad C \cong .046\%. \end{aligned}$$

In the following examples, the areas under the curves obtained by both methods are compared with the terms of the series.

The curve obtained by applying the first method, i.e. from the equation

$$\frac{1}{y} \frac{dy}{dz} = \frac{\Delta y_r}{\frac{c}{2}(y_r + y_{r+1})}$$

(§ 1), will be referred to as curve A, while the one obtained by applying the above method will be referred to as curve B.

Examples.

I. Series $F(30, 30, 111, .75)$; $c = 1$.

Equation of curve A is

$$y = y_0 e^{-\frac{1}{2}z} (z + 4.322,090)^{13.203,457} (z + 83.392,196)^{-64.143,232}.$$

Equation of curve B is

$$y = y_0' e^{-287.662z} (z + 4.408,767)^{13.406,468} (z + 90.876,152)^{-69.792,127}.$$

The origin of both curves is at the start of the histogram of the series.

Terms of Series	Areas under A	Corrected Areas under B	Terms of Series	Areas under A	Corrected Areas under B
1.000	1.866	2.143	43.152	43.638	43.174
6.081	7.539	7.592	31.030	31.496	31.036
19.567	21.166	21.147	21.877	22.299	21.875
44.328	45.503	45.344	15.152	15.517	15.146
79.397	79.826	79.539	10.327	10.632	10.321
119.717	119.421	119.114	6.937	7.183	6.932
158.031	157.252	157.067	4.599	4.937	4.596
187.553	186.573	186.615	3.013	3.160	3.010
203.994	203.042	203.317	1.952	2.062	1.951
206.280	205.490	205.922	1.252	1.334	1.252
196.095	195.535	196.001	.796	.896	.796
176.795	176.484	176.865	.502	.544	.502
152.250	152.178	152.392	.314	.343	.314
125.971	126.107	126.115	.195	.215	.195
100.628	100.928	100.745	.121	.134	.121
77.926	78.342	78.005
58.705	59.176	58.752			

II. Series $F(-30, -50, 100, \frac{1}{2}); c = 1$.

Equation of curve A is

$$y = y_0 e^{-\frac{2}{3}z} \left\{ \left(z + \frac{58}{3} \right)^2 + \frac{1379}{9} \right\}^{-\frac{2172}{27}} e^{\frac{121928}{9\sqrt{(1379)}} \arctan \frac{3}{\sqrt{(1379)}} \left(z + \frac{58}{3} \right)}.$$

Equation of curve B is

$$y = y_0' e^{-693.147z} (z + 6.051,499)^{27.737,858} (36.453,380 - z)^{52.069,623},$$

origin of both curves being at the start of the histogram.

Terms of Series	Areas under A	Corrected Areas under B	Terms of Series	Areas under A	Corrected Areas under B
1.000	2.124	1.506	17.899	18.921	17.909
7.500	9.324	8.157	7.069	8.043	7.072
26.380	27.538	26.523	2.337	2.976	2.341
57.932	57.201	57.502	.650	.974	.653
89.219	86.861	88.893	.153	.285	.154
102.602	99.994	102.742	.030	.076	.031
91.609	89.536	91.946	.005	.018	.005
65.188	64.423	65.408
37.658	38.098	37.728			

III. Series $F(-30, 60, -81, \frac{1}{3}); c = 1$.

Equation of curve A is

$$y = y_0 e^{-z} (z + 7.388,310)^{28.599,623} (61.888,310 - z)^{53.900,372}.$$

Equation of curve B is

$$y = y_0' e^{-1.098,812z} (z + 7.531,867)^{27.629,947} (37.648,329 - z)^{24.035,187},$$

origin of both curves being at the start of the histogram.

Terms of Series	Areas under A	Corrected Areas under B	Terms of Series	Areas under A	Corrected Areas under B
1.000	2.318	1.937	92.107	92.405	92.144
7.407	11.512	10.646	55.036	56.115	55.074
27.299	29.901	30.477	29.462	30.803	29.478
66.655	67.817	70.564	14.189	15.388	14.191
121.133	120.174	123.146	6.165	7.040	6.164
174.515	172.412	174.767	2.421	2.963	2.420
207.300	203.876	206.642	.860	1.152	.859
208.485	205.333	207.814	.276	.416	.276
180.898	178.696	180.567	.080	.138	.080
137.303	136.405	137.235

IV.

Series F (1, -60, -65, .9); c = 1.

Equation of curve A is

$$y = y_0 e^{-105,2632z} (63.631,579 - z)^{4.988,150}.$$

Equation of curve B is

$$y = y_0' e^{-105,3606z} (63.349,729 - z)^{4.988,824},$$

origin of both curves being at the start of the histogram of the series.

Terms of Series	Areas under A	Corrected Areas under B	Terms of Series	Areas under A	Corrected Areas under B
1.000,000	.995,309	.999,980	.003,534	.003,595	.003,534
.830,769	.827,323	.830,630	.002,762	.002,813	.002,762
.689,279	.686,802	.689,193	.002,150	.002,193	.002,150
.571,117	.579,389	.571,058	.001,666	.001,702	.001,666
.472,553	.471,397	.472,519	.001,285	.001,314	.001,285
.390,437	.389,715	.390,421	.000,987	.001,011	.000,987
.322,111	.321,712	.322,105	.000,753	.000,773	.000,753
.265,332	.265,169	.265,334	.000,572	.000,589	.000,572
.218,213	.218,219	.218,221	.000,432	.000,445	.000,432
.179,164	.179,287	.179,174	.000,324	.000,334	.000,324
.146,850	.147,051	.146,863	.000,241	.000,250	.000,241
.120,150	.120,398	.120,162	.000,178	.000,186	.000,178
.098,123	.098,395	.098,136	.000,131	.000,135	.000,131
.079,979	.080,260	.079,990	.000,095	.000,098	.000,095
.065,060	.065,338	.065,071	.000,068	.000,071	.000,068
.052,814	.053,081	.052,823	.000,049	.000,051	.000,049
.042,779	.043,030	.042,786	.000,034	.000,037	.000,034
.034,572	.034,805	.034,579	.000,024	.000,025	.000,024
.027,874	.028,086	.027,879	.000,016	.000,018	.000,016
.022,418	.022,608	.022,422	.000,011	.000,012	.000,011
.017,983	.018,153	.017,987	.000,007	.000,008	.000,007
.014,386	.014,536	.014,389	.000,005	.000,006	.000,005
.011,476	.011,608	.011,479	.000,003	.000,003	.000,003
.009,128	.009,241	.009,129	.000,002	.000,002	.000,002
.007,237	.007,335	.007,238	.000,001	.000,001	.000,001
.005,719	.005,802	.005,720	.000,000	.000,000	.000,000
.004,504	.004,575	.004,504

There is very little difference between the fit of the two curves at points up to the mode, but after the mode and especially towards the tail, the curve B is a distinct improvement on curve A. In the last example, where the curve is J-shaped, the former method gives far better results than that developed in § 1. An added advantage is that the fairly troublesome type G_{IV} does not arise. It must be remembered, however, that this method is applicable only when the series with x replaced by unity is convergent.

§ 3. Evaluation of the Incomplete Integral $\int_{x_1}^{x_2} e^{-\rho z} P(z) dz$.

In the previous paragraphs we have shown that when the number of significant terms is fairly large, a general hypergeometric series $F(\alpha, \beta, \gamma, x)$ may be fitted quite closely by a curve of the form

$$y = y_0 e^{-\rho z} P(z) \dots\dots\dots(1),$$

where $P(z)$ is a Pearson type curve. Hence, a number of terms of this series can be represented by an integral of y taken between the proper limits. When this curve is not very skew, we may often evaluate the integral by using the method outlined by Pearson in *Biometrika*, vol. I. p. 390 *et seq.* This consists in expanding (1) about its mode into a series of the form

$$y = y_m e^{\frac{1}{2} \left(\frac{d^2 u}{dz^2} \right)_m z^2} (1 + b_3 z^3/3! + b_4 z^4/4! + \dots),$$

where $u = \log y$, and $z = m$, the position of the mode of the curve y . The coefficients are expressed by Pearson in the form

$$\begin{aligned} b_3 &= a_3, & b_6 &= (a_6 + 10a_3^2), \\ b_4 &= a_4, & b_7 &= (a_7 + 35a_3a_4), \\ b_5 &= a_5, & b_8 &= (a_8 + 56a_3a_5 + 35a_4^2), \text{ etc.} \end{aligned}$$

where $a_i = \left(\frac{d^i u}{dz^i} \right)_{z=m}$ ($i = 1, 2, \dots$).

These coefficients become increasingly more complicated.

When finding the partial integrals of the curve, i.e., integrals from the start of the curve up to a point z , we may safely take the lower limit to be $-\infty$. This enables us to express the partial areas of (1) in terms of incomplete normal moment functions. These have been tabulated up to the twelfth. (See *Tables for Statisticians and Biometricians*, Pts. I and II.)

If l_1 and l_2 are the limits to the range of the curve y_1 the complete integral is

$$\begin{aligned} N &= \int_{l_1}^{l_2} y dz \cong \int_{-\infty}^{+\infty} y dz \\ &= \frac{y_m \sqrt{(2\pi)}}{\sqrt{(-a_2)}} \left[1 + \frac{b_4}{8a_2^2} - \frac{b_6}{48a_2^3} + \dots \right] \dots\dots\dots(2). \end{aligned}$$

This method will be referred to as Method A.

Example.

$$y = y_0 e^{-z} z^{30} (40 - z)^{60}.$$

The total area of the curve found by quadrature is $79345.112 y_0 10^{110}$.

It is convenient to take $y_0 = 10^{-110}$.

Now $u = \log y$ and $\frac{du}{dz} = -1 + \frac{30}{z} - \frac{60}{40-z}.$

The mode is, therefore, at the point $z = 10$. The constants

$$a_q = \left(\frac{d^q u}{dz^q} \right)_{10} = (q-1)! \left\{ (-1)^q \frac{30}{10^q} - \frac{60}{30^q} \right\} \quad (q = 2, 3, \dots)$$

are readily found. Substitute their values in (2) and we have

$$\begin{aligned} N &= 79668.590 (1 - .017,14876 + .011,51807 \\ &\quad + .004,69824 - .003,31286 + .000,90737 - \dots) \\ &= 79402.661. \end{aligned}$$

This is in excess of the true value by approximately .072 %, which is sufficiently close for most statistical purposes.

Method A has been applied* with varying success to the evaluation of the incomplete Beta function. The results obtained there hold with a slight modification for integrals of the curve $e^{-pz} P(z)$ because this curve and $P(z)$ behave in very much the same manner. It has been shown that the curve obtained by Method A fits a Pearson type curve very well at points lying within $\pm 1.5\sigma$ of the mode, but may deviate appreciably at points further removed from the mode. This is also the case with the more general curves.

In the next three paragraphs we shall develop other and more accurate methods of curve fitting.

§ 4. *The Fitting of Pearson Curves to Generalised Pearson Curves by Equating the Corresponding Logarithmic Differentials about the Mode.*

Method A may be generalised somewhat by expanding

$$y = y_0 e^{-pz} P(z)$$

around the mode into a series of the form

$$y = y_0 u(z) \{1 + b_1 z + b_2 z^2 + \dots\} \dots\dots\dots(i),$$

where $u(z)$ is a Pearson curve of the same type as $P(z)$.

When $P(z)$ is a Type I curve, y may be expressed in the form

$$y = y_0 e^{-pz} z^r (a-z)^s \dots\dots\dots(ii).$$

Let (ii) be represented around its mode " m " by the curve

$$y = y_0' (z+m)^p (a-m-z)^q \dots\dots\dots(iii),$$

where p and q are at our disposal.

* *Biometrika*, Vol. vi. (1908), p. 68; *Introduction to Incomplete Γ -Function Tables* (1922), pp. xviii—xix; *Tables for Statisticians*, Part II. (1931), pp. cxxxix—cxxxix; *Tables of the Incomplete B-Function* (1934), p. vi; Soper, *Tracts for Computers*, VII. (1921), pp. 43—44; Wishart, *Biometrika*, Vol. xix. (1927), p. 488.

Now
$$y = y_0'' e^{-\rho z} (z+m)^r (a-m-z)^s$$

$$= y_0''' (m+z)^p (a-m-z)^q V,$$

where
$$V = e^{-\rho z} (m+z)^{r-p} (a-m-z)^{s-q}$$

and m the distance of the mode of (ii) from the start of the curve.

Let
$$v = \log V,$$

then
$$(dV/dz) = V(dv/dz),$$

$$(d^2V/dz^2) = V(d^2v/dz^2) + V(dv/dz)^2, \text{ etc.}$$

Choose p and q so that
$$(dv/dz)_m = 0 \dots\dots\dots(\text{iv } a),$$

$$(d^2v/dz^2)_m = 0 \dots\dots\dots(\text{iv } b).$$

The two curves (ii) and (iii) have the same range. Condition (iv a) implies that they also have the same mode and condition, (iv b) indicates that the two curves coincide to the second order around the mode.

By Maclaurin's theorem

$$y = y_0^{(iv)} (m+z)^p (a-m-z)^q$$

$$\times \left\{ 1 + \frac{1}{6} \left(\frac{d^2v}{dz^2} \right)_m z^2 + \frac{1}{24} \left(\frac{d^4v}{dz^4} \right)_m z^4 + \frac{1}{120} \left(\frac{d^6v}{dz^6} \right)_m z^6 + \dots \right\}.$$

The coefficients of the successive powers of z have the same form as those for the expansion obtained by Method A. Here, however,

$$\alpha_i = (d^i \log V/dz^i)_m; \quad V = e^{-\rho z} (m+z)^{r-p} (a-m-z)^{s-q}.$$

The partial integral $y_0 \int_0^z e^{-\rho z} z^r (a-z)^s dz$ may now be expressed in terms of the incomplete moments of Type I curves about the mode which may, in turn, be expressed in terms of incomplete Beta functions. These have been tabulated within a certain range, and, for functions lying outside this range, auxiliary tables have been constructed to aid in their evaluation.

The t th moment of $(m+z)^p (a-m-z)^q$ about its mode is

$$\int_{-m}^{a-m} (m+z)^p (a-m-z)^q z^t dz$$

$$= (-)^t \int_0^a z^p (a-z)^q (m-z)^t dz$$

$$= (-)^t \sum_{i=0}^{\infty} (-)^i m^{t-i} \int_0^a z^{p+i} (a-z)^q \{t!/i! (t-i)!\} dz$$

$$= (-)^t a^{p+q+1} \frac{\Gamma(p+1) \Gamma(q+1)}{\Gamma(p+q+2)} \left[m^t - t m^{t-1} \frac{(p+1)a}{(p+q+2)} \right.$$

$$+ \frac{t(t-1)}{2!} m^{t-2} \frac{(p+1)(p+2)}{(p+q+2)(p+q+3)} a^2$$

$$- \dots + (-)^t \frac{(p+1)(p+2) \dots (p+t)a^t}{(p+q+2)(p+q+3) \dots (p+q+t+1)} \left. \right]$$

$$\equiv (-)^t a^{p+q+1} \frac{\Gamma(p+1) \Gamma(q+1)}{\Gamma(p+q+2)} M_t.$$

Hence the total area N is equal to

$$y_m m^{-p} (a-m)^{-q} a^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} [1 - b_3 M_3 + b_4 M_4 - b_5 M_5 + \dots] \quad (v).$$

When evaluating the incomplete integral of y , M_t has to be replaced by the corresponding series of incomplete Beta function ratios.

Examples.

$$\begin{aligned} \text{I.} \quad y &= y_0 e^{-z} z^{30} (40-z)^{60}; \\ &= y_0 z^p (40-z)^q V; \quad y_0 = 10^{-110}, \end{aligned}$$

$$\text{where} \quad V = e^{-z} z^{30-p} (40-z)^{60-q}.$$

Choose those values of p and q which satisfy

$$0 = (d \log V / dz)_m = -1 + \frac{(30-p)}{10} - \frac{(60-q)}{30},$$

$$0 = (d^2 \log V / dz^2)_m = -\frac{(30-p)}{100} - \frac{60-q}{900};$$

$$\text{i.e.} \quad p = 27.5, \quad q = 82.5.$$

The constants $a_t = (d^t \log V / dz^t)_m$ may be readily calculated from

$$\begin{aligned} a_t &= (t-1)! \left[(-)^{t+1} \frac{(30-p)}{10^t} - \frac{(60-q)}{30^t} \right] \\ &= \frac{(t-1)! 2.5}{10^t} [(-)^{t+1} + (\frac{1}{3})^{t-2}]. \end{aligned}$$

Substitute these values in (v) and we have the following series for the total area:

$$\begin{aligned} N &= 79210 (1 + .002,63380 - .001,27338 + .000,22961 \\ &\quad + .000,06981 - .000,02788 + .000,04546 - \dots) \\ &= 79342.867. \end{aligned}$$

This involves an error of .0028%. When we include seven terms in the expansion of y , the resulting curve fits y very well at points lying within the range $\pm 2.5\sigma$ of the mode, but may deviate somewhat at points lying outside this range.

$$\text{II.} \quad y = y_0 e^{-iz} z^{100} (150-z)^{150}, \quad y_0 = 10^{-320} e^{25} \quad (\text{mode at } z = 50).$$

$$\text{We have} \quad y = y_0' (z+50)^p (100-z)^q (1 + a_3 z^3/3! + a_4 z^4/4! \dots),$$

where a_i is the i th logarithmic derivative of V at $z=0$, V being the expression

$$e^{-iz} (z+50)^{100-p} (100-z)^{150-q}.$$

The values of p and q found by equating to zero the first two derivatives of $\log V$ at $z=0$, are

$$p = 275/3, \quad q = 2p = 550/3.$$

The series giving the total area N is

$$\begin{aligned} N &= c (1 + .0005,39144 - .0002,44637 + .0000,12131 \\ &\quad + .0000,44450 - .0000,02193 - \dots) \\ &= c (1.000,353282) \text{ and } c = 8.409,952. \end{aligned}$$

$$\text{Therefore} \quad N = 8.412923.$$

The area of the curve found by quadrature is 8.412901. Hence N differs from the true value by less than .0003%. Had we taken $N=c$, i.e. neglected all terms except the first in the expansion, the error would only have been .03%. This in itself is sufficiently accurate for most statistical purposes. If we include six terms in the expansion of y , the resulting curve fits y remarkably well at points within $\pm 3\sigma$ of the mode.

The series obtained by the application of Method A is

$$N' = c' (1 - .004,33884 + .002,01703 \dots).$$

Its convergence is not nearly as rapid as the series obtained by the method of this section.

Although the latter method gives far greater accuracy than Pearson's, it has the serious drawback in being extremely laborious in application, especially when attempting to find partial areas. For this reason we will develop a method of fitting a Pearson type curve which depends on the equating of the first four logarithmic derivatives at the mode. It is hoped that the resulting fit will be sufficiently good for statistical purposes. An incomplete integral of the original curve would then, with the exception of certain cases, be expressible in terms of a single incomplete Beta function.

§ 5. *A systematic Fitting of Pearson Curves to certain Frequency Distributions.*

Let $f(z)$ be a bell-shaped distribution and let a_1, a_2, a_3, \dots be its successive logarithmic differentials about its mode.

Let y be a Pearson type curve referred to its mode; its differential equation is then

$$\frac{1}{y} \frac{dy}{dz} = \frac{d \log y}{dz} = \frac{z}{b_0 + b_1 z + b_2 z^2} \dots \dots \dots (i).$$

Differentiate both sides of (i) successively with respect to z , we have

$$\frac{d^2 \log y}{dz^2} = \frac{1}{b_0 + b_1 z + b_2 z^2} - \frac{z(b_1 + 2b_2 z)}{(b_0 + b_1 z + b_2 z^2)^2},$$

$$\frac{d^3 \log y}{dz^3} = -\frac{2b_1 + 6b_2 z}{(b_0 + b_1 z + b_2 z^2)^2} + \frac{2z(b_1 + 2b_2 z)^2}{(b_0 + b_1 z + b_2 z^2)^3},$$

$$\frac{d^4 \log y}{dz^4} = -\frac{6b_2}{(b_0 + b_1 z + b_2 z^2)^2} + \frac{2(3b_1^2 + 18b_1 b_2 z + 24b_2^2 z^2)}{(b_0 + b_1 z + b_2 z^2)^3} - \frac{6z(b_1 + 2b_2 z)^3}{(b_0 + b_1 z + b_2 z^2)^4},$$

etc.

Equate now the corresponding logarithmic differentials of $y(z)$ and $f(z)$ at their respective modes. We have the following set of equations to solve for the unknown b 's:

$$1/b_0 = a_1,$$

$$-2b_1/b_0^2 = a_2,$$

$$6(b_1^2 - b_0 b_2)/b_0^3 = a_4.$$

Hence

$$b_0 = 1/a_2,$$

$$b_1 = -a_3/2a_2^2,$$

$$b_2 = (3a_3^2 - 2a_4a_2)/12a_2^3,$$

and

$$\frac{1}{y} \frac{dy}{dz} = \frac{z}{\frac{1}{a_2} - \frac{a_3z}{2a_2^2} + \frac{1}{12a_2^3}(3a_3^2 - 2a_4a_2)z^2}$$

The discriminant of the quadratic in the denominator is

$$D = \frac{a_3^2}{4a_2^4} - \frac{1}{3a_2^4}(3a_3^2 - 2a_4a_2) = \frac{1}{12a_2^4}(8a_4a_2 - 9a_3^2).$$

Hence the condition for a Type IV curve is ($D < 0$)

$$9a_3^2 > 8a_4a_2.$$

Condition for a Type V curve is ($D = 0$)

$$9a_3^2 = 8a_4a_2.$$

Condition for a Type III curve is ($b_2 = 0$)

$$3a_3^2 = 2a_4a_2.$$

Condition for curves lying above the Type IV line is

$$9a_3^2 < 8a_4a_2.$$

For a Type I curve the roots of the quadratic must be real and of opposite sign, i.e.

$$9a_3^2 < 8a_4a_2,$$

and

$$3a_3^2 < 2a_4a_2.$$

The first condition is here implied in the second.

For a Type VI curve the roots of the quadratic must be real and of the same sign, i.e.

$$9a_3^2 < 8a_4a_2,$$

$$3a_3^2 > 2a_4a_2.$$

The curve is symmetrical when a_3 is zero. The differential equation then reduces to

$$\frac{1}{y} \frac{dy}{dz} = \frac{6a_4^2z}{6a_2 - a_4z^2}.$$

a_2 is always negative for bell-shaped curves; hence, the following types arise according to the sign of a_4 :

$a_4 < 0$ gives a Type II curve.

$a_4 > 0$ „ „ „ VII „

$a_4 = 0$ „ the normal curve.

Note that the differential equation (i) can never represent a J-shaped curve, when $f(x)$ is bell-shaped.

The above method may be easily generalised to the fitting of higher order Pearson curves, i.e. curves satisfying the differential equation

$$\frac{1}{y} \frac{dy}{dz} = \frac{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots}{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots},$$

by equating the differentials of $\log y$, at $z=0$, to the corresponding logarithmic differentials of $f(z)$ at its mode. The resulting simultaneous equations may be solved successively for the unknown b 's in terms of the known a 's.

Table to determine the Type of a Curve.

Main Types	Criterion	Transition Types	Criterion
I	$3a_3^2 < 2a_4 a_2$	III	$3a_3^2 = 2a_4 a_2$
VI	$8a_4 a_2 > 9a_3^2 > 6a_4 a_2$	V	$9a_3^2 = 8a_4 a_2$
IV	$9a_3^2 > 8a_4 a_2$	II	$a_3 = 0, a_4 < 0$
—	—	Normal	$a_3 = 0, a_4 = 0$
—	—	VII	$a_3 = 0, a_4 > 0$

Let $u(z)$ be the Pearson type curve fitted by the above method to $f(z)$. Write $f(z) = u(z)V$, where $V = f(z)/u(z)$, both curves being referred to their respective modes. We may expand $\log V$ by Maclaurin's theorem into the following series:

$$\log V = a_0' + a_1' z + a_2' z^2/2! + \dots,$$

where

$$a_i' = (d^i \log V / dz^i) \text{ at mode.}$$

The first four a 's vanish; hence

$$\log V = a_5' z^5/5! + a_6' z^6/6! + \dots,$$

or

$$V = \exp. (a_5' z^5/5! + a_6' z^6/6! + \dots) \\ = 1 + \frac{a_5' z^5}{5!} + \frac{a_6' z^6}{6!} + \dots + \frac{a_9' z^9}{9!} + \left\{ \frac{a_5'^2}{(5!)^2} + \frac{2a_{10}'}{10!} \right\} \frac{z^{10}}{2!} + \dots$$

Therefore

$$f(z) = u(z) [1 + a_5' z^5/5! + a_6' z^6/6! + \dots].$$

Examples.

I. $y = y_0 e^{-z} (z+10)^{30} (30-z)^{60}$ (origin at mode)(i).

The first four logarithmic differential coefficients of y at the mode are, respectively,

$$a_1 = 0; \quad a_2 = -11/30; \\ a_3 = 5/90; \quad a_4 = -166/9000.$$

$3a_3^2$ is less than $2a_4 a_2$, hence the curve to be fitted is of Type I. This curve is readily found to be

$$y = y_0' (9.828,086 + z)^{28.210,489} (38.473,920 - z)^{110.435,344} \dots\dots\dots(ii), \\ \sigma \cong 1.65.$$

y_0' is determined by equating the ordinates of (i) and (ii) at their respective modes. The total area of the curve (ii) is $\cdot 793,2284$, and that of (i), found by mechanical quadrature, is $\cdot 793,4511$ (y_0 is taken to be $10^{-115} \times e^{10}$)—a difference of $\cdot 028\%$.

The following table of ordinates is constructed to examine the goodness of fit of the two curves:

z	Ordinates of (i)	Ordinates of (ii)	z	Ordinates of (i)	Ordinates of (ii)
-7	$\cdot 000,000$	$\cdot 000,000$	2	$\cdot 098,488$	$\cdot 098,493$
-6	$\cdot 000,005$	$\cdot 000,005$	3	$\cdot 045,113$	$\cdot 045,127$
-5	$\cdot 000,277$	$\cdot 000,273$	4	$\cdot 015,926$	$\cdot 015,946$
-4	$\cdot 004,241$	$\cdot 004,227$	5	$\cdot 004,413$	$\cdot 004,429$
-3	$\cdot 026,528$	$\cdot 026,511$	6	$\cdot 000,972$	$\cdot 000,980$
-2	$\cdot 084,595$	$\cdot 084,588$	7	$\cdot 000,171$	$\cdot 000,174$
-1	$\cdot 158,609$	$\cdot 158,608$	8	$\cdot 000,024$	$\cdot 000,025$
0	$\cdot 192,456$	$\cdot 192,456$	9	$\cdot 000,003$	$\cdot 000,003$
1	$\cdot 161,593$	$\cdot 161,593$	10	$\cdot 000,000$	$\cdot 000,000$

II. $y = y_0 e^{-\frac{1}{2}z} (z + 50)^{100} (100 - z)^{150}$, $y_0 = 10^{-320}$ (origin at mode)(i).

The first four logarithmic differentials of y at the origin are, respectively,

$$\begin{aligned} a_1 &= 0, & a_2 &= -55/1000, \\ a_3 &= 13/10^4, & a_4 &= -105/10^6. \end{aligned}$$

Here again $3a_3^2$ is less than $2a_4a_2$, hence the curve having the same first four differentials as y at its mode is of Type I. Its equation is

$$f = y_0' (48 \cdot 73692 + z)^{91 \cdot 740,961} (114 \cdot 94062 - z)^{216 \cdot 360,891} \dots\dots\dots(ii),$$

$$\sigma \cong 4 \cdot 25.$$

The total area of the latter curve is $8 \cdot 41285$, which agrees very well with the area ($8 \cdot 412,902$) of the curve (i) found by quadrature.

In the following table the ordinates of both curves (i) and (ii) are compared at intervals of 2 for the argument z :

z	Ordinates of (i)	Ordinates of (ii)	z	Ordinates of (i)	Ordinates of (ii)
-22	$\cdot 000,000$	$\cdot 000,000$	4	$\cdot 514,603$	$\cdot 514,607$
-20	$\cdot 000,001$	$\cdot 000,001$	6	$\cdot 305,561$	$\cdot 305,569$
-18	$\cdot 000,016$	$\cdot 000,016$	8	$\cdot 149,209$	$\cdot 149,225$
-16	$\cdot 000,195$	$\cdot 000,194$	10	$\cdot 060,260$	$\cdot 060,279$
-14	$\cdot 001,607$	$\cdot 001,603$	12	$\cdot 020,221$	$\cdot 020,236$
-12	$\cdot 009,265$	$\cdot 009,254$	14	$\cdot 005,659$	$\cdot 005,668$
-10	$\cdot 038,581$	$\cdot 038,564$	16	$\cdot 001,324$	$\cdot 001,328$
-8	$\cdot 119,034$	$\cdot 119,018$	18	$\cdot 000,260$	$\cdot 000,261$
-6	$\cdot 277,991$	$\cdot 277,983$	20	$\cdot 000,043$	$\cdot 000,043$
-4	$\cdot 500,464$	$\cdot 500,462$	22	$\cdot 000,006$	$\cdot 000,006$
-2	$\cdot 705,414$	$\cdot 705,414$	24	$\cdot 000,001$	$\cdot 000,001$
0	$\cdot 788,861$	$\cdot 788,861$	26	$\cdot 000,000$	$\cdot 000,000$
2	$\cdot 707,866$	$\cdot 707,869$			

III. $y = y_0 e^{-\frac{1}{2}z} (z + 50)^{40} (z + 150)^{-70} \dots\dots\dots(i),$
 $y_0 = 10^{84}$ (origin at mode).

The first four logarithmic differentials of (i) at the mode are, respectively,

$$a_1 = 0, \quad a_2 = -\frac{29}{2250},$$

$$a_3 = \frac{2020}{27 \times 50^3}, \quad a_4 = -\frac{6340}{27 \times 50^4}.$$

The Pearson curve which has the same first four logarithmic differentials as (i) is of Type VI, namely,

$$y = y_0' (48'456,364 + z)^{34'589,810} (387'406,732 + z)^{-276'544,175} \dots\dots(ii),$$

$$\sigma \cong 9'05.$$

Area of (i) found by quadrature is 9'53324. Area of (ii) is 9'53364.

The series for the total area, N , deduced by Method A is very uncertain; it is

$$N = 9'470,962 (1 - '028,26991 - '031,87332 + \dots).$$

z	Ordinates of (i)	Ordinates of (ii)	z	Ordinates of (i)	Ordinates of (ii)
-32	·000,001	·000,001	20	·059,869	·059,913
-28	·000,051	·000,049	24	·028,607	·028,653
-24	·001,115	·001,100	28	·012,617	·012,657
-20	·010,095	·010,053	32	·005,189	·005,217
-16	·047,650	·047,598	36	·002,007	·002,024
-12	·137,084	·137,056	40	·000,735	·000,745
-8	·267,859	·267,853	44	·000,257	·000,262
-4	·384,302	·384,299	48	·000,086	·000,088
0	·428,955	·428,955	52	·000,028	·000,029
4	·389,261	·389,260	56	·000,009	·000,009
8	·297,170	·297,174	60	·000,003	·000,003
12	·196,083	·196,097	64	·000,001	·000,001
16	·114,276	·114,308	68	·000,000	·000,000

The above examples show that the Pearson curve which has the same first four logarithmic differentials about the mode as the curve $y = y_0 e^{-\frac{1}{2}z} P(z)$ (when not very skew) fits it with sufficient accuracy for most statistical purposes at points lying within $\pm 3\sigma$ of the mode. The range of agreement of the two curves is further extended when a few terms of the expansion is included; but these render the finding of partial areas an extremely laborious task. It is desirable, therefore, to develop accurate and convenient methods for fitting integrable curves to the "tail" of these frequency curves.

§ 6. A method similar to the one developed in the beginning of § 3 may generally be applied satisfactorily to determine the area of parts of the curve lying outside the range mode $\pm 2'5\sigma$. Thus, let $z \geq z_0$ be the tail of the frequency curve

$y = y_0 e^{-z^2} P(z)$ and let a_i be the i th logarithmic differential of y at the "stump" $z = z_0$. Then, by Maclaurin,

$$y = e^{a_0 + a_1 \zeta + a_2 \zeta^2/2! + \dots} \\ = y(z_0) e^{a_1 \zeta + a_2 \zeta^2/2!} [1 + a_3 \zeta^3/3! + a_4 \zeta^4/4! + \dots],$$

where

$$\zeta = z - z_0.$$

The series in brackets is identical in form with the one deduced by Method A (§ 3).

Evidently,

$$y = y(z_0) e^{\frac{1}{2} \frac{z_1^2}{\sigma^2}} e^{-\frac{1}{2\sigma^2} (\zeta + z_1)^2} [1 + a_3 \zeta^3/3! + a_4 \zeta^4/4! + \dots],$$

where

$$\sigma = \frac{1}{\sqrt{(-a_2)}}, \quad z_1 = a_1/a_2.$$

Hence, area of tail $z \geq z_0$ is

$$N'_{z_0} = y(z_0) e^{\frac{1}{2} \frac{z_1^2}{\sigma^2}} \int_0^\infty e^{-\frac{1}{2\sigma^2} (\zeta + z_1)^2} [1 + a_3 \zeta^3/3! + a_4 \zeta^4/4! + \dots] d\zeta.$$

This may be expressed in terms of incomplete normal moment functions. Note that the first approximation is

$$N'_{z_0} = y(z_0) e^{\frac{1}{2} \frac{z_1^2}{\sigma^2}} \int_0^\infty e^{-\frac{1}{2\sigma^2} (\zeta + z_1)^2} d\zeta \\ = y(z_0) \sigma \cdot \frac{\frac{1}{2}(1 - \alpha_{z_1'})}{z_{z_1'}},$$

where $z_1' = \frac{z_1}{\sigma}$, $\frac{1}{2}(1 - \alpha_{z_1'})$ the area of the tail of a normal curve, and $z_{z_1'}$ the ordinate of a normal curve at the point z_1' . This is the expression obtained by Burton H. Camp* for the determination of the tail of a frequency distribution. Only in rare cases is it necessary to go beyond the 6th term in the above expansion of N .

Examples.

I. Fit a curve to the tail $z \geq 4$ of the curve

$$(i) \quad y = y_0 e^{-z} (z + 10)^{80} (30 - z)^{80}, \quad y_0 = e^{10} 10^{-115}, \quad \sigma \cong 1.65 \\ \text{(origin at the mode).}$$

The first six logarithmic differentials of y at the point $z = 5$ are, respectively,

$$a_1 = -1.4, \quad a_2 = -.2293, 33333, \\ a_3 = .0100, 97777, \quad a_4 = -.0044, 77155, \\ a_5 = .0008, 0069215, \quad a_6 = -.0003, 4554058;$$

and the curve is

$$(ii) \quad y = .0044, 1287 e^{-1.4\zeta - .2293\frac{\zeta^2}{2}} [1 + .0016, 8296 \zeta^3 \\ - .0001, 86548 \zeta^4 + .0000, 06672 \zeta^5 \\ + .0000, 00936 \zeta^6 - \dots],$$

where

$$\zeta = z - 5.$$

* *Biometrika*, Vol. xvi. p. 168.

The point $\zeta = -1$ is approximately 2.5σ away from the mode.

The following table compares the ordinates of (i) and (ii) at unit intervals of the argument z :

z	Ordinates of (i)	Ordinates of (ii)
4	·01592,63	·01592,64
5	·00441,29	·00441,29
6	·00097,18	·00097,18
7	·00017,145	·00017,145
8	·00002,433	·00002,435
9	·00000,278	·00000,279
10	·00000,025	·00000,026
11	·00000,002	·00000,002

II. *Fit a curve to the tail $z \geq 10$ of the generalised Pearson curve*

$$(i) \quad y = y_0 e^{-\frac{1}{2}z} (z+50)^{100} (100-z)^{150}, \quad y_0 = 10^{-320}, \quad \sigma \cong 4.25$$

(origin at the mode).

The first five logarithmic differential coefficients of (i) at $z = 10$ are, respectively,

$$\begin{aligned} a_1 &= -\cdot 5, & a_2 &= -\frac{1}{108}, \\ a_3 &= \cdot 00051,44033, & a_4 &= -\cdot 00006,00137, \\ a_5 &= \cdot 00000,24768. \end{aligned}$$

The required curve is

$$(ii) \quad y = \cdot 0602,600 e^{a_1 \zeta + \frac{a_2}{2} \zeta^2} \left[1 + \cdot 00068,58106 \left(\frac{\zeta}{2} \right)^3 - \cdot 00004,00091 \left(\frac{\zeta}{2} \right)^4 + \cdot 00000,06605 \left(\frac{\zeta}{2} \right)^5 + \dots \right],$$

where

$$\zeta = z - 10.$$

z	Ordinates of (i)	Ordinates of (ii)
10	·0602,600	·0602,600
12	·0202,210	·0202,210
14	·0056,585	·0056,585
16	·0013,241	·0013,240
18	·0002,596	·0002,595
20	·0000,427	·0000,426
22	0000,059	·0000,058
24	·0000,007	·0000,007
26	·0000,001	·0000,001

III. *Fit a curve to the tail of the generalised Pearson curve*

$$(i) \quad y = y_0 e^{-\frac{1}{2}z} (z+50)^{40} (z+150)^{-70} : z \geq 20, \quad y_0 = 10^{84}, \quad \sigma \cong 9.05$$

(origin at mode).

The first six logarithmic differentials of (i) at $z = 20$ are

$$\begin{aligned} a_1 &= -173,669,468, & a_2 &= -005,741,120, \\ a_3 &= 000,204,74032, & a_4 &= -000,009,4929675, \\ a_5 &= 000,000,55935840, & a_6 &= -000,000,04045132. \end{aligned}$$

The required curve is

$$\begin{aligned} \text{(ii)} \quad y &= 059,869 e^{a_1 \zeta + \frac{a_2}{2} \zeta^2} \left[1 + 002,183,897 \left(\frac{\zeta}{4} \right)^3 \right. \\ &\quad - 000,101,258 \left(\frac{\zeta}{4} \right)^4 + 000,004,773 \left(\frac{\zeta}{4} \right)^5 \\ &\quad \left. + 000,002,154 \left(\frac{\zeta}{4} \right)^6 - \dots \right], \end{aligned}$$

where

$$\zeta = z - 20.$$

z	Ordinates of (i)	Ordinates of (ii)
20	059,869	059,869
24	028,607	028,607
28	012,617	012,617
32	005,189	005,190
36	002,007	002,011
40	000,735	000,741
44	000,257	000,262
48	000,086	000,089
52	000,028	000,030
56	000,008	000,008

In the above examples, with the possible exception of the last, in which there happens to be an appreciable degree of skewness, the agreement between the two curves is very close indeed. We conclude, therefore, that when the curve

$$G(z) = y_0 e^{-pz} P(z)$$

is not very skew, the tail $z \geq \xi$, where ξ does not lie within mode $\pm 2.5\sigma$, may be fitted very closely by the curve

$$y = G(\xi) e^{a_1(z-\xi) + a_2(z-\xi)^2/2} [1 + a_3(z-\xi)^3/3! + a_4(z-\xi)^4/4! + \dots],$$

where $a_i = \frac{d^i}{dz^i} \{\log G(z)\}$ at $z = \xi$. Hence integrals of $G(z)$ up to any point outside mode $\pm 2.5\sigma$ may be expressed in terms of incomplete normal moment functions. The rapidity of convergence of the sequence $\sigma^i a_i/i!$ ($i = 3, 4, 5, \dots$) will give a good indication as to how many terms of the expansion y need be retained.

It has already been shown (§5) that integrals of $G(z)$ up to points lying within mode $\pm 2.5\sigma$ may be expressed very closely by a single incomplete Beta function.

Finally, integrals of very skew and J-shaped curves are best evaluated by an accurate quadrature formula such as Weddle's or Euler-Maclaurin's.

§7. *Methods for Obtaining Approximations to the Sum of a Tail of a Hypergeometric Series.*

When the standard deviation of a hypergeometric series is sufficiently large, the above methods for finding approximations to the sum of a finite number of terms serve very well indeed for regions within $\pm 2.5\sigma$ of the mode but do not always give satisfactory results for more remote regions. In problems on probability the tail of a distribution is often of greater importance even than the body itself. For this reason it is desirable to develop more accurate methods for approximating to this tail.

Let, then, the following series of constantly decreasing positive quantities, namely,

$$a_0 + a_1 + a_2 + a_3 + \dots,$$

represent the tail of a hypergeometric series $F(\alpha, \beta, \gamma, x)$.

Draw up the following table of the successive ratios of the terms a_i :

Terms	1st ratios	2nd ratios	3rd ratios	4th ratios	5th ratios
a_0					
a_1	$r_1 = \frac{a_1}{a_0}$				
	$r_2 = \frac{a_2}{a_1}$	$p_1 = \frac{r_2}{r_1}$	$q_1 = \frac{p_2}{p_1}$		
a_2	$r_3 = \frac{a_3}{a_2}$	$p_2 = \frac{r_3}{r_2}$	$q_2 = \frac{p_3}{p_2}$	$u_1 = \frac{q_2}{q_1}$	
a_3	$r_4 = \frac{a_4}{a_3}$	$p_3 = \frac{r_4}{r_3}$	$q_3 = \frac{p_4}{p_3}$	$u_2 = \frac{q_3}{q_2}$	$v_1 = \frac{u_2}{u_1}$
a_4	$r_5 = \frac{a_5}{a_4}$	$p_4 = \frac{r_5}{r_4}$			
a_5					
...

Clearly

$$\begin{aligned} \frac{a_1}{a_0} &= r_1, & \frac{a_2}{a_0} &= r_1 r_2, \dots, & \frac{a_n}{a_0} &= r_1 r_2 \dots r_n, \\ r_2 &= r_1 p_1, & r_3 &= r_1 p_1 p_2, \dots, & r_s &= r_1 p_1 p_2 \dots p_{s-1}, \\ p_2 &= p_1 q_1, & p_3 &= p_1 q_1 q_2, \dots, & p_s &= p_1 q_1 q_2 \dots q_{s-1}, \quad \text{etc.} \end{aligned}$$

Hence

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 + \dots &= a_0 (1 + r_1 + r_1^2 p_1 + r_1^3 p_1^2 p_2 + \dots r_1^s p_1^{s-1} p_2^{s-2} \dots p_{s-1} + \dots) \\ &= a_0 (1 + r_1 + r_1^2 p_1 + r_1^3 p_1^2 q_1 + r_1^4 p_1^3 q_1^2 q_2 + \dots) \\ &= a_0 (1 + r_1 + r_1^2 p_1 + r_1^3 p_1^2 q_1 + r_1^4 p_1^3 q_1^2 u_1 + \dots) \\ &= a_0 \sum_{s=0}^{\infty} r_1^s p_1 \frac{s(s-1)}{1 \cdot 2} q_1 \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} u_1 \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4} \dots \end{aligned}$$

If the first ratios are sensibly constant, $p_1 = q_1 = \dots = 1$ and the series reduces to the simple geometrical

$$a_0(1 + r_1 + r_1^2 + r_1^3 + \dots).$$

We may, without fear of appreciable error, take the upper limit to be infinity. Hence a *first approximation* to the sum of the series is

$$S_1 = \frac{a_0}{1 - r_1}.$$

If the second ratios are sensibly constant, i.e. $q_1 = u_1 = v_1 = \dots = 1$, to the figure of accuracy required in the result, the series reduces to

$$a_0 \sum_{s=0}^{\infty} r_1^s p_1 \frac{s(s-1)}{1 \cdot 2} = a_0(1 + r_1 + r_1^2 p_1 + r_1^3 p_1^3 + r_1^4 p_1^6 + \dots).$$

The first three terms of the original series suffice to determine the values of the constants a_0 , r_1 and p_1 . Write $r_1 = r$, $p_1 = 1 - p$. If the tail of the series does not lie too near the mode, p_1 is close to unity, i.e. p is a small quantity. Hence

$$\begin{aligned} a_0 \sum_{s=0}^{\infty} r^s p_1 \frac{s(s-1)}{1 \cdot 2} &= a_0 \sum_{s=0}^{\infty} r^s (1-p) \frac{s(s-1)}{1 \cdot 2} \\ &= a_0 [1 + r + r^2(1-p) + r^3(1-3p+3p^2-p^3) \\ &\quad + r^4(1-6p+15p^2-20p^3+15p^4-6p^5+p^6) \\ &\quad + r^5(1-10p+45p^2-120p^3+210p^4-252p^5+\dots) \\ &\quad + \dots \dots] \\ &= a_0 [1 + r + r^2 + r^3 + \dots \\ &\quad - pr^2(1+3r+6r^2+10r^3+15r^4+\dots) \\ &\quad + 3p^2r^3(1+5r+15r^2+35r^3+\dots) \\ &\quad - p^3r^3(1+20r+120r^2+455r^3+\dots) \\ &\quad + 15p^4r^4(1+14r+91r^2+\dots) \\ &\quad + \dots \dots]. \end{aligned}$$

The above series may all be summed quite readily. Thus

$$\begin{aligned} 1 + r + r^2 + r^3 + \dots &= \frac{1}{1-r}, \\ 1 + 3r + 6r^2 + 10r^3 + \dots &= \frac{1}{(1-r)^2}, \\ 1 + 5r + 15r^2 + 35r^3 + \dots &= \frac{1}{(1-r)^3}, \\ 1 + 20r + 120r^2 + 455r^3 + \dots &= \frac{1+13r+r^2}{(1-r)^4}, \\ \dots &= \dots \end{aligned}$$

Hence a *second approximation* to the sum of the tail is

$$S_2 = \frac{a_0}{1-r} \left[1 - p \left(\frac{r}{1-r} \right)^2 + 3p^2 \frac{r^3}{(1-r)^4} - \frac{r^3 p^3 (1+13r+r^2)}{(1-r)^6} \right].$$

The first three terms are generally sufficient for computing purposes. The next approximation will involve an additional constant q_1 .

$$\begin{aligned} S_3 &= a_0 (1 + r + r^2 p_1 + r^3 p_1^3 q_1 + r^4 p_1^6 q_1^4 + \dots) \\ &= a_0 \sum_{s=0}^{\infty} r^s p_1^s \frac{s(s-1)}{1 \cdot 2} q_1 \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \dots \end{aligned}$$

Write $q_1 = 1 + q$. Generally q is very much smaller than p so that q^2 and higher powers may be safely neglected.

$$\begin{aligned} S_3 &= a_0 [1 + r + r^2 p_1 + r^3 p_1^3 (1 + q) + r^4 p_1^6 (1 + q)^4 + \dots] \\ &\cong a_0 [(1 + r + r^2 p_1 + r^3 p_1^3 + r^4 p_1^6 + \dots) \\ &\quad + q r^3 p_1^3 (1 + 4r p_1^3 + 10r^2 p_1^7 + 20r^3 p_1^{12} + \dots)]. \end{aligned}$$

The sign \cong denotes "approximately equal to." We will now obtain an approximation to the sum of the series

$$S' = 1 + 4r p_1^3 + 10r^2 p_1^7 + 20r^3 p_1^{12} + \dots$$

p_1 is less than unity because the convergence of the tail of the hypergeometric series is more rapid than that of the geometrical $a_0 (1 + r + r^2 + r^3 + \dots)$ whose first two terms agree with those of the tail of the hyperseries. Hence

$$S' < 1 + 4r + 10r + \dots = (1 - r)^{-4}.$$

Moreover,

$$S' < 1 + 4r p_1^3 + 10r^2 p_1^6 + 20r^3 p_1^9 + \dots = (1 - r p_1^3)^{-4} < (1 - r)^{-4}.$$

Hence $(1 - r p_1^3)^{-4}$ is a better upper limit to the sum S' than $(1 - r)^{-4}$. There is, however, very little difference between these two expressions because p_1 is close to unity.

When $p = (1 - p_1)$ is sufficiently small, say of the order .01, we may obtain a still closer approximation and, incidentally, a lower limit to S' .

$$\begin{aligned} S' &= 1 + 4r p_1^3 + 10r^2 p_1^7 + 20r^3 p_1^{12} + \dots \\ &\cong 1 + 4r (1 - 3p) + 10r^2 (1 - 7p) + 20r^3 (1 - 12p) + \dots \\ &= (1 + 4r + 10r^2 + 20r^3 + \dots) - r p (12 + 70r + 240r^2 + \dots) \\ &= [(1 - r)^{-4} - 2r p (6 - r) (1 - r)^{-6}]. \end{aligned}$$

Evidently

$$(1 - r p_1^3)^{-4} > S' > [(1 - r)^{-4} - 2r p (6 - r) (1 - r)^{-6}].$$

Hence a third approximation to the sum of the tail is

$$\begin{aligned} S_3 &= a_0 (1 - r)^{-1} [(1 - p r^2 (1 - r)^{-2} + 3p^2 r^3 (1 - r)^{-4} \\ &\quad - r^3 p^3 (1 + 13r + r^2) (1 - r)^{-6}) \\ &\quad + q r^3 p_1^3 (1 - r)^{-3} \{1 - 2p r (6 - r) (1 - r)^{-2}\}], \end{aligned}$$

where

$$p = 1 - p_1, \quad q = q_1 - 1.$$

Examples.

I. *Sum the tail of the series $F(30, 30, 111, \cdot 75)$ starting with the 28th term.*

The tail of the series is

$$\cdot 796,454 + \cdot 502,267 + \cdot 314,369 + \dots$$

We readily find

$$\begin{aligned} r_1 &= \cdot 6306,2901, \\ p_1 &= \cdot 9925,014, & p &= \cdot 007,4986, \\ q_1 &= 1\cdot 000,64443, & q &= \cdot 000,64443, \\ u_1 &= \cdot 999,93951, & 1 - u_1 = u &= \cdot 000,06049, \end{aligned}$$

The true sum of the tail found by computing each term separately is $2\cdot 117,841$.

Further,

$$\begin{aligned} S_1 &= 2\cdot 156,244, \\ S_2 &= 2\cdot 156,244(1 - \cdot 021,8576 + \cdot 002,27275 - \cdot 000,39944) \\ &= 2\cdot 113,153, \\ S_3 &= 2\cdot 113,153 + \cdot 004,2444 \\ &= 2\cdot 117,397. \end{aligned}$$

This differs from the true value by only 4 in the fifth place.

II. *Sum the tail of the series $F(50, 50, 100, \frac{1}{2})$ starting with the 81st term.*

The tail of the series is

$$a_0(1 + \cdot 578,123 + \cdot 333,418 + \cdot 191,836 + \dots).$$

Its true sum is $2\cdot 360,158a_0$.

We readily find

$$\begin{aligned} r &= \cdot 578,12289, \\ p_1 &= \cdot 997,581112, & p &= \cdot 002,418888, \\ q_1 &= 1\cdot 0000,604352, & q &= \cdot 0000,604352, \\ S_1 &= 2\cdot 370,3585a_0, \\ S_2 &= 2\cdot 370,3585a_0(1 - \cdot 004,54239 + \cdot 0000,7138 - \cdot 00000,429) \\ &= 2\cdot 359,7505a_0, \\ S_3 &= (2\cdot 359,7505 + \cdot 000,33479)a_0 \\ &= 2\cdot 360,085a_0. \end{aligned}$$

The last approximation differs from the true value by 7 in the sixth place.

III. *Sum the tail of the series $F(50, 50, 100, \frac{1}{2})$ starting with the 61st term.*

The tail of the series is

$$S = a_0'(1 + \cdot 617,161 + \cdot 379,270 + \cdot 232,118 + \dots).$$

True value of S is $2\cdot 585,789a_0'$.

We find the following values for the constants:

$$r = \cdot 617,160890,$$

$$p_1 = \cdot 995,7540,680, \quad p = \cdot 004,245932,$$

$$q_1 = 1\cdot 000,13065, \quad q = \cdot 000,13065,$$

and $S_1 = 2\cdot 612,0623a_0',$

$$S_2 = 2\cdot 612,0623a_0(1 - \cdot 011,03411 + \cdot 000,59183 - \cdot 0000,2058) \\ = 2\cdot 584,7335a_0',$$

$$S_3 = (2\cdot 584,7335 + \cdot 001,1399)a_0' \\ = 2\cdot 585,873a_0'.$$

S_3 differs from the true value by 1 in the fifth place.

IV. *Sum the tail of the series $F(-100, -100, 1, \frac{1}{2})$ starting with the 10th term after the mode.*

Modal term is the 42nd. We find

$$r = \cdot 443,97189,$$

$$p_1 = \cdot 923,73046, \quad p = \cdot 076,26954,$$

$$q_1 = 1\cdot 000,156141, \quad q = \cdot 000,156141.$$

True sum of tail is $\cdot 751,702$ multiplied by a constant which will be dropped in the subsequent work.

$$S_1 = \cdot 781,4122,$$

$$S_2 = \cdot 781,4122(1 - \cdot 048,626 + \cdot 015,9773 - \cdot 005,09085).$$

p is too large to apply formula S_2 with any great degree of accuracy. It is interesting, however, to examine how closely the successive approximations to S_2 represent the true value of S . These approximations are

$$(i) \cdot 781,412, \quad (ii) \cdot 743,415,$$

$$(iii) \cdot 755,900, \quad (iv) \cdot 748,746.$$

The formulae S_2 is not entirely satisfactory when p is larger than $\cdot 01$. For such cases, we may obtain a very close approximation to S_2 by the application of the Euler-Maclaurin theorem to the summing of the series

$$\sum_{s=0}^{\infty} r^s p_1^s \frac{s(s-1)}{1 \cdot 2}$$

The problem, however, will be approached from a wider angle, namely, by applying the Euler-Maclaurin theorem to the summing of the more general series

$$\sum_{s=0}^{\infty} r^s p_1^s \frac{s(s-1)}{1 \cdot 2} \quad q_1 \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \quad u_1 \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

§ 8. *Application of the Euler-Maclaurin Theorem to the summing of the Series*

$$S = a_0 \sum_{s=0}^{\infty} r_1^s p_1^s \frac{s(s-1)}{1 \cdot 2} \quad q_1 \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \quad \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

Write

$$\begin{aligned}\log_e r_1 &= a, \\ \log_e p_1 &= b, \\ \log_e q_1 &= c, \text{ etc.,}\end{aligned}$$

then

$$S = a_0 \sum_{s=0}^{\infty} e^{as+b \frac{s(s-1)}{1 \cdot 2} + c \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} + \dots} \dots\dots\dots(i).$$

The quantities a, b, c, \dots generally tend rapidly to zero. Clearly,

$$\begin{aligned}S &= a_0 \sum_{s=0}^{\infty} e^{as+b \frac{s(s-1)}{2}} \left[1 + \left\{ c \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} + d \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \dots \right\} \right. \\ &\quad \left. + \frac{1}{2} \left\{ c \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} + d \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \dots \right\}^2 + \dots \right] \dots(ii)\end{aligned}$$

$$= a_0 \sum_{s=0}^{\infty} e^{as+b \frac{s(s-1)}{2}} [1 + c_0 s + c_1 s^2 + c_2 s^3 + \dots] \dots\dots\dots(iii)$$

$$= a_0 \sum e^{-c_2' s^2 - c_1' s} [1 + c_0 s + c_1 s^2 + c_2 s^3 + \dots] \dots\dots\dots(iv),$$

where

$$c_2' = -b/2, \quad c_1' = -(a - \frac{1}{2}b)$$

$$= a_0 \sum_{s=0}^{\infty} e^{\frac{1}{2}x_1^2} e^{-\frac{1}{2\sigma^2}(s+x_1\sigma)^2} [1 + c_0 s + c_1 s^2 + \dots],$$

where

$$x_1 = c_1' \sigma, \quad \sigma = \frac{1}{\sqrt{(2c_2')}}.$$

Hence, by the Euler-Maclaurin theorem,

$$\begin{aligned}S &\cong a_0 e^{\frac{1}{2}x_1^2} \left[\int_0^{\infty} e^{-\frac{1}{2\sigma^2}(x+x_1\sigma)^2} \{1 + c_0 x + c_1 x^2 + \dots\} dx + E_s \right] \\ &= a_0 e^{\frac{1}{2}x_1^2} \left[\sigma \int_{x_1}^{\infty} e^{-\frac{1}{2}x^2} \{1 + \xi_1 x + \xi_2 x^2 + \dots\} dx + E_s \right] \dots\dots\dots(ivb),\end{aligned}$$

where E_s is the Euler-Maclaurin correction*.

* We may also write

$$S \cong a_0 e^{\frac{1}{2}x_1^2} \left[\sigma \int_{\left(x_1 - \frac{1}{2\sigma}\right)}^{\infty} e^{-\frac{x^2}{2}} \{1 + \xi_1 x + \xi_2 x^2 + \dots\} dx + E_s' \right],$$

where E_s' is approximately equal to

$$.0387,153a_0 - .1578,127a_1 + .2411,459a_2 - .1220,458a_3. \quad (\text{See } \S 10, \text{ p. 106.})$$

More accurate expressions for E_s' are the following:

${}_6E_s'$	${}_8E_s'$
$\frac{1}{2}a_0 + 0.195,7755a_0$	$\frac{1}{2}a_0 + 0.213,0246a_0$
$-0.460,3838a_1$	$-0.589,0198a_1$
$+0.546,5364a_2$	$+0.964,0148a_2$
$-0.471,4289a_3$	$-1.240,8904a_3$
$+0.260,6070a_4$	$+1.140,5644a_4$
$-0.082,4786a_5$	$-0.720,9438a_5$
$+0.011,3674a_6$	$+0.297,8547a_6$
	$-0.072,4970a_7$
	$+0.007,8926a_8$

These last two expressions are obtained from a form of the Euler-Maclaurin theorem due to Professor Karl Pearson, the former based on sixth differences and the latter on eighth differences.

The integral $\int_{x_1}^{\infty} e^{-\frac{1}{2}x^2} x^s dx$ is the s th incomplete normal moment function and has been tabulated up to the twelfth. S can, therefore, be expressed in terms of incomplete normal moment functions.

As is frequently the case, the constants c, d, e, \dots are sufficiently small to be neglected altogether. Then

$$S \cong a_0 e^{\frac{1}{2}x_1^2} \left(\sigma \int_{x_1}^{\infty} e^{-\frac{1}{2}x^2} dx + E_1 \right) \dots\dots\dots (v).$$

This is, in effect, the expression obtained by Professor Burton H. Camp* for the sum of the tail of a discrete frequency distribution. He approached the problem differently, namely, by fitting the tail of a normal curve to the tail of the distribution. (v) has been put in the convenient form†

$$S \cong a_0 \left\{ \frac{1 - a_{x_1}}{z_{x_1}} \sigma + \psi \right\} \dots\dots\dots (vi),$$

where
$$\psi = \frac{1}{2} + \frac{1}{12} c_1 \left[1 + \frac{3 - x_1^2}{60\sigma^2} + \frac{15 - 10x_1^2 + x_1^4}{2520\sigma^4} \right] \dots\dots\dots (vi b).$$

This formula has been applied with a large measure of success to the summing of the tail of a binomial and hypergeometric series in which the last element x is unity. It fails, however, when the stump lies within 2σ of the mode, and is only reliable for points outside mode $\pm 3\sigma$. Similarly, when applied to the general hypergeometric series, the results are reliable only when the stump lies outside the range mode $\pm 3\sigma$. When the stump lies within mode $\pm 3\sigma$ it may be found necessary to include a few terms in the expansion (iv b). The formula (vi) is essentially equivalent to the approximation S_2 obtained for the series

$$a_0 \sum_{s=0}^{\infty} r_1^s p_1^{s(s-1)}$$

When, as is very frequently the case, $p = 1 - p_1$ is not appreciably greater than .01, S_2 holds several advantages over formula (vi); it may be applied more easily, the contribution of each successive term is made clear, and, since S_2 is concerned only with the fitting of a series to another, no correction such as the Euler-Maclaurin need be applied.

Retaining the constant c in (i), we find

$$S \cong a_0 \sum_{s=0}^{\infty} e^{\frac{1}{2}x_1^2} e^{-\frac{1}{2\sigma^2}(s+x_1\sigma)^2} \left\{ 1 + c \frac{s(s-1)(s-2)}{1.2.3} \right\}.$$

Note that
$$c = \log_e q_1 = \log_e (1 + q) = q - \frac{q^2}{2} + \frac{q^3}{3} - \dots$$

Therefore
$$\left\{ 1 + c \frac{s(s-1)(s-2)}{1.2.3} \right\} < \left\{ 1 + q \frac{s(s-1)(s-2)}{1.2.3} \right\}.$$

Moreover,
$$(1 + q) \frac{s(s-1)(s-2)}{1.2.3} > 1 + \frac{s(s-1)(s-2)}{1.2.3} \cdot q.$$

* *Biometrika*, Vol. xvi. p. 168.

† *Tables for Statisticians and Biometricians*, Part II, Introduction, pp. xxx—xl.

Hence, a better approximation to S is

$$a_0 e^{\frac{1}{2}x_1^2} \sum_{s=0}^{\infty} e^{-\frac{1}{2\sigma^2}(s+x_1\sigma)^2} \left\{ 1 + q \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \right\}.$$

This differs from (vi) by the quantity

$$M = a_0 e^{\frac{1}{2}x_1^2} \sum_{s=0}^{\infty} \left[e^{-\frac{1}{2\sigma^2}(s+x_1\sigma)^2} q \cdot \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \right] = a_0 e^{\frac{1}{2}x_1^2} \sum_{s=3}^{\infty} [\text{same expression}].$$

This is essentially the quantity added to S_2 (p. 94) to obtain the closer approximation S_3 . It is identical with the series

$$a_0 q r^3 p_1^3 (1 + 4rp_1^3 + 10r^2 p_1^7 + 20r^3 p_1^{12} + \dots).$$

M accordingly satisfies the inequalities

$$\begin{aligned} a_0 q r^3 p_1^3 (1-r)^{-4} &> a_0 q r^3 p_1^3 (1-rp_1^3)^{-4} > M \\ &> a_0 q r^3 p_1^3 (1-r)^{-4} \left\{ 1 - \frac{2r(1-p_1)(6-r)}{(1-r)^2} \right\} \dots\dots\dots(\text{vii}). \end{aligned}$$

These inequalities give a good indication of the accuracy of the result obtained by the application of Camp's formula (vi) and whether it is necessary to go a stage further in the approximation.

M may also be expressed in the form

$$a_0 q \left[e^{\frac{1}{2}x_1^2} \sigma \int_{\frac{x}{\sigma}+x_1}^{\infty} e^{-\frac{1}{2}x^2} \{ \sigma(x-x_1) \} \{ \sigma(x-x_1)-1 \} \{ \sigma(x-x_1)-2 \} dx + E \right] \quad (\text{viii}).$$

If

$$t_s = e^{as+bs(s-1)/2} \cdot \frac{s(s-1)(s-2)}{6},$$

then

$$t_3 = r^3 p_1^3,$$

$$t_3' = r^3 p_1^3 \left(\frac{11}{6} + a + \frac{5b}{2} \right),$$

$$t_3''' = r^3 p_1^3 \left(1 + 6a + \frac{41b}{2} \right),$$

neglecting squares and products of a and b .

$$\text{Hence} \quad E \cong r^3 p_1^3 \left[\frac{1}{2} + \frac{11+6a+15b}{72} - \frac{2+12a+41b}{1440} \right].$$

Since q is itself small, one term only of E generally suffices. The evaluation of (viii) is often a little troublesome. However, in most cases the right-hand side of inequality (vii) gives sufficiently accurate results.

The most satisfactory way so far, then, of finding the sum of a tail of a hypergeometric series when the "stump" does not lie within the mode $\pm 2.5\sigma$ is to

(a) obtain the approximate sum by the application of formula (vi),

(b) correct this result, when necessary, by evaluating M .

When $p = (1-p_1)$ is not appreciably larger than .01, the right-hand side of inequality (vii) may be taken as a good approximation to M . In all cases, however, M is less than $a_0 q r^3 p_1^3 (1-rp_1^3)^{-4}$.

Examples.

I. (See above.) *Sum the tail of $F(30, 30, 111, \cdot 75)$ starting with the 28th term.*

The approximation obtained by using (vi) is 2.113,316.

The correction (taking into account the value of q) obtained from the right-hand side of inequality (vii) is .004,2444. Hence,

$$\left. \begin{array}{l} \text{Corrected sum} = 2.117,560 \\ \text{True sum} \quad \quad = 2.117,841 \end{array} \right\}.$$

II. *Sum the tail of $F(50, 50, 100, \frac{1}{2})$ starting with the 81st term.*

$$\left. \begin{array}{l} \text{Formula (vi) gives} \quad 2.359,836, \\ \text{Correction} \quad \quad \quad = .000,335, \\ \text{Corrected sum} \quad \quad = 2.360,171 \\ \text{True sum} \quad \quad \quad = 2.360,158 \end{array} \right\}.$$

III. *Sum the tail of $F(50, 50, 100, \frac{1}{2})$ starting with the 61st term.*

$$\left. \begin{array}{l} \text{Formula (vi) gives} \quad 2.584,6630, \\ \text{Correction from (vii)} = .001,1399, \\ \text{Corrected sum} \quad \quad = 2.585,803 \\ \text{True sum} \quad \quad \quad = 2.585,789 \end{array} \right\}.$$

IV. *Sum the tail of $F(-100, -100, 1, \frac{1}{2})$ starting with the 52nd term.*

$$\left. \begin{array}{l} \text{Formula (vi) gives} \quad .751,738 \\ \text{True sum is} \quad \quad \quad .751,702 \end{array} \right\}.$$

In this case it is not necessary to apply any correction, formula (vi) gives sufficient accuracy. The result cannot be trusted in the fifth place because ψ does not converge sufficiently rapidly for the approximate value (vi b) to give five or more figure accuracy. Applying the left-hand side of inequality (vii) we readily find that the correction is less than .0000,3325, i.e. the result differs from the true value by less than 3.3 in the fifth place. The right-hand side of (vii) does not give a closer result because $p = .07$ is rather large.

§ 9. *On the Fitting of the Tail of a Type III Curve to the Tail of a Frequency Distribution.*

The above examples shew clearly that Camp's method for the evaluation of the tail of a discrete frequency distribution by fitting to it the tail of a normal curve is not always adequate, especially in those cases in which the third ratios q_i differ appreciably from unity. In general, the correction given by formula (vii) makes a considerable difference in the results; the accuracy being improved by one, two and sometimes three figures.

The fitting of the tail of a normal curve to the tail of a frequency distribution $F(z)$ depends on the evaluation of three constants. We can, alternatively, fit the tail of the Type III curve

$$y = y_0 e^{-\rho z} (a + z)^t. \dots\dots\dots(i).$$

This will depend on the evaluation of four independent constants and, accordingly, we should expect a certain amount of improvement in the results.

The cases, $F(z)$ continuous and $F(z)$ discontinuous, will be treated separately.

$F(z)$ continuous.

The problem will be approached by fitting (i) to the tail $z \geq z_0$ of $F(z)$ by equating the corresponding logarithmic differentials at the stump.

Let $u = \log F(z)$ and $a_s = \frac{d^s}{dz^s} F(z)$ at $z = z_0$. Equate these to the logarithmic differentials of (i) at $z = 0$. There result the following equations, which have to be solved for the constants of the Type III curve:

$$\begin{aligned} -\rho + t/a &= a_1, \\ -t/a^2 &= a_2, \\ 2t/a^3 &= a_3. \end{aligned}$$

Clearly,

$$\begin{aligned} a &= -2a_2/a_3, \\ t &= -4a_2^3/a_3^2, \\ \rho &= (2a_2^2/a_3 - a_1). \end{aligned}$$

y_0 may be found by equating the ordinates at the respective stumps.

Area of the tail $z \geq z_0$ of $F(z)$

$$\begin{aligned} &= \frac{1}{a^t} F(z_0) \int_0^\infty e^{-\rho z} (a+z)^t dz \\ &= \frac{1}{a^t} F(z_0) e^{\rho a} \int_a^\infty e^{-\rho z} z^t dz \\ &= F(z_0) \frac{\alpha_a}{y_a}, \end{aligned}$$

where α_a is the area of the tail $z \geq a$ of the Type III curve $y = y_0 e^{-\rho z} z^t$ and y_a the ordinate at $z = a$. The quantity α_a may be found from the *Tables of Incomplete Γ -Function*.

$F(z)$ discontinuous $\equiv (a_0, a_1, a_2, \dots)$.

I. Consider the discrete terms a_i ($i = 0, 1, 2, \dots$) as being the ordinates of a continuous function $f(z)$ at unit intervals of the argument z , the first term a_0 being given by $z = 0$.

Let a_s be the first term of the tail of $F(z)$. We require the first three logarithmic differentials of $f(z)$ at $z = s$. These may be found as follows. Calculate (to ten figures) the natural logarithms of the first six terms of the tail, calculate the differences and substitute in either of the following sets of formulae giving the differentials in terms of the differences.

Forward Difference Formulae:*

$$\begin{aligned} hf_0' &= (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \tfrac{1}{5}\Delta^5 - \tfrac{1}{6}\Delta^6 + \dots)f_0, \\ h^2f_0'' &= (\Delta^2 - \Delta^3 + \tfrac{1}{2}\Delta^4 - \tfrac{5}{6}\Delta^5 + \tfrac{13}{24}\Delta^6 - \dots)f_0, \\ h^3f_0''' &= (\Delta^3 - \tfrac{3}{2}\Delta^4 + \tfrac{7}{4}\Delta^5 - \tfrac{15}{8}\Delta^6 + \dots)f_0. \end{aligned}$$

* Whitaker and Robinson, *Calculus of Observations*.

*Central Difference Formulae**:

$$\begin{aligned} hf'_0 &= \frac{1}{2}(f_1 - f_{-1}) - \frac{1}{12}(\delta^2 f_1 - \delta^2 f_{-1}) + \frac{1}{60}(\delta^4 f_1 - \delta^4 f_{-1}) - \frac{1}{420}(\delta^6 f_1 - \delta^6 f_{-1}) + \dots, \\ h^2 f''_0 &= \delta^2 f_0 - \frac{1}{12}\delta^4 f_0 + \frac{1}{60}\delta^6 f_0 - \frac{1}{420}\delta^8 f_0 + \dots, \\ h^3 f'''_0 &= \frac{1}{2}(\delta^3 f_1 - \delta^3 f_{-1}) - \frac{1}{8}(\delta^5 f_1 - \delta^5 f_{-1}) + \frac{7}{240}(\delta^7 f_1 - \delta^7 f_{-1}) + \dots \end{aligned}$$

The constants of the Type III curve are then found as in the previous case.

If S is the sum of the tail of $F(z)$, then

$$S \cong \left[y_0 \int_0^\infty e^{-\rho z} (a+z)^t + E \right],$$

where E is the Euler-Maclaurin correction†. Furthermore,

$$S \cong a_s \left[a^{-t} e^{a\rho} \int_a^\infty e^{-\rho z} z^t dz + \psi \right] \dots\dots\dots(ii),$$

where

$$\begin{aligned} \psi &= \frac{1}{2} + \frac{1}{12a} \left[(t - \rho a) - \frac{1}{60a^2} \{ t(t-1)(t-2) - 3t(t-1)a\rho + 3ta^2\rho^2 - a^3\rho^3 \} \right. \\ &\quad + \frac{1}{2520a^4} \{ t(t-1)(t-2)(t-3)(t-4) - 5t(t-1)(t-2)(t-3)a\rho \\ &\quad \left. + 10t(t-1)(t-2)a^2\rho^2 - 10t(t-1)a^3\rho^3 + 5ta^4\rho^4 - a^5\rho^5 \} - \dots \right]. \end{aligned}$$

If a_s is the area of the tail of the Type III curve $y = y_0 e^{-\rho z} z^t$, $z \geq a$, and z_a the ordinate at $z = a$, S may be thrown into the convenient form

$$S \cong a_s \left(\frac{a_s}{z_a} + \psi \right).$$

II. Alternatively, we may find the Type III curve which passes through the first four terms of the tail of $F(z)$. Thus, let $y = y_0 e^{-\rho z} (a+z)^t$ be the equation of a Type III curve, and let d_1, d_2, d_3 be the first three logarithmic forward differences of the first term, a_s , of the tail of $F(z)$. The logarithmic differences of y at $z = 0$ are, respectively,

$$\begin{aligned} \text{1st} & -\rho + t \log(a+1)/a, \\ \text{2nd} & t \log(a+2)a/(a+1)^2, \\ \text{3rd} & t \log(a+3)(a+1)^2/(a+2)^3 a. \end{aligned}$$

Equate these to the corresponding differences of a_s . This results in three equations to solve for the constants ρ , a and t . Eliminate t from the last two equations and we have

$$d_2 \log \frac{(a+3)(a+1)^3}{(a+2)^3 a} = d_3 \log \frac{(a+2)a}{(a+1)^2},$$

$$\begin{aligned} \text{i.e.} \quad d_2 \log(a+3) - (3d_2 + d_3) \log(a+2) \\ + (3d_2 + 2d_3) \log(a+1) - (d_2 + d_3) \log a = 0 \{ \equiv X(a) \}. \end{aligned}$$

* Sheppard, *Proc. Lond. Math. Soc.* Vol. xxxi. p. 465.

† The correction E may, perhaps, be obtained more readily by using the formulae in the footnote,

This equation may be solved quite readily by forming a small table of the values of the function $X(a)$ and interpolating for the value of a which makes X zero.

Having found a , the other constants may be determined by simple substitution. The sum S is then given by formula (ii) above.

Examples.*

I. *Fit a Type III curve to the tail of $F(50, 50, 100, \frac{1}{2})$ starting with the 81st term.*

The first six forward differences of the common logarithms of the first term of the tail are, respectively,

$$\begin{aligned}\Delta &= -\cdot237,9798,316, & \Delta^2 &= -\cdot00105,17821, \\ \Delta^2 &= +\cdot000,0263,063, & \Delta^4 &= -\cdot00000,08886, \\ \Delta^4 &= +\cdot000,0000,383, & \Delta^6 &= -\cdot00000,00021.\end{aligned}$$

Converting into natural logarithms and substituting in the forward difference formulae, the first three logarithmic differentials reduce to

$$\begin{aligned}D &= -\cdot546,737,414, \\ D^2 &= -\cdot002,484,344, \\ D^3 &= \cdot000,0638,021.\end{aligned}$$

The Type III curve which has the same differentials at $z = 0$, is

$$y = y_0 e^{-740,2066z} (77\cdot876,5589 + z)^{15\cdot066,946} \dots\dots\dots(iii).$$

y_0 is found by equating y (at $z = 0$) to the first term of the tail of the series. Hence $y_0 = (77\cdot876,559)^{-15\cdot066,946}$.

Terms of tail	Ordinates of (i)	Terms of tail	Ordinates of (i)
1·000,000	1·000,000	·000,678	·000,678
·578,123	·578,123	·000,381	·000,381
·333,418	·333,417	·000,214	·000,214
·191,836	·191,836	·000,120	·000,120
·110,121	·110,122	·000,067	·000,067
·063,073	·063,072	·000,038	·000,038
·036,046	·036,046	·000,021	·000,021
·020,556	·020,556	·000,012	·000,012
·011,698	·011,698	·000,006	·000,006
·006,644	·006,644	·000,004	·000,004
·003,766	·003,766	·000,002	·000,002
·002,130	·002,130	·000,001	·000,001
·001,203	·001,203		

The fit of the Type III curve to the tail of the series is surprisingly good, and the area of the tail, corrected by the Euler-Maclaurin theorem, agrees to six figures with the true sum of the tail. This is a decided improvement on any of the previous methods.

* In the first two examples the actual terms of the series have been multiplied throughout by a constant to reduce the first term of the tail to unity.

II. *Fit a Type III curve to the tail of $F(50, 50, 100, \frac{1}{2})$ starting with the 61st term.*

Using the forward difference formula, the first three logarithmic differentials are found to be

$$D = -\cdot480,453,0646, \quad D^2 = -\cdot004,390,8871, \\ D^3 = \cdot000,139,3941,$$

and the required Type III curve is

$$y = y_0 e^{-\cdot787,0772z} (z + 62\cdot999,6028)^{17\cdot427,2117} \dots\dots\dots(\text{iv}),$$

where

$$y_0 = (62\cdot999,6028)^{-17\cdot427,2117}.$$

Terms of tail	Ordinates of (i)	Terms of tail	Ordinates of (i)
1\cdot000,000	1\cdot000,000	\cdot000,824	\cdot000,824
\cdot617,161	\cdot617,161	\cdot000,484	\cdot000,484
\cdot379,270	\cdot379,270	\cdot000,284	\cdot000,283
\cdot232,118	\cdot232,117	\cdot000,166	\cdot000,165
\cdot141,492	\cdot141,491	\cdot000,096	\cdot000,096
\cdot085,915	\cdot085,914	\cdot000,056	\cdot000,056
\cdot051,972	\cdot051,971	\cdot000,032	\cdot000,031
\cdot031,325	\cdot031,324	\cdot000,019	\cdot000,018
\cdot018,813	\cdot018,812	\cdot000,011	\cdot000,010
\cdot011,260	\cdot011,259	\cdot000,006	\cdot000,006
\cdot006,117	\cdot006,116	\cdot000,004	\cdot000,004
\cdot003,994	\cdot003,993	\cdot000,002	\cdot000,002
\cdot002,367	\cdot002,366	\cdot000,001	\cdot000,001
\cdot001,399	\cdot001,398		

Here again the fit is surprisingly close. The corrected area of the tail of the Type III curve agrees to five figures with the true sum of the series. It is out by only 2 in the sixth place.

III. *Fit a Type III curve to the tail of $F(30, 30, 111, \frac{3}{4})$ starting with the 28th term.*

The first three forward differences of the common logarithm of the first term are, respectively,

$$d_1 = -\cdot200,2261,429, \quad d_2 = -\cdot003,2694,047, \\ d_3 = \cdot000,2809,485.$$

The equation of the Type III curve which passes through the first four terms of the tail is readily found to be

$$y = y_0 e^{-\cdot628,746z} (20\cdot78533 + z)^{3\cdot569,066} \dots\dots\dots(\text{v}),$$

where

$$y_0 = \cdot796,454/y \text{ (at } z = 0\text{)}.$$

The true sum of the tail of the series is 2\cdot117,841 and the area of the tail of the Type III curve (i) corrected by the Euler-Maclaurin theorem is 2\cdot117,662. The fit of the curve to the series is not so good as in the previous two examples, but still, it is an improvement on any of the previous methods.

Terms of tail	Ordinates of (i)	Terms of tail	Ordinates of (i)
·796,454	·796,454	·001,283	·001,272
·502,267	·502,267	·000,762	·000,752
·314,369	·314,369	·000,451	·000,444
·195,414	·195,414	·000,267	·000,261
·120,708	·120,706	·000,157	·000,153
·074,135	·074,127	·000,093	·000,090
·045,291	·045,279	·000,054	·000,052
·027,537	·027,520	·000,032	·000,030
·016,669	·016,649	·000,019	·000,018
·010,049	·010,029	·000,011	·000,011
·006,036	·006,017	·000,006	·000,006
·003,614	·003,597	·000,002	·000,002
·002,157	·002,142	·000,001	·000,001

§ 10. It was shown in § 1 that when the number of significant terms of the hypergeometric series $F(\alpha, \beta, \gamma, x)$ is not small, the curve for which $\frac{1}{y} \frac{dy}{dz}$ is equal to $\Delta y_r / \{\frac{1}{2}c(y_r + y_{r+1})\}$, $r = 1, 2, 3, \dots$, fits the series quite well around the mode but may deviate a little towards the extreme tails. The probability integral of the series may be replaced by that of a continuous curve of the type $G(z) = y_0 e^{-\rho z} P(z)$ with sufficient accuracy for most statistical purposes.

An examination of the tables in the latter part of § 1 will reveal that the ordinates of the curve $G(z)$ correspond more closely to the terms of the series than the areas under the curve. If, therefore, the constant of integration, y_0 , is calculated by equating the maximum term of the series to the corresponding mid-ordinate of the curve, the mid-ordinates around the mode will agree even more closely with the terms of the series. The deviation towards the tails, however, will be more pronounced. If, therefore, a more accurate determination of the terms *around the mode* is required, the following method may be used:

Let $G(z)$ be the curve obtained from the differential equation $\frac{1}{y} \frac{dy}{dz} = \frac{2}{c} \frac{\Delta y_r}{(y_r + y_{r+1})}$, the constant of integration being determined by equating the modal term of the series to the corresponding mid-ordinate of the curve (note that this is not necessarily the mode of $G(z)$). The approximate value of the sum of a number of terms near the mode is, therefore,

$$S = \int_{z_1}^{z_2} G(z) dz + E,$$

where z_1 and z_2 are the appropriate limits and E the Euler-Maclaurin correction. A close approximation to E may be found as follows:

If y_r is the $(r+1)$ st mid-ordinate of the curve, the area under this part of the curve is given approximately by

$$\frac{1}{5760} [5178y_r + 308(y_{r+1} + y_{r-1}) - 17(y_{r+2} + y_{r-2})];$$

the grouping unit c is taken to be unity.

Hence, if we wish to sum the terms from the s th to the t th,

$$\int_s^t G(z) dz = \sum_{r=s+1}^t y_r - \cdot 0387,153 (y_{s+4} + y_{t-3}) \\ + \cdot 1578,127 (y_{s+3} + y_{t-2}) - \cdot 2411,459 (y_{s+2} + y_{t-1}) + \cdot 1220,485 (y_{s+1} + y_t).$$

Therefore,

$$\sum_{r=s+1}^t y_r = \int_s^t G(z) dz + \cdot 0387,153 (y_{s+4} + y_{t-3}) \\ - \cdot 1578,127 (y_{s+3} + y_{t-2}) + \cdot 2411,459 (y_{s+2} + y_{t-1}) - \cdot 1220,485 (y_{s+1} + y_t).$$

The terms to the right of the integral make up the correction. $\sum y_r$ is approximately equal to the required sum of the terms of the series near the mode.

When the fourth element x is not small, the method given in §2 for finding the partial sums of $F(\alpha, \beta, \gamma, x)$ is quite satisfactory for all parts of the series except the start. The curve deduced by this method fits the series very closely at points beyond the mode but may deviate appreciably at the start. The range of disagreement extends rapidly towards the mode as x becomes small.

The particular method to adopt for finding the sum of a number of terms, when large, of the hypergeometric series depends on the type of series and the position of the terms relative to the mode. It will be sufficient to consider sums of the type S_t (= sum of the first t terms) because the sum of any number of successive terms may be expressed as a difference of two S 's.

1. When the last term of S_t (i.e. the t th) falls outside the range mode $\pm 2.5\sigma$, one of the methods of §§7—9 may be used for evaluating S_t . The list of methods in order of accuracy is as follows:

(i) Fit a geometrical series $a_0 \sum_{s=0}^{\infty} r^s$ to the tail and obtain the approximation S_1 (p. 93).

(ii a) Fit the series $a_0 \sum_{s=0}^{\infty} r^s p_1^{s(s-1)/2}$ and obtain the approximation S_2 (p. 93).

(ii b) Use the equivalent method of p. 98 (Camp's method).

(iii a) Fit the series $a_0 \sum_{s=0}^{\infty} r^s p_1^{\frac{s(s-1)}{1.2}} q_1^{\frac{s(s-1)(s-2)}{1.2.3}}$ and obtain the approximation S_3 (p. 94).

(iii b) Correct the result obtained by (iv b) as in p. 97.

(iv) Fit a Type III curve to the tail of the series by either

(a) equating the logarithmic differences at the stumps, or

(b) equating the logarithmic differentials at the stumps.

The required sum may then be expressed by an incomplete Gamma function (pp. 103—104).

(v) Any degree of accuracy may be obtained by taking a sufficient number of constants in the series $a_0 \sum_{s=0}^{\infty} r^s p_1 \frac{s(s-1)}{1 \cdot 2} q_1 \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} u_1 \frac{s(s-1)(s-2)(s-3)}{1 \cdot 2 \cdot 3 \cdot 4} \dots$ and expressing the sum of the tail in terms of incomplete normal moment functions.

2. When the t th term lies near the mode, then for

(i) x small ($< \frac{1}{2}$), use method of § 1,

(ii) $x \geq \frac{1}{2}$, use method of § 2.

3. For J-shaped series, use method developed in § 2 whatever the value of x .

The partial areas of any bell-shaped continuous distribution, $f(z)$, whose logarithmic differentials at any point are known, may be found by applying the methods of §§ 3—6. In addition to those given for finding the area of the tail, we may also use the method of § 9 where the tail of a Type III curve is fitted to the tail of a continuous distribution. Clearly if $u(z)$ be the Type III curve fitted by this method to the tail $z \geq z_0$ of $f(z)$, then

$$f(\zeta) = u(\zeta) \exp. (a_4 \zeta^4/4! + a_5 \zeta^5/5! + \dots),$$

where $\zeta = z - z_0$ and a_i the i th logarithmic differential of $f(z)/u(z)$. Moreover,

$$f(\zeta) = u(\zeta) [1 + a_4 \zeta^4/4! + a_5 \zeta^5/5! + \dots].$$

The area $\int_{z_0}^{\infty} f(z) dz$ of the tail of $f(z)$ may then be expressed in terms of incomplete Gamma functions.

In conclusion I wish to record my indebtedness to Professor Karl Pearson for suggesting this problem to me and for his helpful advice and criticism. I wish also to thank Miss Kirby for preparing my diagrams.

DIE STATISTIK DER SELTENEN EREIGNISSE*.

VON ROLF LÜDERS.

I. TEIL. THEORIE.

§ 1. *Einleitung und Problemstellung.* Werden n Individuen eines Kollektivs auf das Vorhandensein eines gewissen Merkmals geprüft und ist bei jedem Individuum die Wahrscheinlichkeit für das Vorhandensein dieses Merkmals q , so ist die Wahrscheinlichkeit dafür, dass genau r Individuen das Merkmal besitzen, bekanntlich

$$P_r = \binom{n}{r} q^r (1 - q)^{n-r}.$$

Diese Formel, die häufig Newtonsche Formel genannt wird, lässt sich aus dem binomischen Satz leicht herleiten†.

Wird die Anzahl n der Individuen unendlich gross und konvergiert q gegen Null derart, dass $n \cdot q = m$ endlich bleibt, so erhält man die sogenannte Poissonsche Formel

$$P_r = \frac{e^{-m} \cdot m^r}{r!}.$$

Diese Formel stellt zugleich eine Näherungsformel für den Fall dar, dass n sehr gross und q sehr klein ist. Die Grösse m bedeutet dabei den Mittelwert, d. h. die durchschnittliche Zahl der Individuen, die das Merkmal besitzen.

Die Poissonsche Formel lässt sich in zahlreichen Fällen auf die Statistik der seltenen Ereignisse, wie der Unfälle, Krankheiten oder Selbstmorde anwenden. Bei Unfällen ist das Kollektiv die einem Unfall ausgesetzte Bevölkerung, das Merkmal ist das Eintreten eines Unfalls. Stellt man die Zahl der Unfälle etwa für eine Reihe von Monaten fest, so kann man die beobachtete Häufigkeit einer bestimmten Zahl von Unfällen im Monat mit der aus der Poissonschen Formel berechneten Häufigkeit vergleichen.

Die Poissonsche Formel gilt aber nur unter folgenden Voraussetzungen:

- (1) Die Wahrscheinlichkeit für den Eintritt des Ereignisses muss in allen beobachteten Monaten dieselbe sein, sie darf nicht von der Zeit abhängen.
- (2) Die Wahrscheinlichkeit muss sehr klein sein, da unter dieser Voraussetzung die Poissonsche Formel abgeleitet worden ist.

* Dissertation zur Erlangung der Doktorwürde der Mathematisch-Naturwissenschaftlichen Fakultät der Hamburgischen Universität, 1933.

† Mises, *Vorlesungen aus dem Gebiete der angewandten Mathematik*, I. Bd., Leipzig-Wien, 1931, S. 128.

(3) Die einzelnen Ereignisse müssen unabhängig von einander sein, durch das Eintreten eines Ereignisses darf die Wahrscheinlichkeit nicht geändert werden. Es darf also keine Wahrscheinlichkeitsansteckung bestehen.

Lässt man die letzte Voraussetzung fallen, so kommt man zu der von Eggenberger und Pólya aufgestellten Formel, die im folgenden stets Eggenbergersche Formel genannt wird*. Nach dieser Formel ist die Wahrscheinlichkeit dafür, dass genau r Ereignisse eintreten,

$$P_r = \frac{1}{r!} h(h+d)(h+2d) \dots (h+(r-1)d) \cdot (1+d)^{-\frac{h}{d}-r} \quad (r=1, 2, 3, \dots);$$

und es ist $P_0 = (1+d)^{-\frac{h}{d}}$.

Dabei ist h der Mittelwert, die Grösse d ergibt sich aus der Beziehung

$$h(1+d) = \mu_2, \quad \text{wo} \quad \mu_2 = \sum_{r=0}^{\infty} (r-m)^2 P_r,$$

das Mittel zweiter Ordnung (Quadrat der mittleren Abweichung oder Streuungsquadrat) ist.

Die Eggenbergersche Formel findet Verwendung in der Statistik der ansteckenden Krankheiten. Ihre Ableitung aus dem Urnenschema erscheint aber, worauf in einem Artikel von H. Pollaczek-Geiringer† hingewiesen wird, sehr gekünstelt. In demselben Artikel gibt nun die Verfasserin eine Formel an, die einen weit allgemeineren Ansatz für die Theorie der seltenen ansteckenden Ereignisse bietet.

Diese Formel, zur Abkürzung als "Formel I" bezeichnet, soll im folgenden von einem speziellen Standpunkt aus hergeleitet werden. Darauf sind einige Spezialfälle zu diskutieren, ausserdem ist der Zusammenhang mit der Eggenbergerschen Formel zu klären. Ferner soll die Frage der Mittelwerte einer kurzen Untersuchung unterzogen werden.

Die Ergebnisse der Theorie sollen im zweiten Teil an Hand einiger Beispiele nachgeprüft werden.

§ 2. *Ableitung der Formel I.* In der Statistik der seltenen Ereignisse ist die Poissonsche Formel häufig deswegen nicht gültig, weil nicht nur Ereignisse beobachtet werden, die einzeln und unabhängig von einander eintreten, sondern auch Ereignisse, die zu je zweien oder dreien eintreten. Das gilt etwa bei Unfällen: Durch *eine* Ursache kann eine Person zu Schaden kommen, es können aber auch zwei, drei oder noch mehr Todesfälle durch *eine* Ursache eintreten.

Es sei h_1 die mittlere Anzahl (der Mittelwert) der einzeln auftretenden Ereignisse, h_2 die mittlere Anzahl der Paare von Ereignissen, h_3 die mittlere Anzahl der Tripel von Ereignissen, allgemein h_n die mittlere Anzahl der n -fachheit von Ereignissen. Dann ist die mittlere Anzahl der Ereignisse überhaupt

$$m = h_1 + 2h_2 + 3h_3 + \dots + nh_n + \dots$$

* Pólya und Eggenberger, *Zeitschrift f. angew. Math. u. Mech.* III. 279, 1923.

† *Zeitschr. f. angew. Math. u. Mech.* VIII. 292, 1928.

Nun sei die fundamentale Annahme gemacht, dass die Wahrscheinlichkeiten für das Auftreten von 1, 2, 3 bzw. allgemein n zusammengehörigen Ereignissen durch die Poissonsche Formel bestimmt werden und von einander unabhängig sind. Dann ist die Wahrscheinlichkeit dafür, dass genau ν_1 Einzelereignisse vorkommen,

$$P_{\nu_1}^{(1)} = \frac{e^{-h_1} \cdot h_1^{\nu_1}}{\nu_1!},$$

entsprechend die Wahrscheinlichkeit für das Vorkommen von ν_2 Paaren von Ereignissen

$$P_{\nu_2}^{(2)} = \frac{e^{-h_2} \cdot h_2^{\nu_2}}{\nu_2!},$$

und ganz allgemein für das Vorkommen von ν_n n -fachheiten von Ereignissen

$$P_{\nu_n}^{(n)} = \frac{e^{-h_n} \cdot h_n^{\nu_n}}{\nu_n!}.$$

Wegen obiger Voraussetzung der Unabhängigkeit der Verteilungen von einander ist die Wahrscheinlichkeit für das Auftreten von r Ereignissen überhaupt

$$P_r = \sum_{\nu_1 + 2\nu_2 + \dots + r\nu_r = r} P_{\nu_1}^{(1)} \cdot P_{\nu_2}^{(2)} \dots P_{\nu_r}^{(r)},$$

$$P_r = e^{-h_1 - h_2 - \dots - h_r - \dots} \sum_{\nu_1 + 2\nu_2 + \dots + r\nu_r = r} \frac{h_1^{\nu_1} \cdot h_2^{\nu_2} \dots h_r^{\nu_r}}{\nu_1! \nu_2! \dots \nu_r!}.$$

Aus dieser im folgenden als "Formel I" bezeichneten Formel ergibt sich speziell

$$P_0 = e^{-h_1 - h_2 - h_3 - \dots}.$$

$$P_1 = P_0 \cdot h_1, \quad P_2 = P_0 \left(\frac{h_1^2}{2!} + h_2 \right), \quad P_3 = P_0 \left(\frac{h_1^3}{3!} + h_1 h_2 + h_3 \right).$$

§ 3. *Der Mittelwert und die Mittel höherer Ordnung bei der Formel I.* Zunächst ist die erzeugende Funktion der Formel des vorigen Paragraphen zu bestimmen. Diese Funktion erhält man aus

$$f(z) = \sum_{r=0}^{\infty} P_r z^r = \sum_{r=0}^{\infty} z^r e^{-h_1 - h_2 - \dots} \sum_{\nu_1 + 2\nu_2 + \dots = r} \frac{h_1^{\nu_1} h_2^{\nu_2} \dots}{\nu_1! \nu_2! \dots}$$

$$= e^{-h_1 - h_2 - \dots} \sum_{r=0}^{\infty} \sum_{\nu_1 + 2\nu_2 + \dots = r} \frac{(h_1 z)^{\nu_1} (h_2 z^2)^{\nu_2} \dots}{\nu_1! \nu_2! \dots}$$

da $z^r = z^{\nu_1 + 2\nu_2 + \dots}$ ist.

$$\text{Weiter folgt} \quad f(z) = e^{-h_1 - h_2 - \dots} \sum_{\nu_1=0}^{\infty} \frac{(h_1 z)^{\nu_1}}{\nu_1!} \sum_{\nu_2=0}^{\infty} \frac{(h_2 z^2)^{\nu_2}}{\nu_2!} \dots$$

Die erzeugende Funktion der Formel I ist also

$$f(z) = e^{-h_1 - h_2 - h_3 - \dots} \cdot e^{h_1 z + h_2 z^2 + h_3 z^3 + \dots}.$$

Zunächst ist $f(1) = 1$, d. h. $\sum_{r=0}^{\infty} P_r = 1$,

eine Bedingung, die bekanntlich notwendig erfüllt sein muss.

Um die Mittelwerte zu erhalten, hat man zu differenzieren. Setzt man zur Abkürzung

$$h(z) = -h_1 - h_2 - h_3 - \dots + h_1 z + h_2 z^2 + h_3 z^3 + \dots,$$

so erhält man

$$f(z) = e^{h(z)}, \quad f'(z) = h'(z) e^{h(z)},$$

$$f''(z) = \{h'(z)^2 + h''(z)\} e^{h(z)}, \quad f'''(z) = \{h'(z)^3 + 3h'(z)h''(z) + h'''(z)\} e^{h(z)}.$$

Nun ist

$$h(1) = 0, \quad h'(1) = h_1 + 2h_2 + 3h_3 + 4h_4 + \dots,$$

$$h''(1) = 2 \cdot 1h_2 + 3 \cdot 2h_3 + 4 \cdot 3h_4 + \dots, \quad h'''(1) = 3 \cdot 2 \cdot 1h_3 + 4 \cdot 3 \cdot 2h_4 + \dots$$

Man erhält also zunächst $f'(1) = h_1 + 2h_2 + 3h_3 + \dots$

Da nun der Mittelwert $m = \sum_{r=0}^{\infty} r P_r = f'(1)$

ist, so ergibt sich wie im vorigen Paragraphen

$$m = h_1 + 2h_2 + 3h_3 + \dots$$

Ferner ist $f''(1) = m^2 + 2 \cdot 1h_2 + 3 \cdot 2h_3 + 4 \cdot 3h_4 + \dots$

Andererseits gilt für das Mittel zweiter Ordnung

$$\begin{aligned} \mu_2 &= \sum_{r=0}^{\infty} (r-m)^2 P_r = \sum_{r=0}^{\infty} r(r-1) P_r + m - m^2 \\ &= f''(1) + m - m^2; \end{aligned}$$

also wird

$$\begin{aligned} \mu_2 &= h_1 + 2h_2 + 3h_3 + 4h_4 + \dots \\ &\quad + 2 \cdot 1h_2 + 3 \cdot 2h_3 + 4 \cdot 3h_4 + \dots \\ &= h_1 + 4h_2 + 9h_3 + 16h_4 + \dots, \end{aligned}$$

oder

$$\mu_2 = h_1 + 2^2 h_2 + 3^2 h_3 + 4^2 h_4 + \dots$$

Endlich ist

$$f'''(1) = m^3 + 3m(2 \cdot 1h_2 + 3 \cdot 2h_3 + 4 \cdot 3h_4 + \dots) + 3 \cdot 2 \cdot 1h_3 + 4 \cdot 3 \cdot 2h_4 + \dots$$

Nun gilt für das Mittel dritter Ordnung

$$\begin{aligned} \mu_3 &= \sum_{r=0}^{\infty} (r-m)^3 P_r = \sum_{r=0}^{\infty} r(r-1)(r-2) P_r + 3(1-m) \sum_{r=0}^{\infty} r(r-1) P_r + m - 3m^2 + 2m^3 \\ &= f'''(1) + 3(1-m)f''(1) + m - 3m^2 + 2m^3; \end{aligned}$$

also wird

$$\begin{aligned} \mu_3 &= m^3 + 3mh''(1) + h'''(1) + 3m^2 + 3h''(1) - 3m^3 - 3mh''(1) + m - 3m^2 + 2m^3 \\ &= h'''(1) + 3h''(1) + m \\ &= h_1 + 2h_2 + 3h_3 + 4h_4 + \dots \\ &\quad + 3 \cdot 2 \cdot 1h_2 + 3 \cdot 3 \cdot 2h_3 + 3 \cdot 4 \cdot 3h_4 + \dots \\ &\quad + 3 \cdot 2 \cdot 1h_3 + 4 \cdot 3 \cdot 2h_4 + \dots \\ &\quad \dots\dots\dots, \end{aligned}$$

d. h. aber

$$\mu_3 = h_1 + 8h_2 + 27h_3 + 64h_4 + \dots$$

oder

$$\mu_3 = h_1 + 2^3 h_2 + 3^3 h_3 + 4^3 h_4 + \dots$$

Für die Berechnung der Parameter stehen also zunächst die folgenden drei Gleichungen zur Verfügung:

$$m = h_1 + 2h_2 + 3h_3 + \dots,$$

$$\mu_2 = h_1 + 2^2 h_2 + 3^2 h_3 + \dots,$$

$$\mu_3 = h_1 + 2^3 h_2 + 3^3 h_3 + \dots$$

Die Mittel vierter und höherer Ordnung dürften für die praktische Verwendung kaum in Frage kommen, da sie mit zu hohen Fehlern behaftet sind.

§ 4. *Die Eggenbergersche Formel als Spezialfall der Formel I.* Es ist jetzt zu zeigen, dass die Formel von Eggenberger sich als Spezialfall der Formel I ansehen lässt. Sollen die Verteilungen nach diesen Formeln übereinstimmen, so muss die Gleichung

$$e^{-h_1 - h_2 - h_3 - \dots} = (1 + d)^{-\frac{h}{d}},$$

und ausserdem die Gleichung

$$e^{-h_1 - h_2 - h_3 - \dots} \sum_{\nu_1 + 2\nu_2 + 3\nu_3 + \dots = r} \frac{h_1^{\nu_1} h_2^{\nu_2} h_3^{\nu_3} \dots}{\nu_1! \nu_2! \nu_3! \dots} = (1 + d)^{-\frac{h}{d}} \cdot \frac{h(h+d) \dots (h+(r-1)d)}{r! (1+d)^r},$$

für $r = 1, 2, 3, \dots$ erfüllt sein. Es müssen demnach die folgenden Gleichungen gelten:

$$h_1 = \frac{h}{1+d} \dots \dots \dots (1),$$

$$\frac{h_1^2}{2!} + h_2 = \frac{h(h+d)}{2!(1+d)^2} \dots \dots \dots (2),$$

$$\frac{h_1^3}{3!} + h_1 h_2 + h_3 = \frac{h(h+d)(h+2d)}{3!(1+d)^3} \dots \dots \dots (3),$$

usw.

Aus (2) erhält man unter Benutzung von (1)

$$\frac{h^2}{2!(1+d)^2} + h_2 = \frac{h^2}{2!(1+d)^2} + \frac{hd}{2(1+d)^2},$$

$$h_2 = \frac{hd}{2(1+d)^2}.$$

Analog ergibt sich aus (3) unter Benutzung der Gleichungen (1) und (2)

$$\frac{h^3}{3!(1+d)^3} + \frac{h^2 d}{2(1+d)^3} + h_3 = \frac{h^3}{3!(1+d)^3} + \frac{h^2 d}{2(1+d)^3} + \frac{hd^2}{3(1+d)^3},$$

$$h_3 = \frac{hd^2}{3(1+d)^3}.$$

Setzt man allgemein für $n = 1, 2, 3, \dots$

$$h_n = \frac{hd^{n-1}}{n(1+d)^n} \dots \dots \dots (4),$$

so wird aus der erzeugenden Funktion der Formel I (§ 3)

$$f(z) = e^{-h_1 - h_2 - h_3 - \dots} \cdot e^{h_1 z + h_2 z^2 + h_3 z^3 + \dots},$$

wegen
$$h_1 z + h_2 z^2 + h_3 z^3 + \dots = \frac{h}{d} \left\{ \frac{dz}{1+d} + \frac{d^2 z^2}{2(1+d)^2} + \frac{d^3 z^3}{3(1+d)^3} + \dots \right\}$$

$$= -\frac{h}{d} \ln \left(1 - \frac{dz}{1+d} \right),$$

die Funktion
$$f(z) = e^{\frac{h}{d} \ln \left(1 - \frac{dz}{1+d} \right)} \cdot e^{-\frac{h}{d} \ln \left(1 - \frac{dz}{1+d} \right)}$$

$$= \left(1 - \frac{dz}{1+d} \right)^{\frac{h}{d}} \cdot \left(1 - \frac{dz}{1+d} \right)^{-\frac{h}{d}},$$

$$f(z) = (1+d)^{-\frac{h}{d}} \cdot \left(1 - \frac{dz}{1+d} \right)^{-\frac{h}{d}} \dots\dots\dots(5).$$

Andererseits ist bei der Eggenbergerschen Verteilung, wie Eggenberger nachgewiesen hat*, die erzeugende Funktion

$$\begin{aligned} \bar{f}(z) &= \sum_{r=0}^{\infty} \bar{P}_r z^r = (1+d)^{-\frac{h}{d}} \left\{ 1 + \sum_{r=1}^{\infty} \frac{h(h+d) \dots (h+(r-1)d)}{r! (1+d)^r} z^r \right\} \\ &= (1+d)^{-\frac{h}{d}} \left\{ 1 + \sum_{r=1}^{\infty} \frac{1}{r!} \left(-\frac{h}{d} \right) \left(-\frac{h}{d} - 1 \right) \dots \left(-\frac{h}{d} - r + 1 \right) \cdot \left(-\frac{dz}{1+d} \right)^r \right\} \\ &= (1+d)^{-\frac{h}{d}} \left\{ 1 + \sum_{r=1}^{\infty} \left(-\frac{h}{d} \right)_r \left(-\frac{dz}{1+d} \right)^r \right\}, \\ \bar{f}(z) &= (1+d)^{-\frac{h}{d}} \left(1 - \frac{dz}{1+d} \right)^{-\frac{h}{d}} \dots\dots\dots(6). \end{aligned}$$

Hieraus folgt wegen (5), dass die erzeugenden Funktionen identisch sind, d. h. dass $f(z) = \bar{f}(z)$,

oder
$$\sum_{r=0}^{\infty} P_r z^r = \sum_{r=0}^{\infty} \bar{P}_r z^r \text{ ist.}$$

Setzt man $z=0$, so folgt
$$P_0 = \bar{P}_0.$$

Ferner ist für $k=1, 2, 3, \dots$

$$\frac{d^k}{dz^k} \left(\sum_{r=0}^{\infty} P_r z^r \right) = \frac{d^k}{dz^k} \left(\sum_{r=0}^{\infty} \bar{P}_r z^r \right);$$

setzt man wieder $z=0$, so erhält man die Gleichungen

$$P_k = \bar{P}_k \quad (k=1, 2, 3, \dots).$$

Sind also die erzeugenden Funktionen zweier Verteilungen identisch, so stimmen beide Verteilungen überein. Damit ist bewiesen, dass die Eggenbergersche Formel ein Spezialfall der Formel I ist. Man erhält die Formel von Eggenberger, wenn man bei der Formel I

$$h_n = \frac{h d^{n-1}}{n(1+d)^n} \quad (n=1, 2, 3, \dots)$$

setzt.

* Die Wahrscheinlichkeitsansteckung, Bern, 1924.

§ 5. *Spezialfälle der Formel I.* Die Formel I ist zunächst nicht zu verwenden, da unendlich viele Parameter h_1, h_2, h_3, \dots bestimmt werden müssen. Es lassen sich aber spezielle Annahmen über diese Größen machen, sodass nur eine endliche Anzahl von Parametern zu berechnen ist.

Die Poissonsche Formel erhält man, wenn man

$$h_3 = h_4 = h_5 = \dots = 0$$

setzt; denn dann ist

$$h_1 = m$$

und

$$P_r = \frac{e^{-m} \cdot m^r}{r!} \quad (r = 0, 1, 2, \dots).$$

Man kann auch annehmen, dass h_1 und h_2 von Null verschieden, dagegen $h_3 = h_4 = \dots = 0$ ist, d. h. dass die Ereignisse entweder isoliert oder zu Paaren auftreten können, jedoch nicht mehr zu dreien, vierten usw. Unter dieser Voraussetzung ergibt sich die Formel Ia:

$$P_r = e^{-h_1 - h_2} \sum_{\nu_1 + 2\nu_2 = r} \frac{h_1^{\nu_1} h_2^{\nu_2}}{\nu_1! \nu_2!}$$

Dabei ist

$$h_1 + 2h_2 = m, \quad h_1 + 4h_2 = \mu_2,$$

also

$$h_1 = 2m - \mu_2, \quad h_2 = \frac{\mu_2 - m}{2}.$$

Die Parameter lassen sich auf eine recht einfache Weise berechnen.

Analog erhält man unter der Voraussetzung, dass höchstens drei zusammengehörige Ereignisse beobachtet werden,

$$h_4 = h_5 = h_6 = \dots = 0.$$

Die entsprechende Formel Ib lautet:

$$P_r = e^{-h_1 - h_2 - h_3} \sum_{\nu_1 + 2\nu_2 + 3\nu_3 = r} \frac{h_1^{\nu_1} h_2^{\nu_2} h_3^{\nu_3}}{\nu_1! \nu_2! \nu_3!}.$$

Dabei ist

$$h_1 + 2h_2 + 3h_3 = m,$$

$$h_1 + 4h_2 + 9h_3 = \mu_2,$$

$$h_1 + 8h_2 + 27h_3 = \mu_3,$$

woraus folgt

$$h_1 = 3m - \frac{5}{2}\mu_2 + \frac{\mu_3}{2},$$

$$h_2 = -\frac{3}{2}m + 2\mu_2 - \frac{\mu_3}{2},$$

$$h_3 = \frac{m}{3} - \frac{\mu_2}{2} + \frac{\mu_3}{6}.$$

§ 6. *Die Formel II, eine Erweiterung der Eggenbergerschen Formel.* Man erhält, wie in § 4 gezeigt worden ist, die Eggenbergersche Formel aus Formel I, wenn man die h_1, h_2, h_3, \dots in der folgenden Progression abnehmen lässt:

$$h_1, \quad h_2 = \frac{h_1 q}{2}, \quad h_3 = \frac{h_1 q^2}{3}, \quad h_4 = \frac{h_1 q^3}{4}, \quad \dots, \quad h_n = \frac{h_1 q^{n-1}}{n}, \quad \dots,$$

wo $q = \frac{h}{1+d}$ ist. Nun hat die Formel von Eggenberger aber nur zwei Parameter (h und d). Um eine Formel mit drei Parametern zu bekommen, liegt es nahe, die h_1, h_2, h_3, \dots in der Weise zu spezialisieren, dass man nach Festsetzung von h_1 und h_2 erst die h_3, h_4, h_5, \dots in entsprechender Progression abnehmen lässt:

$$h_1, h_2 = \frac{p}{2}, h_3 = \frac{pq}{3}, h_4 = \frac{pq^2}{4}, \dots, h_n = \frac{pq^{n-2}}{n}, \dots$$

Nun ist
$$m = h_1 + 2h_2 + 3h_3 + 4h_4 + \dots,$$

also erhält man
$$m = h_1 + p(1 + q + q^2 + \dots) = h_1 + \frac{p}{1-q}.$$

Ferner ist
$$\begin{aligned} \mu_2 &= h_1 + 4h_2 + 9h_3 + 16h_4 + \dots \\ &= h_1 + p(2 + 3q + 4q^2 + 5q^3 + \dots). \end{aligned}$$

Es ist jedoch
$$2 + 3q + 4q^2 + \dots = \frac{1}{1-q} + \frac{1}{(1-q)^2}.$$

Daher ist
$$\mu_2 = h_1 + p\left(\frac{1}{1-q} + \frac{1}{(1-q)^2}\right).$$

Endlich gilt
$$\begin{aligned} \mu_3 &= h_1 + 8h_2 + 27h_3 + 64h_4 + 125h_5 + \dots \\ &= h_1 + p(4 + 9q + 16q^2 + 25q^3 + \dots). \end{aligned}$$

Nun ist aber
$$4 + 9q + 16q^2 + 25q^3 + \dots = \frac{1}{1-q} + \frac{1}{(1-q)^2} + \frac{1}{(1-q)^3}.$$

Man hat also das Gleichungssystem:

$$m = h_1 + \frac{p}{1-q} \dots\dots\dots(1),$$

$$\mu_2 = h_1 + p\left(\frac{1}{1-q} + \frac{1}{(1-q)^2}\right) \dots\dots\dots(2),$$

$$\mu_3 = h_1 + p\left(\frac{1}{1-q} + \frac{1}{(1-q)^2} + \frac{2}{(1-q)^3}\right) \dots\dots\dots(3).$$

Hieraus folgt
$$\frac{p}{(1-q)^2} = \mu_2 - m,$$

$$\frac{2p}{(1-q)^3} = \mu_3 - \mu_2;$$

durch Division erhält man
$$1 - q = 2 \cdot \frac{\mu_2 - m}{\mu_3 - \mu_2},$$

ferner
$$p = (\mu_2 - m)(1 - q)^2,$$

und
$$h_1 = m - \frac{p}{1-q}.$$

Zur Berechnung der P_r hat man zunächst

$$P_0 = e^{-h_1 - h_2 - h_3 - \dots}$$

auszuwerten. Nun ist

$$\begin{aligned} h_1 + h_2 + h_3 + \dots &= h_1 + \frac{p}{2} + \frac{pq}{3} + \frac{pq^2}{4} + \dots \\ &= h_1 - \frac{p}{q} + \frac{p}{q^2} \left(q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \dots \right) \\ &= h_1 - \frac{p}{q} - \frac{p}{q^2} \ln(1-q). \end{aligned}$$

Also wird

$$\begin{aligned} P_0 &= e^{-h_1 + \frac{p}{q} + \frac{p}{q^2} \ln(1-q)} \\ &= (1-q)^{\frac{p}{q^2}} \cdot e^{-h_1 + \frac{p}{q}}, \end{aligned}$$

und man erhält

$$P_r = P_0 \sum_{\nu_1 + 2\nu_2 + 3\nu_3 + \dots = r} \frac{h_1^{\nu_1} h_2^{\nu_2} h_3^{\nu_3} \dots}{\nu_1! \nu_2! \nu_3! \dots},$$

wobei man für die h_1, h_2, h_3, \dots die zu Beginn des Paragraphen angegebenen Werte einzusetzen hat.

Die hergeleitete Formel (im folgenden als Formel II bezeichnet) hat gegenüber der Eggenbergerschen Formel den Vorzug, dass die P_r von *drei* Parametern, nämlich h, p, q , abhängen. Sie hat allerdings den Nachteil, dass zur Berechnung der Parameter auch das Mittel dritter Ordnung μ_3 gebraucht wird, das bekanntlich mit einem ziemlich hohen wahrscheinlichen Fehler behaftet ist.

Für

$$p = h_1 q$$

erhält man wieder die Eggenbergersche Formel; es ist dann

$$\begin{aligned} h_1 &= \frac{h}{1+d}, \\ q &= \frac{d}{1+d}, \end{aligned}$$

also

$$h_n = \frac{p \cdot q^{n-2}}{n} = \frac{h d^{n-1}}{n(1+d)^n} \quad (n = 1, 2, 3, \dots).$$

Die Formel II stellt daher zugleich eine Verallgemeinerung der Formel von Eggenberger dar.

§7. *Modifizierte Mittelwerte.* Die betrachteten Formeln, vor allem die Formel II, haben den Nachteil, dass zur Berechnung der Parameter neben dem Mittelwert m noch die höheren Mittel μ_2 und μ_3 benötigt werden. Nun ist bekanntlich der mittlere Fehler von μ_2

$$\epsilon_{\mu_2} = \sqrt{\frac{2m^2 + m}{N}},$$

der von μ_3

$$\epsilon_{\mu_3} = \sqrt{\frac{15m^3 + 24m^2 + m}{N}},$$

wo N die Zahl der beobachteten Zeiteinheiten ist. (Diese Ausdrücke gelten, streng genommen, nur im Falle der Poissonschen Verteilung; sie sind aber auch bei anderen Verteilungen anwendbar, wenn es nur auf die Grössenordnung der Fehler ankommt.)

Für grössere Werte von m sind die obigen mittleren Fehler offenbar recht hoch. Die Parameter der Formeln, die aus den Mitteln berechnet werden, sind dann so ungenau, dass auf diesem Wege die theoretische Verteilung nicht gefunden werden kann.

Der grosse Fehler der höheren Mittel kommt daher, dass in der Formel

$$\mu_k = \sum_{r=0}^{\infty} (r-m)^k P_r,$$

schon von $k=3$ an die Faktoren $(r-m)^k$ zu gross und deshalb die Summanden, mithin auch die Summe, zu gross werden. Dasselbe gilt von den Ausdrücken $f'(1)$, $f''(1)$ und $f'''(1)$ des § 3. Aus diesem Grunde liegt es nahe, jeden Summanden mit einem Faktor zu multiplizieren, der für grosse r sehr klein wird.

Es soll die folgende Funktion eingeführt werden:

$$\phi(z) = \sum_{r=0}^{\infty} P_r x^r z^r \dots\dots\dots(1),$$

die der in § 3 definierten erzeugenden Funktion entspricht. Dabei ist x eine Zahl zwischen 0 und 1. Für $z=1$ erhält man die folgenden Ausdrücke:

$$\phi = \sum_0^{\infty} P_r x^r \dots\dots\dots(2),$$

$$\phi' = \sum_0^{\infty} r P_r x^r \dots\dots\dots(3),$$

$$\phi'' = \sum_0^{\infty} r(r-1) P_r x^r \dots\dots\dots(4),$$

$$\phi''' = \sum_0^{\infty} r(r-1)(r-2) P_r x^r \dots\dots\dots(5).$$

Die drei letzten Ausdrücke werden im folgenden modifizierte Mittelwerte genannt. Durch entsprechende Wahl von x (z. B. $x=\frac{1}{2}$) lässt sich erreichen, dass die Summanden nicht zu hoch werden und so die mittleren Fehler obiger Ausdrücke gering bleiben.

Die Formel I lautet nun

$$P_r = \sum_{\nu_1+2\nu_2+3\nu_3+\dots=r} \frac{h_1^{\nu_1} h_2^{\nu_2} h_3^{\nu_3} \dots}{\nu_1! \nu_2! \nu_3! \dots} \cdot e^{-h_1-h_2-h_3-\dots}.$$

Also wird ähnlich wie in § 3

$$\phi(z) = e^{-h_1-h_2-h_3-\dots+h_1xz+h_2x^2z^2+h_3x^3z^3+\dots} \dots\dots\dots(6),$$

oder

$$\phi(z) = e^{h(z)},$$

wo

$$h(z) = -h_1-h_2-h_3-\dots+h_1xz+h_2x^2z^2+h_3x^3z^3+\dots \dots\dots(7)$$

ist. Für $z = 1$ erhält man $\phi = e^h$,
 $\phi' = \phi \cdot h' \dots\dots\dots(8)$,

$$\phi'' = \phi (h'' + h'^2) \dots\dots\dots(9),$$

$$\phi''' = \phi (h''' + 3h'h'' + h'^3) \dots\dots\dots(10).$$

Dabei ist $h = -h_1 - h_2 - h_3 - \dots + h_1x + h_2x^2 + h_3x^3 + \dots$,
 $h' = h_1x + 2h_2x^2 + 3h_3x^3 + \dots \dots\dots(11)$,

$$h'' = 2h_2x^2 + 6h_3x^3 + \dots \dots\dots(12),$$

$$h''' = 6h_3x^3 + \dots \dots\dots(13).$$

Aus den Gleichungen (8), (9), (10) folgt

$$h' = \frac{\phi'}{\phi} \dots\dots\dots(14),$$

$$h'' = \frac{\phi''}{\phi} - h'^2 \dots\dots\dots(15),$$

$$h''' = \frac{\phi'''}{\phi} - 3h'h'' - h'^3 \dots\dots\dots(16).$$

Beschränkt man sich auf die Formel I b, so kann man aus diesen Gleichungen die h' , h'' , h''' berechnen und dann aus (11), (12), (13) die Werte für h_1 , h_2 , h_3 .

§ 8. Die Methode der modifizierten Mittelwerte in Anwendung auf Formel II. Zur Berechnung der Parameter der Formel II kann man sogar auf die Ausdrücke (4) und (5) des vorigen Paragraphen verzichten, indem man drei verschiedene Werte für x , nämlich x_1 , x_2 und x_3 , benutzt. Durch Einsetzung dieser Werte in (11) ergeben sich dann drei Gleichungen zur Berechnung von h_1 , p und q .

Nun ist wegen $h_2 = \frac{p}{2}$, $h_3 = \frac{pq}{3}$, $h_4 = \frac{pq^2}{4}$, ...

$$\begin{aligned} h' &= h_1x + 2h_2x^2 + 3h_3x^3 + 4h_4x^4 + \dots \\ &= h_1x + px^2 + pqx^3 + pq^2x^4 + \dots \\ &= x \left\{ h_1 + \frac{px}{1 - qx} \right\}, \end{aligned}$$

oder $h_1 + \frac{px}{1 - qx} = \frac{h'}{x} \dots\dots\dots(1).$

Für die rechte Seite dieser Gleichung gilt wegen Gleichung (14) des vorigen Paragraphen

$$d = \frac{h'}{x} = \frac{1}{x} \cdot \frac{\phi'}{\phi} = \frac{1}{x} \cdot \frac{\sum_{r=0}^{\infty} r P_r x^r}{\sum_{r=0}^{\infty} P_r x^r} \dots\dots\dots(2).$$

Die drei Gleichungen für h_1 , p und q lauten daher

$$h_1 + \frac{px_1}{1 - qx_1} = d_1 \dots\dots\dots(3),$$

$$h_1 + \frac{px_2}{1 - qx_2} = d_2 \dots\dots\dots(4),$$

$$h_1 + \frac{px_3}{1 - qx_3} = d_3 \dots\dots\dots(5).$$

Aus (3) und (4) folgt

$$\frac{px_1}{1 - qx_1} - \frac{px_2}{1 - qx_2} = d_1 - d_2,$$

d. h. $p(x_1 - x_2) = (d_1 - d_2)(1 - qx_1)(1 - qx_2).$

Analog ergibt sich aus (4) und (5)

$$p(x_2 - x_3) = (d_2 - d_3)(1 - qx_2)(1 - qx_3).$$

Durch Division erhält man

$$\frac{1 - qx_1}{1 - qx_3} = \frac{x_1 - x_2}{x_2 - x_3} \cdot \frac{d_2 - d_3}{d_1 - d_2} \dots\dots\dots(6).$$

Aus dieser Gleichung lässt sich q leicht berechnen, nachdem die Werte der d_1 , d_2 , d_3 entsprechend der Gleichung (2) festgestellt worden sind.

II. TEIL. ANWENDUNGEN.

§ 9. *Vorbemerkungen zu den Beispielen.* Die im ersten Teil besprochenen Formeln werden im folgenden Teil auf verschiedene Gebiete der Statistik der seltenen Ereignisse angewandt.

Zur Berechnung der Parameter der Formeln werden benutzt:

der Mittelwert $m = \sum_{r=0}^{\infty} \frac{rP_r}{N} \dots\dots\dots(1),$

das Mittel zweiter Ordnung $\mu_2 = \sum_{r=0}^{\infty} \frac{(r-m)^2 P_r}{N} \dots\dots\dots(2),$

und das Mittel dritter Ordnung $\mu_3 = \sum_{r=0}^{\infty} \frac{(r-m)^3 P_r}{N} \dots\dots\dots(3).$

Dabei ist N die Anzahl der Zeitintervalle, über die sich die Beobachtung ausdehnt, und P_r die Anzahl der Zeitintervalle, in denen r Fälle beobachtet worden sind, sodass

$$\sum_{r=0}^{\infty} P_r = N \text{ ist.}$$

Zu bemerken ist hier, dass der wahrscheinlichste Wert des Mittels zweiter Ordnung

$$\sum_{r=0}^{\infty} \frac{(r-m)^2 P_r}{N-1}$$

ist. Gleichwohl ist das Mittel zweiter Ordnung stets nach Gleichung (2) berechnet worden. Der Fehler kann für grosse N vernachlässigt werden. Bei dieser Methode vereinfacht sich die Rechnung durch folgende Gleichungen:

$$\mu_2 = \sum_{r=0}^{\infty} \frac{(r-m+\delta)^2 P_r}{N} - \delta^2,$$

$$\mu_3 = \sum_{r=0}^{\infty} \frac{(r-m+\delta)^3 P_r}{N} - 3\delta\mu_2 - \delta^3.$$

Hier ist δ eine Zahl zwischen 0 und 1, die so gewählt ist, dass $m - \delta$ eine ganze Zahl wird ($m - \delta$ ist ein "vorläufiger Mittelwert," nämlich die dem Mittelwert m nächstkommende ganze Zahl).

Was die Genauigkeit anbelangt, so ist der mittlere Fehler des Mittelwertes m

$$= \sqrt{\frac{\mu_2}{N}},$$

der des Mittels zweiter Ordnung

$$= \sqrt{\frac{2m^2 + m}{N}},$$

und der des Mittels dritter Ordnung

$$= \sqrt{\frac{15m^3 + 24m^2 + m}{N}}.$$

(Betreffs der beiden letzten Ausdrücke gilt das in § 7 Gesagte.)

Um die Güte der Anpassung der theoretischen Häufigkeiten an die beobachteten zu prüfen, ist das Pearsonsche χ^2 -Kriterium* benutzt worden. Aus der Grösse

$$\chi^2 = \sum_r \frac{(\bar{P}_r - P_r)^2}{\bar{P}_r},$$

wo die P_r die beobachteten, die \bar{P}_r die theoretischen Häufigkeiten sind, erhält man mit Hilfe der Pearsonschen Tabelle die Wahrscheinlichkeit P dafür, dass die Abweichungen von den theoretischen Werten ebenso gross oder grösser als beobachtet sind. (Die Häufigkeiten für grössere r sind bei der Anwendung des Kriteriums zusammenzufassen.) Je näher P an 1 liegt, um so besser ist die Anpassung der Theorie an die Beobachtung.

Grundsätzlich ist noch zu bemerken, dass eine Formel sich um so besser der Beobachtung anpasst, je mehr Parameter sie besitzt. Das gilt insbesondere bei Verteilungen mit sehr kleinen Mittelwerten. In solchen Fällen ist die gute Anpassung einer Formel mit drei Parametern noch kein Beweis dafür, dass das Schema der Formel (etwa die Ansteckung) den tatsächlichen Verhältnissen entspricht.

Die in den folgenden Tabellen angeführten theoretischen Häufigkeiten sind im allgemeinen auf eine Stelle nach dem Komma abgerundet. Dadurch ist es zu

* Pearson, *Tables for Statisticians and Biometricians*, Cambridge, 1914, Table XII.

erklären, dass die Summe dieser Zahlen in einigen Fällen von der Summe N der beobachteten Häufigkeiten um eine oder zwei Einheiten der ersten Dezimale abweicht.

§ 10. *Todesfälle durch Eisen- und Strassenbahn im Saargebiet.* Das Statistische Amt des Saargebietes veröffentlicht die monatliche Anzahl der durch Eisen- und Strassenbahn verursachten Todesfälle*. Es sollen die Monate Januar 1925 bis Dezember 1929, also insgesamt 60 Monate, und zwar nur die Todesfälle von *männlichen* Personen untersucht werden. Tabelle I gibt in Spalte II die Anzahl der Monate an, in denen die in Spalte I angegebene Zahl von Todesfällen vorgekommen ist, also die beobachtete Häufigkeit der Todesfälle. Der Mittelwert beträgt $m = 1.383$, d. h. es sind monatlich im Durchschnitt 1.383 Todesfälle eingetreten.

Nach der Formel von Poisson ist die Wahrscheinlichkeit dafür, dass genau r Todesfälle während eines Monats vorkommen,

$$\frac{e^{-m} \cdot m^r}{r!},$$

also die theoretische Häufigkeit, da 60 Monate beobachtet worden sind,

$$P_r = 60 \frac{e^{-m} \cdot m^r}{r!} \quad (r = 0, 1, 2, \dots).$$

Spalte III der Tabelle I enthält die auf diese Weise berechneten Häufigkeitszahlen.

Um die Güte der Anpassung an die Beobachtung zu prüfen, hat man nach dem Pearsonschen χ^2 -Kriterium die Grösse

$$\chi^2 = \sum_{r=0}^3 \frac{(P_r - \bar{P}_r)^2}{\bar{P}_r} = 3.72$$

zu berechnen. Dabei sind die Häufigkeiten von $r=4$ ab zusammengefasst; die Anzahl der Summanden ist demnach $n' = 5$. Auf Grund der Pearsonschen Tabelle besteht dann die Wahrscheinlichkeit $P = 0.45$ dafür, dass die Abweichungen von der theoretischen Verteilung mindestens ebenso gross wie beobachtet sind.

Bei der Poissonschen Formel müsste

$$m = \mu_2 = \mu_3$$

sein, hier ist aber $m = 1.383$, $\mu_2 = 2.036$, $\mu_3 = 3.585$.

Unter Berücksichtigung der mittleren Fehler

$$\epsilon_m = 0.18, \quad \epsilon_{\mu_2} = 0.29, \quad \epsilon_{\mu_3} = 1.2,$$

dürften die Differenzen zwischen den Mitteln zu gross sein, als dass man sie noch als zufällig ansehen kann.

Was nun die Eggenbergersche Formel angeht, so ist

$$h = m = 1.383, \\ 1 + d = \frac{\mu_2}{m} = 1.472, \text{ also } d = 0.472.$$

* Bericht des Statistischen Amtes des Saargebietes, Saarbrücken; 5. Heft (1927), bis 9. Heft (1929).

Man erhält
$$P_0 = 60 (1 + d)^{-\frac{h}{d}} = 19.31$$

und die weiteren in Spalte IV angegebenen Häufigkeitszahlen. Die Übereinstimmung mit der Beobachtung ist hier besser als bei der Poissonschen Formel. Denn nach dem Pearsonschen χ^2 -Kriterium ist wegen $\chi^2 = 0.97$ die Wahrscheinlichkeit dafür, dass die Abweichungen mindestens ebenso gross wie beobachtet sind, $P = 0.91$.

In dem vorliegenden Beispiel besteht Ansteckung in dem Sinne, dass durch *einen* Unfall auch zwei oder mehr Personen in Mitleidenschaft gezogen werden können. Nimmt man eine maximale Zahl von drei Toten bei einem Unfall an, so hat man die Formel I b anzuwenden. Dann ist

$$h_1 = 3m - \frac{5}{2}\mu_2 + \frac{\mu_3}{2} = 0.852,$$

$$h_2 = -\frac{3}{2}m + 2\mu_2 - \frac{\mu_3}{2} = 0.205,$$

$$h_3 = \frac{m}{3} - \frac{\mu_2}{2} + \frac{\mu_3}{6} = 0.041.$$

Man erhält
$$P_0 = 60 \cdot e^{-h_1 - h_2 - h_3} = 20.01$$

und die weiteren Zahlen der Spalte V. Diese Werte stimmen ausgezeichnet mit den empirischen überein. Auf Grund des Pearsonschen χ^2 -Kriteriums ist $\chi^2 = 0.65$ und $P = 0.96$.

Zur Kontrolle sei noch die Formel II benutzt. Nach dieser Formel ist

$$1 - q = 2 \cdot \frac{\mu_2 - m}{\mu_3 - \mu_2} = 0.843, \quad q = 0.157;$$

ferner ist $p = (\mu_2 - m)(1 - q)^2 = 0.464$.

Es wird
$$h_1 = m - \frac{p}{1 - q} = 0.833,$$

$$h_2 = \frac{p}{2} = 0.232,$$

$$h_3 = \frac{pq}{3} = 0.024,$$

$$h_4 = \frac{pq^2}{4} = 0.003 \text{ usw.}$$

Bedenkt man, dass insbesondere μ_3 mit einem recht hohen mittleren Fehler behaftet ist, so sind die Abweichungen der obigen Werte der h_1, h_2, h_3 von den nach Formel I b berechneten als gering anzusehen. h_4 ist fast gleich Null, womit die oben gemachte Annahme über die Höchstzahl der Toten bei einem Unfall sich als berechtigt erwiesen hat. Die aus dieser Formel zu berechnende Verteilung würde sich daher nur unwesentlich von der Verteilung in Spalte V unterscheiden. Nach Formel II verlaufen von allen tödlichen Unfällen durch Eisen- oder Strassenbahn rd. 76 % mit einem Todesfall, 21 % mit zwei Todesfällen und 2 % mit drei Todesfällen. Eine Nachprüfung dieser Daten durch die Beobachtung ist auf Grund des

zitierten Berichtes nicht möglich, da Angaben über die Zahl der durch die gleiche Ursache Verunglückten fehlen.

Die Eggenbergersche Formel stimmt hier nicht so gut mit der Beobachtung überein wie die Formel I b. Das kommt daher, dass nach der Eggenbergerschen Formel

$$h_1 = \frac{h}{1+d} = 0.940,$$

$$h_2 = \frac{1}{2} \frac{d}{1+d} \cdot h_1 = 0.150,$$

$$h_3 = \frac{1}{3} \left(\frac{d}{1+d} \right)^2 h_1 = 0.032,$$

$$h_4 = \frac{1}{4} \left(\frac{d}{1+d} \right)^3 h_1 = 0.008$$

ist. Man sieht, dass diese Zahlen von den entsprechenden der Formel I b oder II abweichen, dass hier insbesondere h_1 zu gross und h_2 zu klein ist.

Die gute Übereinstimmung zwischen Theorie und Beobachtung ist hier vor allem dadurch begründet, dass das Material homogen ist. Es beträgt nämlich

im Jahre	die Zahl der Todesfälle
1925	12
1926	19
1927	15
1928	18
1929	19

Die Schwankungen von Jahr zu Jahr sind also nicht zu gross.

TABELLE I.

Verteilung der männlichen Todesfälle durch Eisen- und Strassenbahn im Saargebiet.

I. Anzahl der Fälle	II. Beobacht. Verteil.	III. Verteil. nach Poisson	IV. Verteil. nach Eggenberger	V. Verteil. nach Formel I b
0	20	15.1	19.3	20.0
1	17	20.8	18.1	17.1
2	11	14.4	11.4	11.4
3	8	6.6	6.0	6.4
4	2	2.3	2.9	3.0
5	—	0.6	1.3	1.3
6	2	0.1	0.5	0.5
7	—	0.0	0.3	0.3
und mehr				

§ 11. *Verteilung der weiblichen Diphteriefälle in Böhmen 1921 bis 1929.* In einem kürzlich erschienenen Artikel der *Aktuárské Vědy* gibt M. Vacek bei der Untersuchung der Eggenbergerschen Formel als Beispiel u. a. die weiblichen *Diphterie*-Fälle in Böhmen für die 108 Monate der Jahre 1921 bis 1929 an*.

Spalte II der folgenden Tabelle II zeigt die beobachtete, Spalte III die von Vacek auf Grund der Eggenbergerschen Formel berechnete Häufigkeit. Dabei sind der Mittelwert

$$m = h = 15 \cdot 8796$$

und die Grösse

$$d = 13 \cdot 6274$$

zugrundegelegt worden. Was die Übereinstimmung mit der beobachteten Verteilung angeht, so ist

$$\chi^2 = 24 \cdot 4$$

also

$$P = 0 \cdot 08,$$

d. h. die Anpassung der Eggenbergerschen Formel ist hier sehr schlecht.

Das Eggenbergersche Schema ist nicht geeignet, entweder weil das Mittel zweiter Ordnung, aus dem d berechnet worden ist, mit einem zu hohen Fehler behaftet ist, oder weil die speziellen Voraussetzungen, die über die Art der Ansteckung zu machen sind, hier nicht zutreffen. Um beide Fehler möglichst auszuschalten, soll die Formel II benutzt werden, wobei die Parameter durch die Methode der modifizierten Mittelwerte berechnet werden sollen. Dadurch wird der hohe Fehler durch das Mittel dritter Ordnung vermieden.

Setzt man

$$x_1 = 1,$$

$$\log x_2 = -0 \cdot 1 \dagger, \text{ d. h. } x_2 = 0 \cdot 7943,$$

$$\log x_3 = -0 \cdot 2, \text{ d. h. } x_3 = 0 \cdot 6310,$$

so erhält man entsprechend der Gleichung (2) des § 8

$$d_1 = 15 \cdot 880, \quad d_2 = 6 \cdot 401, \quad d_3 = 5 \cdot 254.$$

Aus der Gleichung (6) des § 8 wird hier

$$\frac{1-q}{1-0 \cdot 631 q} = 0 \cdot 1524, \text{ also } q = 0 \cdot 9378, \quad p = 0 \cdot 7315,$$

$$h_1 = 4 \cdot 124, \quad h_2 = 0 \cdot 366, \quad h_3 = 0 \cdot 229, \quad h_4 = 0 \cdot 161 \text{ usw.}$$

Die h_i nehmen nur sehr langsam ab, was auf eine starke Ansteckung hindeutet. Man erhält

$$P_0 = 108 (1-q)^{\frac{p}{q^2}} e^{-h+\frac{p}{q}} = 0 \cdot 378,$$

und die weiteren in Spalte IV angegebenen Zahlen.

Das Pearsonsche χ^2 -Kriterium liefert hier wegen $\chi^2 = 8 \cdot 9$

$$P = 0 \cdot 9,$$

* "Sur la loi de Pólya régissant les faits corrélatifs," *Akt. Vědy*, III, 18, 49, 1932.

† x_2 und x_3 sind so gewählt worden, dass ihre Logarithmen auf volle Zehntel lauten, dass sich also ihre Potenzen leicht berechnen lassen.

d. h. die Anpassung an die Beobachtung ist als befriedigend anzusehen. Gleichwohl fällt auf, dass auch hier die theoretische maximale Häufigkeit hinter der beobachteten zurückbleibt.

Bei ansteckenden Krankheiten scheint die Verwendung von drei Parametern unter Umgehung der höheren Mittel zu brauchbaren Resultaten zu führen. Das gilt vor allem für Verteilungen mit einem grossen Mittelwert wie in dem vorliegenden Fall, wo $m = 15.8796$ ist.

TABELLE II.

Verteilung der weiblichen Diphteriefälle in Böhmen 1921 bis 1929.

I. Anzahl der Fälle	II. Beobacht. Verteil.	III. Verteil. nach Eggenberger	IV. Verteil. nach Formel II
0	1	4.7	0.4
1	1	5.1	1.6
2	3	5.2	3.4
3	4	5.1	5.1
4	7	5.0	6.2
5	6	4.8	6.5
6	9	4.6	6.4
7	10	4.4	5.9
8	7	4.1	5.5
9	5	3.9	5.0
10	3	3.7	4.5
11	4	3.5	4.2
12	1	3.3	3.8
13	3	3.1	3.5
14	3	3.0	3.2
15	2	2.8	3.0
16	39	41.8	39.8
und mehr			

§ 12. *Verteilung der Blutkörperchen.* In einem 1907 erschienenen Artikel der *Biometrika** wird die Poissonsche Formel benutzt, um die Verteilung der Blutkörperchen auf die 400 Quadrate eines Hämacytometers zu untersuchen. Falls die Blutkörperchen unabhängig von einander auf die Fläche verteilt sind, ist auch hier die Poissonsche Formel gültig.

Der Verfasser des erwähnten Artikels hat vier verschiedene Konzentrationen untersucht. In der ersten Konzentration ist die mittlere Anzahl der Blutkörperchen pro Quadrat

$$m = 0.6825.$$

Tabelle III enthält in Spalte II die Zahl der Quadrate, in denen die in Spalte I angegebene Anzahl von Blutkörperchen beobachtet worden ist. "Student" hat die Zahlen der Poissonschen Formel berechnet (Spalte III).

Die höheren Mittel werden angegeben als

$$\mu_2 = 0.8117, \quad \mu_3 = 1.0876.$$

* "Student," "On the Error of Counting with a Haemacytometer," *Biometrika*, Vol. v. p. 351, 1907.

Da die mittleren Fehler

$$\epsilon_m = 0.045, \quad \epsilon_{\mu_2} = 0.064, \quad \epsilon_{\mu_3} = 0.20$$

sind, kann man die Abweichungen der höheren Mittel von m nicht als zufällig ansehen. Aus diesem Grunde sind die Differenzen zwischen Theorie und Beobachtung recht gross. Der Verfasser gibt nämlich an

$$\chi^2 = 9.92 \quad \text{und} \quad P = 0.04,$$

wobei die Häufigkeiten für 4 und mehr Blutkörperchen zusammengefasst sind.

Zur Erklärung wird angenommen, dass die Blutkörperchen das Bestreben haben, zusammenzuhaften ("...there is also probably a tendency to stick together in groups which was not altogether abolished even by vigorous shaking"). Wenn diese Annahme richtig ist, muss Formel II mit der Beobachtung besser übereinstimmen. Man erhält

$$1 - q = 0.9366; \quad q = 0.0634, \quad p = 0.113;$$

$$h_1 = 0.562, \quad h_2 = 0.057, \quad h_3 = 0.0024, \quad h_4 = 0.0001 \text{ usw.}$$

Die Wahrscheinlichkeit für Gruppen von 4 oder mehr Blutkörperchen ist also verschwindend klein. Spalte IV der Tabelle erhält die nach der Formel II berechneten Werte, die hier auf ganze Zahlen abgerundet sind. Was die Güte der Anpassung angeht, so ist

$$\chi^2 = 3.33, \quad \text{d. h.} \quad P = 0.51.$$

Diese weit bessere Anpassung an die Erfahrung spricht für die erwähnte Gruppenbildung.

Formel I b liefert die Konstanten

$$h_1 = 0.562, \quad h_2 = 0.056, \quad h_3 = 0.0029,$$

die fast identisch mit denen der Formel II sind.

Die hier geschilderten Verhältnisse treffen nicht für alle untersuchten Konzentrationen zu. In einigen Fällen passt sich nämlich die Poissonsche Formel der Beobachtung gut an.

TABELLE III.

Verteilung der Blutkörperchen auf 400 Quadrate eines Hämacytometers.

I. Zahl der Blutkörperchen pro Quadrat	II. Beobacht. Verteil.	III. Verteil. nach Poisson	IV. Verteil. nach Formel II
0	213	202	215
1	128	138	121
2	37	47	46
3	18	11	14
4	3	2	3
5	1	0	1
und mehr			

§ 13. *Zahlen der Blumenkronblätter von Ranunculus bulbosus.* Zur Illustration der Poisson-Charlierschen B -Kurve sind von Riebesell* die Zahlen der Blumenkronblätter von *Ranunculus bulbosus* benutzt worden. Die mittlere Anzahl der Blumenkronblätter für die 222 untersuchten Exemplare ist 5·631, das Streuungsquadrat

$$\mu_2 = 0\cdot918.$$

Spalte II der Tabelle IV zeigt die beobachtete Häufigkeit der in Spalte I angegebenen Zahlen x der Blumenkronblätter.

In dem zitierten Artikel wird zunächst der Wert $x = 5$ als "Nullpunkt" der Verteilung gewählt, d. h. es ist

$$m = 0\cdot631.$$

Dann werden aus der Charlierschen Formel

$$P_2 = 222 \{ \psi(r) + c \cdot \Delta^2 \psi(r) \},$$

wo
$$\psi(r) = \frac{e^{-m} \cdot m^r}{r!},$$

$$c = \frac{\mu_2 - m}{2},$$

$$\Delta^2 \psi(r) = \psi(r) - 2\psi(r-1) + \psi(r-2)$$

sind, die in Spalte IV angeführten theoretischen Werte berechnet, wobei $r = x - 5$ ist. Spalte III gibt zum Vergleich die Poissonschen Zahlen an.

Nun ist aber
$$\mu_3 = 1\cdot644.$$

Die mittleren Fehler sind

$$\epsilon_m = 0\cdot064, \quad \epsilon_{\mu_2} = 0\cdot080, \quad \epsilon_{\mu_3} = 0\cdot25.$$

Die Abweichungen der Mittel von einander sind also grösser, als die mittleren Fehler zulassen.

Die Formel II liefert die folgenden Zahlen:

$$1 - q = 0\cdot7906; \quad q = 0\cdot2094, \quad p = 0\cdot1794,$$

$$h_1 = 0\cdot4037, \quad h_2 = 0\cdot0897, \quad h_3 = 0\cdot0125, \quad h_4 = 0\cdot00197 \text{ usw.}$$

Man erhält die theoretische Häufigkeit für das Vorkommen von 5 Blumenkronblättern:

$$P_0 = 222 (1 - q)^{\frac{p}{q^2}} \cdot e^{-h + \frac{p}{q}} = 133\cdot5,$$

und die weiteren in Spalte V angegebenen Zahlen.

Um die Güte der Anpassung zu prüfen, seien die Fälle von 9 und 10 Blumenkronblättern zusammengefasst. Für die Poissonsche Formel ist

$$\chi^2 = 18\cdot5,$$

d. h. wegen $n' = 5,$
$$P = 0\cdot001,$$

* Riebesell, "Biometrik und Variationsstatistik," *Handbuch der biologischen Arbeitsmethoden*, Abt. v. Teil 2, 1. Hälfte, Wien, 1928, S. 759.

die Formel ist unbrauchbar. Für die Charliersche Verteilung ist

$$\chi^2 = 0.99, \text{ d. h. } P = 0.91.$$

Für die Formel II ist sogar

$$\chi^2 = 0.25 \text{ und } P = 0.99;$$

diese Formel passt sich am besten der Erfahrung an, wobei jedoch zu beachten ist, dass drei Parameter zur Berechnung gebraucht worden sind. Es scheint, als ob sich statt ursprünglich *eines* Blumenkronblattes zwei oder mehr entwickeln können (vielleicht durch Zellteilung) und so das Schema der Formel II gültig ist.

TABELLE IV.

Zahlen der Blumenkronblätter von Ranunculus bulbosus.

I. Zahl der Blumenkron- blätter x	II. Beobacht. Häufigkeit*	III. Häufigkeit nach Poisson	IV. Häufigkeit nach Char- liers B -Kurve	V. Häufigkeit nach Formel II
5	133	118.2	134.9	133.5
6	55	74.5	51.6	53.9
7	23	23.5	22.5	22.9
8	7	4.9	9.5	8.0
9	2	0.8	2.9	2.6
10	2	0.1	0.6	0.8

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* De Vries, *Berichte der deutschen botanischen Gesellschaft*, Jahrg. 12, pp. 203—204, 1894.

ON CERTAIN NON-NORMAL SYMMETRICAL FREQUENCY DISTRIBUTIONS.

By G. H. HANSMANN, M.Sc. (Stellenbosch), B.A. (Oxon.).

INTRODUCTION.

IN 1889 Professor Karl Pearson introduced a series of curves which enabled in all cases a mathematical curve to be found which had the same first four moments as any given frequency distribution, thus providing for the more usual types of frequency an adequate description in mathematical terms of their distribution as judged by the test of goodness of fit. He reached these curves by integrating

$$\frac{1}{y} \frac{dy}{dx} = \frac{-(x+a)}{c_0 + c_1x + c_2x^2 + \dots c_nx^n},$$

retaining only the first three terms of the denominator on the right-hand side.

Beside the need to graduate data, there is another aspect of frequency curves. Mathematically—generally with certain limiting hypotheses—the distribution of certain frequency constants, such as the correlation coefficient, the regression coefficient, the variance, and “Student’s” z from normal distributions, has been found. In some of these cases the distributions are exact Pearson curves, in other cases these curves suffice when the sample reaches 25–30 individuals. In a further series of cases it is possible to determine the theoretical moment coefficients, although we cannot find the mathematical form of the curve. Examples are the distribution of the third moment and that of the ordinates of the regression line at a given abscissa, both from a normal population. Both curves are symmetrical and neither curve is exactly described by a Pearson curve with the same first four moments. Furthermore, in the case of the distribution of μ_3 , a $\beta_2 = 12.5$ has been found. Here the corresponding Pearson curve may give us a very bad fit, because the values* of β_4 and of β_6 , as determined from the difference equation, are negative when $\beta_2 > 6$ and $4.5 < \beta_2 < 6$ respectively.

We now ask: To what extent shall we better matters by taking into account higher moments? What are the types of curves to which higher moments lead us in such cases? Shall we be able to find a higher order symmetrical Pearson curve when $\beta_2 > 6$ and $4.5 < \beta_2 < 6$, for which β_4 and β_6 are respectively positive?

Dr David Heron has already discussed the third order Pearson curves which correspond to the differential equation

$$\frac{1}{y} \frac{dy}{dx} = \frac{-(x+a)}{c_0 + c_1x + c_2x^2 + c_3x^3},$$

$$* \quad \beta_4 = \frac{5\beta_2^3}{6 - \beta_2}, \quad \beta_6 = \frac{35\beta_2^3}{(6 - \beta_2)(9 - 2\beta_2)}.$$

130 *On certain Non-Normal Symmetrical Frequency Distributions*

but his manuscript* has remained unpublished because the addition of a single c_3 applied to actually occurring frequencies did not as a rule materially improve the goodness of fit.

The aim of the present paper is to include the following term in c_4 , but to consider only the symmetrical curves, for which a , c_1 , c_3 may be taken to be zero.

This leads us to the differential equation

$$\frac{1}{y} \frac{dy}{dx} = \frac{-x}{c_0 + c_2 x^2 + c_4 x^4} \dots\dots\dots(1).$$

It is to the discussion of the integrals of this equation and their applications that this paper is devoted.

I. THE DIFFERENT FOURTH ORDER PEARSON CURVES.

(a) *Discussion of Types.*

Without the loss of generality we can write the differential equation (1) as

$$\frac{1}{y} \frac{dy}{dx} = \frac{x/\sigma}{b_0 + b_2 \frac{x^2}{\sigma^2} + b_4 \frac{x^4}{\sigma^4}},$$

where the constants b_0 , b_2 and b_4 are now mere numerics.

Therefore

$$\frac{1}{y} \frac{dy}{dx} = \frac{x}{b_0 \sigma^2 + b_2 x^2 + b_4 \frac{x^4}{\sigma^2}} \dots\dots\dots(2).$$

Multiply both sides by $y x^{2s+1}$ and integrate with respect to x between the limits a_1 and a_2 . Divide by N .

$$\text{Hence} \quad \frac{1}{N} \int_{a_1}^{a_2} x^{2s+1} \left(b_0 \sigma^2 + b_2 x^2 + b_4 \frac{x^4}{\sigma^2} \right) dy = \frac{1}{N} \int_{a_1}^{a_2} x^{2s+2} y dx.$$

Integrate the left-hand side by parts. We find

$$\begin{aligned} & \frac{1}{N} x^{2s+1} \left\{ b_0 \sigma^2 + b_2 x^2 + \frac{b_4}{\sigma^2} x^4 \right\} y \Big|_{a_1}^{a_2} \\ & - \frac{1}{N} \int_{a_1}^{a_2} \left\{ (2s+1) b_0 \sigma^2 x^{2s} + (2s+3) b_2 x^{2s+2} + (2s+5) \frac{b_4}{\sigma^2} x^{2s+4} \right\} y dx \\ & = \frac{1}{N} \int_{a_1}^{a_2} x^{2s+2} y dx. \end{aligned}$$

Take the origin of x at the mean of the distribution. Now since the distribution is symmetrical the mean and the mode correspond. Hence the limits of integration are from $a_1 = -a$ to $a_2 = +a$ and the integral $\frac{1}{N} \int_{-a}^{+a} x^s y dx = \mu_s = s$ th moment about the mean of the distribution.

* I am very grateful to Prof. K. Pearson for drawing my attention to this manuscript.

Therefore

$$(2s+1)\mu_{2s}b_0\sigma^2 + (2s+3)b_2\mu_{2s+2} + (2s+5)\frac{b_4}{\sigma^2}\mu_{2s+4} = c_a - \mu_{2s+2},$$

where
$$c_a = \frac{2a^{2s+1}}{N} \left\{ b_0\sigma^2 + b_2a^2 + \frac{b_4}{\sigma^2}a^4 \right\} y_a \dots\dots\dots(3).$$

Now c_a vanishes if

- (i) a is zero; but a cannot vanish or there would be no range;
- (ii) a^2 is a root of $b_0\sigma^2 + b_2a^2 + \frac{b_4}{\sigma^2}a^4 = 0$ and if y_a is finite;
- (iii) y_a is zero and if $\left(b_0\sigma^2 + b_2a^2 + \frac{b_4}{\sigma^2}a^4\right)$ is zero or remains finite.

By means of (ii) and (iii) we are able to fix the limits of integration. Put $\frac{x}{\sigma} = u$ in the equation (2). We obtain

$$\frac{1}{y} \frac{dy}{du} = \frac{u}{b_0 + b_2u^2 + b_4u^4} \dots\dots\dots(4).$$

Now write $m_1 = \frac{1}{2b_4}$, $z = u^2 = \frac{x^2}{\sigma^2}$. We obtain

$$\frac{1}{y} \frac{dy}{dz} = \frac{m_1}{\frac{b_0}{b_4} + \frac{b_2}{b_4}z + z^2} \dots\dots\dots(5).$$

The expression $\left(\frac{b_0}{b_4} + \frac{b_2}{b_4}z + z^2\right)$ can assume any of the forms

- (1) $(p_1 + z)(q_1 + z)$, real and unequal factors;
- (2) $(b_1 + z)^2 + c_1^2$, imaginary factors;
- (3) $(p_1 + z)^2$, equal factors.

Let us integrate the differential equation (5) in these three cases and consider what happens to the expression for c_a .

$$\begin{aligned} (1) \quad \frac{1}{y} \frac{du}{dz} &= \frac{m_1}{(p_1 + z)(q_1 + z)} \\ &= \frac{m_1}{q_1 - p_1} \left(\frac{1}{p_1 + z} - \frac{1}{q_1 + z} \right). \end{aligned}$$

Integrating, we find

$$\log y = \text{const.} + \left(\frac{m_1}{q_1 - p_1} \right) \log \left(\frac{p_1 + z}{q_1 + z} \right),$$

therefore
$$y = y_0 \left[\frac{p_1 + z}{q_1 + z} \right]^{\left(\frac{m_1}{q_1 - p_1} \right)},$$

where y_0 is the constant of integration

132 On certain Non-Normal Symmetrical Frequency Distributions

Notation. I shall use letters with suffix $_1$ to represent the quantities m, q, p, k and k' , when they can be positive or negative and unsuffixed letters to represent their absolute values. Write

$$q = |q_1|, \quad p = |p_1|, \quad m = |m_1|,$$

$$k_1 = \frac{m_1}{q - p}, \quad k_1' = \frac{m_1}{q + p},$$

$$k = |k_1|, \quad k' = |k_1'|.$$

I shall always take $q > p$.

(i) p_1 and q_1 both positive. Hence writing $p = p_1$ and $q = q_1$ we find

$$c_a = \text{const.} \times a^{2s+1} (a^2 + p)^{k_1+1} (a^2 + q)^{1-k_1}.$$

Hence $c_a = 0$ if $a^2 = -p$, when we have an imaginary range, or if $a^2 = \infty$, provided in addition $(2s + 1) + 2(k_1 + 1) < 2(k_1 - 1)$, i.e. $2s < -5$. Hence c_a can only be zero for negative moments.

(ii) $p_1 < 0$ and $q_1 > 0$. Write $p = -p_1$ and $q = q_1$. Clearly the equation for y is now

$$y = y_0 \left(\frac{p - u^2}{q + u^2} \right)^{k_1'}.$$

Accordingly, $c_a = \text{const.} \times a^{2s+1} (p - a^2)^{k_1'+1} (q + a^2)^{1-k_1'}$ and becomes zero at $a = \pm \sqrt{p}$, provided $k_1' \geq -1$. If k_1' be < -1 , c_a can never be zero.

We get a similar discussion for $p_1 > 0$ and $q_1 < 0$.

(iii) $p_1 < 0$ and $q_1 < 0$. Write $p_1 = -p$ and $q_1 = -q$. The equation for y reduces to

$$y = y_0 \left(\frac{q - u^2}{p - u^2} \right)^{k_1}, \quad \text{where } k_1 = \frac{m_1}{q - p}.$$

We now find $c_a = \text{const.} \times [p - a^2]^{1-k_1} [q - a^2]^{k_1+1} a^{2s+1}$ and accordingly $c_a = 0$ at $a = \pm \sqrt{p}$, provided $k_1 < 1$, but can never be $= 0$ when $k_1 \geq 1$.

$$(2) \quad \frac{1}{y} \frac{dy}{du} = \frac{2m_1 u}{(u^2 + b_1)^2 + c_1^2}.$$

Integrating with respect to u^2 we obtain

$$\log y = \text{const.} + \frac{m_1}{c_1} \tan^{-1} \frac{b_1 + u^2}{c_1},$$

therefore

$$y = y_0 e^{\frac{m_1}{c_1} \tan^{-1} \frac{b_1 + u^2}{c_1}}.$$

$$\text{Hence } c_a = \text{const.} \times a^{2s+1} e^{\frac{m_1}{c_1} \tan^{-1} \frac{b_1 + a^2}{c_1}} [(a^2 + b_1)^2 + c_1^2].$$

Clearly c_a can only be zero if $a = \pm \infty$ and if in addition we have the condition $(2s + 5) < 0$ or $s < -\frac{5}{2}$. In other words, for positive moments c_a cannot be zero.

(3)

$$\frac{1}{y} \frac{dy}{du} = \frac{2m_1 u}{(p_1 + u^2)^2},$$

therefore

$$\log y = \text{const.} - \frac{m_1}{p_1 + u^2},$$

therefore

$$y = y_0 e^{-\frac{m_1}{p_1 + u^2}}.$$

Accordingly

$$c_a = \text{const.} \times a^{2s+1} e^{-\frac{m_1}{p_1 + a^2} (p_1 + a^2)^2}.$$

Hence $c_a = 0$ at $a^2 = -p_1$ provided $m_1 > 0$, but can never tend to zero as $a^2 \rightarrow \pm \infty$ when $s > 0$. In other words, of all the possible forms of $y_0 e^{-\frac{m_1}{p_1 + u^2}}$, the only equation which can represent a real frequency distribution is

$$y = y_0 e^{-\frac{m}{p - u^2}},$$

where we must take the range for u from $-\sqrt{p}$ to $+\sqrt{p}$. This solution corresponds to $m_1 < 0$, $\frac{b_0}{b_4} > 0$ and $\frac{b_2}{b_4} < 0$ in the differential equation (5).

In the present paper I shall confine myself only to those equations which satisfy the condition $c_a = 0$.

Consider the equation

$$y = y_0 \left(\frac{q - u^2}{p + u^2} \right)^{k'},$$

where

$$p = |p_1|,$$

$$q = |q_1|,$$

$$m = |m_1|,$$

$$k' = |k_1'| = \frac{m}{q + p}.$$

I shall take the limits of the range, viz. a , to be defined by the relation $c_a = 0$, and hence in the same way as above we find $a = \pm \sqrt{q}$. The corresponding differential equation is

$$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{pq - (q - p)u^2 + u^4}.$$

Compare this equation with the Equation (5). Hence

$$m_1 = m, \quad \text{and therefore } m_1 > 0,$$

$$\frac{b_0}{b_4} = pq, \quad \text{,,} \quad \text{,,} \quad \frac{b_0}{b_4} > 0,$$

$$\frac{b_2}{b_4} = -(q - p), \quad \text{,,} \quad \text{,,} \quad \frac{b_2}{b_4} < 0,$$

134 *On certain Non-Normal Symmetrical Frequency Distributions*

which gives us the criterion for this type of curve, which I shall call the Type (i) curve. In this way we find the following types of curves and their criterion :

TABLE I.

Type	Form of Curve	Range for u	Criterion		
			b -ratios	m_1	k or k'
(i)	$y = y_0 \left(\frac{q - u^2}{p + u^2} \right)^{k'}$	$-\sqrt{q}$ to $+\sqrt{q}$	$\frac{b_2}{b_4} < 0$ $\frac{b_0}{b_4} < 0$	$m_1 > 0$...
(ii)	$y = y_0 \left(\frac{p - u^2}{q + u^2} \right)^{k'}$	$-\sqrt{p}$ to $+\sqrt{p}$	$\frac{b_2}{b_4} > 0$ $\frac{b_0}{b_4} < 0$	$m_1 > 0$...
* (iii)	$y = y_0 \left(\frac{q - u^2}{p - u^2} \right)^k$	$-\sqrt{p}$ to $+\sqrt{p}$	$\frac{b_2}{b_4} < 0$ $\frac{b_0}{b_4} > 0$	$m_1 > 0$	$k < 1$
* (iv)	$y = y_0 \left(\frac{p + u^2}{q - u^2} \right)^k$	$-\sqrt{q}$ to $+\sqrt{q}$	$\frac{b_2}{b_4} < 0$ $\frac{b_0}{b_4} < 0$	$m_1 < 0$	$k' < 1$
* (v)	$y = y_0 \left(\frac{q + u^2}{p - u^2} \right)^k$	$-\sqrt{p}$ to $+\sqrt{p}$	$\frac{b_2}{b_4} > 0$ $\frac{b_0}{b_4} < 0$	$m_1 < 0$	$k' < 1$
(vi)	$y = y_0 \left(\frac{p - u^2}{q - u^2} \right)^k$	$-\sqrt{p}$ to $+\sqrt{p}$	$\frac{b_2}{b_4} < 0$ $\frac{b_0}{b_4} > 0$	$m_1 < 0$...
(vii)	$y = y_0 e^{-\frac{m}{p - u^2} +}$	$-\sqrt{p}$ to $+\sqrt{p}$	$\frac{b_2}{b_4} < 0$ $\frac{b_0}{b_4} > 0$	$m_1 < 0$	$\left(\frac{b_2}{b_4} \right)^2 = 4 \frac{b_0}{b_4}$

* The Types (iii), (iv) and (v) are u-shaped curves.

+ $m = |m_1|$.

Note that for the Types (i)—(vi) we must also have the condition $\left(\frac{b_2}{b_4} \right)^2 > 4 \frac{b_0}{b_4}$,

so that p and q may be real.

The general differential equation for the Types (i)—(vi) is

$$\frac{1}{y} \frac{dy}{du} = \frac{2m_1 u}{p_1 q_1 + (p_1 + q_1) u^2 + u^4}.$$

Compare this with the Equation (5).

Therefore .

$$\left. \begin{aligned} m_1 &= \frac{1}{2b_4} \\ p_1 q_1 &= \frac{b_0}{b_4} \\ p_1 + q_1 &= \frac{b_2}{b_4} \end{aligned} \right\} \dots\dots\dots (6).$$

Let us express the quantities b_0 , b_2 and b_4 , and therefore m_1 , $p_1 q_1$ and $p_1 + q_1$, in terms of β_2 , β_4 and β_6 .

Put $c_a = 0$ in (3) and the difference equation reduces to

$$(2s+1)\mu_{2s}b_0\mu_2 + (2s+3)b_2\mu_{2s+2} + (2s+5)\frac{b_4}{\mu_2}\mu_{2s+4} = -\mu_{2s+2}.$$

$$\text{Put } s=2: \quad b_0\mu_2 + 3b_2\mu_2 + 5\frac{b_4}{\mu_2^2}\mu_4 = -\mu_2,$$

$$s=1: \quad 3b_0\mu_2^2 + 5b_2\mu_4 + 7\frac{b_4}{\mu_2}\mu_6 = -\mu_4,$$

$$s=2: \quad 5b_0\mu_2\mu_4 + 7b_2\mu_6 + 9\frac{b_4}{\mu_2}\mu_8 = -\mu_6.$$

Divide the equations above by μ_2 , μ_2^2 , μ_2^3 respectively. Write

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \quad \beta_4 = \frac{\mu_6}{\mu_2^3} \quad \text{and} \quad \beta_6 = \frac{\mu_8}{\mu_2^4}.$$

We obtain

$$\left. \begin{aligned} b_0 + 3b_2 + 5b_4\beta_2 &= -1 \\ 3b_0 + 5b_2\beta_2 + 7b_4\beta_4 &= -\beta_2 \\ 5b_0\beta_2 + 7b_2\beta_4 + 9b_4\beta_6 &= -\beta_4 \end{aligned} \right\}.$$

Solving, for b_0 , b_2 and b_4 , we get

$$\begin{aligned} b_0 &= \frac{2\{14\beta_4^2 - 9\beta_2\beta_6 - 5\beta_2^2\beta_4\}}{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3} \\ b_2 &= \frac{\{9\beta_6(3 - \beta_2) + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3\}}{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3}, \dots\dots\dots (7). \\ b_4 &= \frac{2\{5\beta_2^2 - \beta_4(6 - \beta_2)\}}{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3} \end{aligned}$$

The solutions are valid, provided that

$$\Delta = 9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3$$

is not zero.

$$\text{If } \Delta = 0, \text{ we solve for } \frac{b_0}{b_4}, \frac{b_2}{b_4}, \frac{1}{b_4},$$

and find

$$\left. \begin{aligned} \frac{1}{b_4} &= \frac{5\beta_2^2 - (6 - \beta_2)\beta_4}{5\beta_2^2 - (6 - \beta_2)\beta_4} \\ \frac{b_2}{b_4} &= \frac{9\beta_6(3 - \beta_2) + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3}{2\{5\beta_2^2 - (6 - \beta_2)\beta_4\}} \\ \frac{1}{b_4} &= \frac{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3}{2\{5\beta_2^2 - (6 - \beta_2)\beta_4\}} \end{aligned} \right\} \dots\dots\dots (8).$$

136 *On certain Non-Normal Symmetrical Frequency Distributions*

Since $\Delta = 0$, i.e.

$$9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3 = 0,$$

$\frac{1}{b_4} = 0$, but $\frac{b_0}{b_4}$ and $\frac{b_2}{b_4}$ remain finite.

Hence our differential equation reduces to

$$\frac{1}{y} \frac{dy}{dx} = 0.$$

The integral is now $y = \text{constant}$, or the frequency curve is a rectangle, which I shall call the Type (viii) curve.

The solutions (8) above are valid only if $5\beta_2^2 - (6 - \beta_2)\beta_4 \neq 0$. If

$$5\beta_2^2 - (6 - \beta_2)\beta_4 = 0 \quad \text{and} \quad 9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3 = 0$$

are simultaneously true, i.e. if $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ and $\beta_6 = \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$, each of the expressions

$$5\beta_2^2 - (6 - \beta_2)\beta_4,$$

$$9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3,$$

$$14\beta_4^2 - 9\beta_2\beta_6 - 5\beta_2^2\beta_4,$$

$$9\beta_6(3 - \beta_2) + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3$$

becomes zero and we cannot find the values of b_1 , b_2 and b_4 , or the corresponding type of frequency distribution from the difference equation. In Section $h(\beta)$ I shall show that the frequency type corresponding to the values of

$$\left. \begin{aligned} \beta_4 &= \frac{5\beta_2^2}{6 - \beta_2} \\ \beta_6 &= \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} \end{aligned} \right\} \dots\dots\dots (9)$$

is a special form of the Type (vii) curve, viz. $y = y_0 e^{p - \frac{M}{x^2/\sigma^2}}$.

(b) *General Discussion of the Constants of the Curve.*

The following values were found :

$$p_1 q_1 = \frac{b_0}{b_4} = - \frac{2(9\beta_2\beta_6 - f_0)}{2K},$$

where $f_0 = 14\beta_4^2 - 5\beta_2^2\beta_4$ and $K = 5\beta_2^2 - (6 - \beta_2)\beta_4$,

$$(q_1 + p_1) = \frac{b_2}{b_4} = \frac{9\beta_6(3 - \beta_2) - f_2}{2K},$$

where $f_2 = -(7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3)$,

$$m_1 = \frac{1}{2b_4} = \frac{9\beta_6(5\beta_2 - 9) - f_4}{4K},$$

where $f_4 = 49\beta_4^2 - 210\beta_2\beta_4 + 125\beta_2^3$.

I have divided the curves into the different types according to

(i) the sign of $\frac{b_0}{b_4}$, $\frac{b_2}{b_4}$, and m_1 , hence the sign of p_1q_1 , $p_1 + q_1$, and m_1 ; in other words we want the signs of the expressions

$$\left. \begin{aligned} K &= 5\beta_2^2 - (6 - \beta_2)\beta_4 \\ F_0 &= 9\beta_2\beta_6 - f_0 \\ F_2 &= 9\beta_6(3 - \beta_2) - f_2 \\ F_4 &= 9\beta_6(5\beta_2 - 9) - f_4 \end{aligned} \right\} \dots\dots\dots(10);$$

(ii) whether p_1 and q_1 are equal, real or imaginary; hence we require the sign of

$$(q_1 - p_1)^2 = \frac{81\beta_6^2(3 - \beta_2)^2 - 18\beta_6b + c}{4K^2},$$

where $b = (3 - \beta_2)f_2 - 8\beta_2K$, $c = f_2^2 - 16f_0K$, and $L = 7\beta_4(\beta_2 - 3) + \beta_2^2(19 - 5\beta_2)$. On simplification

$$(q_1 - p_1)^2 = \frac{81(\beta_2 - 3)^2}{4K^2} \left\{ \beta_6 - \frac{b + 4|K|\sqrt{L}}{9(\beta_2 - 3)^2} \right\} \left\{ \beta_6 - \frac{b - 4|K|\sqrt{L}}{9(\beta_2 - 3)^2} \right\}.$$

I shall write this equation as

$$F_6 \equiv \{F_6'\} \{F_6''\} \frac{81(\beta_2 - 3)^2}{4K^2},$$

where
$$F_6' = \beta_6 - \frac{b + 4|K|\sqrt{L}}{9(3 - \beta_2)^2}, \quad F_6'' = \beta_6 - \frac{b - 4|K|\sqrt{L}}{9(3 - \beta_2)^2};$$

(iii) whether k or $k' \geq 1$, hence we require the sign of $[m_1 - (q_1 - p_1)]$. The zone where $c_a \neq 0$, where p_1q_1 , $p_1 + q_1$, and m_1 have other values than those we found, is bounded by $F_6 = 0$ and $m_1 = (q_1 - p_1)$. We therefore want to write the condition $[m_1 - (q_1 - p_1)] \geq 0$ in a workable form. To avoid the difficulty of the square root and also of sign, I shall consider $F_8 = m_1^2 - (q_1 - p_1)^2$.

On simplification we find:

$$\begin{aligned} F_8 &= \frac{243\beta_6^2(\beta_2 - 1)(7\beta_2 - 15) - 18\beta_6B + C}{16K^2} \\ &= \frac{243(\beta_2 - 1)(7\beta_2 - 15)}{16K^2} \{F_8'\} \{F_8''\}, \end{aligned}$$

where
$$F_8' = \beta_6 - \frac{B + 4|K|\sqrt{Q}}{27(\beta_2 - 1)(7\beta_2 - 15)},$$

$$F_8'' = \beta_6 - \frac{B - 4|K|\sqrt{Q}}{27(\beta_2 - 1)(7\beta_2 - 15)},$$

$$B = (5\beta_2 - 9)f_4 - 4b,$$

$$C = f_4^2 - 4c,$$

$$Q = 49\beta_4^2 - 14\beta_4(37\beta_2 - 30) + 7\beta_2^2(60\beta_2 - 53).$$

Hence $F_x = 0$ ($x = 0, 2, 4$) and $F_y = 0$ ($y = 6, 8$) represent surfaces of revolution in the $\beta_2, \beta_4, \beta_6$ space such that sections by the plane $\beta_2 = \text{constant}$ of $F_x = 0$ represent parabolas and of $F_y = 0$ two intersecting parabolas.

138 On certain Non-Normal Symmetrical Frequency Distributions

The position of any point $(\beta_2, \beta_4, \beta_6)$ with respect to these surfaces of revolution determines the signs of the expressions F_x and F_y , and the signs of p_1q_1 , $p_1 + q_1$, m_1 , $(q_1 - p_1)^2$, and $[m_1^2 - (q_1 - p_1)^2]$. Hence, theoretically, a model of these surfaces of revolution will enable us to determine what type of curve corresponds to a given set of values of β_2 , β_4 and β_6 .

(c) The Transition Types.

From the previous Section it is clear that when β_2 , β_4 and β_6 satisfy $F_x = 0$ ($x = 0, 2, 4, 6, 8$) either b_0 , b_2 or b_4 vanishes or there is a relation between these b 's, and the general differential equation is simplified. To each variation in the form of the differential equation there corresponds an integral. Here again some of the types so obtained are not admissible because we are confining ourselves to those types of frequency curves which satisfy $c_a = 0$. The discussion is similar to that given in Section (a) and is omitted here for the sake of brevity. The results are summarised below in Table II.

Notes. (1) When $\beta_4 = \frac{5\beta_2^2}{6-\beta_2}$, $b_4 = 0$ and the general differential equation reduces to

$$\frac{1}{y} \frac{dy}{du} = \frac{u}{b_0 + b_2 u^2}.$$

This is the equation Professor Pearson integrates to obtain his symmetrical types. Let us see if by proceeding to the limit as $\beta_4 \rightarrow \frac{5\beta_2^2}{6-\beta_2}$ we get the same values as he deduces for his curve.

$$b_2 = \frac{9\beta_6(3-\beta_2) - f_2}{9\beta_6(5\beta_2-9) - f_4} = \frac{9\beta_6(3-\beta_2) + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3}{9\beta_6(5\beta_2-9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3},$$

therefore

$$\lim_{\beta_4 \rightarrow \frac{5\beta_2^2}{6-\beta_2}} b_2 = \frac{\frac{3-\beta_2}{(6-\beta_2)^2} \{9\beta_6(6-\beta_2)^2 - 25\beta_2^3(\beta_2+8)\}}{\frac{5\beta_2-9}{(6-\beta_2)^2} \{9\beta_6(6-\beta_2)^2 - 25\beta_2^3(\beta_2+8)\}}$$

$$= \frac{3-\beta_2}{5\beta_2-9}, \quad \left(\beta_6 \neq \frac{25\beta_2^3(\beta_2+8)}{9(6-\beta_2)^2} \right).$$

Similarly

$$\lim_{\beta_4 \rightarrow \frac{5\beta_2^2}{6-\beta_2}} b_0 = -\frac{2\beta_2}{5\beta_2-9}.$$

It would seem as if we should have Professor Pearson's types at every point along the curve $\beta_4 = \frac{5\beta_2^2}{6-\beta_2}$, no matter what the value of β_6 is. But this is not so. When $\beta_4 = \frac{5\beta_2^2}{6-\beta_2}$, $b_4 = 0$, b_2 and b_0 can be expressed in terms of β_2 and are fixed for constant β_2 . We get a relation between β_6 , β_4 and β_2 , and find

$$\beta_6 = \frac{35\beta_2^3}{(6-\beta_2)(9-2\beta_2)}.$$

TABLE II. Table showing the various Types of Frequency Curves.

	Type	Equation of Curve	Limits of the Range	Differential Equation	Criterion					Remarks
					m_1	$p_1 q_1$	$q_1 + p_1$	$(q_1 - p_1)^2$	$m_1^2 - (q_1 - p_1)^2$	
Main Types	(i)	$y = y_0 \left[\frac{q - u^2}{p + u^2} \right]^k$	$\pm \sqrt{q}$	$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{-pq - (q-p)u^2 + u^4}$	+	-	-	+	+ or -	
	(ii)	$y = y_0 \left[\frac{p - u^2}{q + u^2} \right]^k$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{-pq + (q-p)u^2 + u^4}$	+	-	+	+	+ or -	
	(iii)	$y = y_0 \left[\frac{q - u^2}{p - u^2} \right]^k$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{pq - (q+p)u^2 + u^4}$	+	+	-	+	-	
	(iv)	$y = y_0 \left[\frac{p + u^2}{q - u^2} \right]^k$	$\pm \sqrt{q}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{-pq - (q-p)u^2 + u^4}$	-	-	-	+	-	
	(v)	$y = y_0 \left[\frac{q + u^2}{p - u^2} \right]^k$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{-pq + (q-p)u^2 + u^4}$	-	-	+	+	-	
	(vi)	$y = y_0 \left[\frac{p - u^2}{q - u^2} \right]^k$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{pq - (q+p)u^2 + u^4}$	-	+	-	+	+ or -	
	(vii)	$y = y_0 e^{-\frac{m}{p-u^2}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{p^2 - 2p + u^4}$	-	+	-	0	+ or -	β_2, β_4 and β_6 satisfy $F_6 = 0$
	(viii)	$y = y_0 = \text{const.}$...	$\frac{1}{y} \frac{dy}{du} = 0$	For all values of p_1 and q_1 provided $p_1 q_1, p_1 + q_1$ and m_1 are not simultaneously zero. β_2, β_4 and β_6 satisfy $F_4 = 0$ or simultaneously $F_4 = 0$ and $F_2 = 0, F_4 = 0$ or $F_6 = 0, F_4 = 0$ and $F_8 = 0$.					
Transition Types	(ix)	$y = y_0 \left[\frac{p - u^2}{p + u^2} \right]^{\frac{m}{2p}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{-p^2 + u^4}$	+	-	0	+	+ or -	β_2, β_4 and β_6 satisfy $F_2 = 0$
	(x)	$y = y_0 \left[\frac{p + u^2}{p - u^2} \right]^{\frac{m}{2p}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{-p^2 + u^4}$	-	-	0	+	-	
	(xi)	$y = y_0 \left[\frac{q - u^2}{u^2} \right]^{\frac{m}{q}}$	$\pm \sqrt{q}$	$\frac{1}{y} \frac{dy}{du} = \frac{+2mu}{-qu^2 + u^4}$	+	0	-	+	$m < q$	β_2, β_4 and β_6 satisfy $F_0 = 0$
	(xii)	$y = y_0 \left[\frac{u^2}{q - u^2} \right]^{\frac{m}{q}}$	$\pm \sqrt{q}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{-qu^2 + u^4}$	-	0	-	+	$m < q$	
	(xiii)	$y = y_0 \left[\frac{q - u^2}{p + u^2} \right]^{\frac{m}{2p}}$	$\pm \sqrt{q}$	$\frac{1}{y} \frac{dy}{du} = \frac{2(p+q)u}{-pq - u^2(q-p) + u^4}$	+	-	-	+	0	β_2, β_4 and β_6 satisfy $F_8 = 0$
	(xiv)	$y = y_0 \left[\frac{p - u^2}{q + u^2} \right]^{\frac{m}{2p}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{2(p+q)u}{-qp + (q-p)u^2 + u^4}$	+	-	+	+	0	
	(xv)	$y = y_0 \left[\frac{p - u^2}{q - u^2} \right]^{\frac{m}{2p}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{-2(q-p)u}{pq - (p+q)u^2 + u^4}$	-	+	-	+	0	
	(xvi)	$y = y_0 (p^2 - u^2)^m$	$\pm p$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{p^2 - u^2}$	$\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ $\beta_6 = \frac{35\beta_2^3}{(9 - 2\beta_2)(6 - \beta_2)}$					Type II-curve
	(xvii)	$y = y_0 (p^2 + u^2)^{-m}$	$\pm \infty$	$\frac{1}{y} \frac{dy}{du} = \frac{-2mu}{p^2 + u^2}$						Type VII-curve
	(xviii)	$y = y_0 (p^2 - u^2)^{-m}$	$\pm p$	$\frac{1}{y} \frac{dy}{du} = \frac{2mu}{p^2 - u^2}$						Type II (u-shaped)
	(xix)	$y = y_0 e^{-\frac{1}{u^2}}$	$\pm \infty$	$\frac{1}{y} \frac{dy}{du} = -u$	$\beta_2 = 3, \beta_4 = 15, \beta_6 = 105$					Normal curve
	(xx)	$y = y_0 \left[\frac{p - u^2}{p + u^2} \right]^{\frac{4pu}{p^2 - u^2}}$	$\pm \sqrt{p}$	$\frac{1}{y} \frac{dy}{du} = \frac{4pu}{p^2 - u^2}$	-	-	0	+	0	β_2, β_4 and β_6 simultaneously satisfy $F_4 = 0$ and $F_8 = 0$

$$m = |m_1|, \quad p = |p_1|, \quad q = |q_1|, \quad k = \frac{m}{q-p}, \quad k' = \frac{m}{q+p}.$$

Hence there is only one point on the curve $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ for each β_2 which corresponds to Professor Pearson's symmetrical curves. At other points we can have other types only.

The normal curve is a special case of the 2nd order symmetrical curves and corresponds to the point $\beta_2 = 3$, $\beta_4 = 15$ and $\beta_6 = 105$.

(2) Consider the equation $y = y_0 e^{\frac{m_1}{u^2}}$.

This equation is obtained by integrating the differential equation

$$\frac{1}{y} \frac{dy}{du} = \frac{2m_1 u}{u^4} = \frac{2m_1}{u^3}.$$

Hence it is the type we find along the curves of section of the surfaces

(i) $F_0 = 0$ and $F_2 = 0$, (ii) $F_0 = 0$ and $F_6 = 0$, (iii) $F_2 = 0$ and $F_6 = 0$.

Note that since $F_0 = p_1 q_1$, $F_2 = q_1 + p_1$, $F_6 = (q_1 - p_1)^2$, the surfaces $F_0 = 0$, $F_2 = 0$ and $F_6 = 0$ in general cut each other where $p_1 = 0 = q_1$, and hence we have a common curve of intersection.

For the equation above

$$c_a = \text{const.} \times a^{2a+4} e^{\pm \frac{m\sigma^2}{a^2}},$$

which is never zero and accordingly the equation

$$y = y_0 e^{\pm \frac{m\sigma^2}{x^2}}$$

does not represent a complete frequency distribution.

(3) We can write the Type (xiii) curve as

$$y = y_0 \left[\frac{(q+p)\sigma^2}{p\sigma^2 + x^2} - 1 \right], \text{ i.e. } y + y_0 = y_0 (q+p) \sigma^2 \frac{1}{p\sigma^2 + x^2}.$$

Hence the Type (xiii) curve is the part above the line $y = y_0$ of Pearson's Type VII curve $y = y_0 \frac{(q+p)\sigma^2}{p\sigma^2 + x^2}$. Similarly the Types (xiv) and (xv) represent the parts above the line $y = y_0$ of the curve $y = y_0 \frac{(p+q)\sigma^2}{q\sigma^2 + x^2}$ and the inverted u -shaped curve

$$y = y_0 \frac{(q-p)\sigma^2}{q\sigma^2 - x^2}.$$

(d) and (e). *The Inequalities satisfied by β_2 , β_4 and β_6 ; Note on the Surfaces $F_0 = 0$ and $F_2 = 0$.*

Before we can discuss the intersections of the transition surfaces with each other we must know something about the relations between β_2 , β_4 and β_6 , and also when the branches of $F_0 = 0$ and $F_2 = 0$ become imaginary.

In this way we introduce the further conditions, viz.

$$(i) \beta_6 > \beta_2 \beta_4 > \beta_2^3 \text{ and } \beta_4 > \beta_2^{2*},$$

and

$$(ii) L > 0, Q > 0.$$

It is easy to show that

$$I. (i) \text{ for } \beta_2 > 3, F_6' = 0 \text{ and } F_6'' = 0 \text{ cut at } \beta_4 = \frac{5\beta_2^2}{6 - \beta_2} \text{ and are always real ;}$$

$$(ii) \text{ for } \beta_2 < 3, F_6' = 0 \text{ and } F_6'' = 0 \text{ cut at } \beta_4 = \frac{5\beta_2^2}{6 - \beta_2} \text{ and again at}$$

$$\beta_4 = \frac{\beta_2^2(19 - 5\beta_2)}{7(3 - \beta_2)}.$$

For values of $\beta_4 > \frac{\beta_2^2(19 - 5\beta_2)}{7(3 - \beta_2)}$ the branches $F_6' = 0$ and $F_6'' = 0$ become imaginary.

$$II. (i) \text{ for } \beta_2 > \frac{15}{7}, F_8' = 0 \text{ and } F_8'' = 0 \text{ cut at } \beta_4 = \frac{5\beta_2^2}{6 - \beta_2} \text{ and are always real ;}$$

(ii) for $\beta_2 < \frac{15}{7}$, $F_8' = 0$ and $F_8'' = 0$ cut at $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$, $\beta_4 = d'$, $\beta_4 = d''$, the curves becoming imaginary for values of β_4 between d' and d'' , where

$$d' = \frac{1}{7}[(37\beta_2 - 30) + 2(\beta_2 - 1)\sqrt{15(15 - 7\beta_2)}],$$

$$d'' = \frac{1}{7}[(37\beta_2 - 30) - 2(\beta_2 - 1)\sqrt{15(15 - 7\beta_2)}].$$

(f) *The Intersections of the Transition Surfaces with each other.*

Let ϕ_x denote the value of β_6 on the surface $F_x = 0$. Then by finding the values of β_4 in terms of β_2 , which make $(\phi_r - \phi_s) = 0$, ($r = 0, 2, 4, 6, 8$), and by substitution in $F_r = 0$ or $F_s = 0$ the value of β_6 , also expressed in terms of β_2 , we can determine the curve of section of the surfaces $F_r = 0$ and $F_s = 0$.

We have found

$$\phi_0 = \frac{f_0}{9\beta_2} = (14\beta_4^2 - 5\beta_2^2\beta_4)/9\beta_2,$$

$$\phi_2 = \frac{f_2}{9(3 - \beta_2)} = -\{7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3\}/9(3 - \beta_2),$$

$$\phi_4 = \frac{f_4}{9(5\beta_2 - 9)} = \{49\beta_4^2 - 210\beta_2\beta_4 + 125\beta_2^3\}/9(5\beta_2 - 9),$$

$$\phi_6'' = \frac{b - 4|K|\sqrt{L}}{9(3 - \beta_2)^2},$$

$$\phi_8'' = \frac{B - 4|K|\sqrt{Q}}{27(\beta_2 - 1)(7\beta_2 - 15)},$$

* J. Shohat, *Biometrika*, Vol. xxi. For a discontinuous distribution the proof is straightforward and similar to that given by the Editorial Note on Shohat's memoir.

142 On certain Non-Normal Symmetrical Frequency Distributions

where

$$K = 5\beta_2^2 - (6 - \beta_2)\beta_4,$$

$$b = (3 - \beta_2)f_2 - 8\beta_2K,$$

$$L = \beta_2^2(19 - 5\beta_2) - 7(3 - \beta_2)\beta_4,$$

$$B = (5\beta_2 - 9)f_4 - 4b,$$

$$Q = 49\beta_4^2 - 14\beta_4(37\beta_2 - 30) + 7\beta_2^2(60\beta_2 - 53).$$

(a) The following table gives the values of β_4 and β_6 , in terms of β_2 , at the intersection of F_0 , F_2 and $F_4 = 0$ with each other:

TABLE III. Values of β_4 and β_6 .

Surfaces	$\phi_r - \phi_s$	Values of β_4	Values of β_6
$F_0=0$ and $F_4=0$	$\frac{(21\beta_4 - 25\beta_2^2)(5\beta_2^2 - 6 - \beta_2\beta_4)}{9\beta_2(5\beta_2 - 9)}$	$\frac{5\beta_2^2}{6 - \beta_2}$ $\frac{25\beta_2^2}{21}$	$\frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$ $\frac{125\beta_2^3}{81}$
$F_2=0$ and $F_4=0$	$\frac{2(7\beta_4 - 15\beta_2)(5\beta_2^2 - 6 - \beta_2\beta_4)}{9(3 - \beta_2)(5\beta_2 - 9)}$	$\frac{5\beta_2^2}{6 - \beta_2}$ $\frac{15\beta_2}{7}$	$\frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$ $\frac{25\beta_2^3}{9}$
$F_2=0$ and $F_0=0$	$\frac{(7\beta_4 - 5\beta_2^2)(5\beta_2^2 - 6 - \beta_2\beta_4)}{9\beta_2(3 - \beta_2)}$	$\frac{5\beta_2^2}{6 - \beta_2} *$	$\frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$

(b) The discussion of the values of β_4 and β_6 , in terms of β_2 , at the intersections of $F_x = 0$ and $F_y = 0$ $\left(\begin{matrix} x=0, 2, 4 \\ y=6, 8 \end{matrix} \right)$ is more complicated. We know that all these surfaces have a common curve of section, viz. $\beta_2 = \beta_2$, $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$, $\beta_6 = \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$ and this enables us to simplify the lengthy algebra since $(\phi_r - \phi_s) \left(\begin{matrix} r \\ s=0, 2, 4, 6, 8 \end{matrix} \right)$ must have $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ as a factor. Furthermore, the surfaces $F_4 = 0$, $F_6 = 0$ and $F_8 = 0$ have a common curve of section, which by considering the expression for $(\phi_6 - \phi_4)$ is found to be

$$\beta_2 = \beta_2, \quad \beta_4 = \frac{1}{7} \{ 9(5\beta_2 - 6) \pm 2(9 - 5\beta_2)^{\frac{1}{2}} \}, \quad \beta_6 = \frac{1}{3} \{ \pm 8(9 - 5\beta_2)^{\frac{1}{2}} + (180\beta_2 - 25\beta_2^2 - 216) \}.$$

Similarly the curve of section of $F_8 = 0$ and $F_0 = 0$ is

$$\beta_2 = \beta_2, \quad \beta_4 = \frac{15\beta_2^2}{7}, \quad \beta_6 = \frac{125\beta_2^3}{21},$$

and finally that of $F_8 = 0$ and $F_2 = 0$

$$\beta_2 = \beta_2, \quad \beta_4 = \frac{1}{7} \{ (7\beta_2 + 24) + 4|\beta_2 - 3|\sqrt{5\beta_2 + 4} \},$$

$$\beta_6 = \frac{1}{21} \{ (85\beta_2^2 + 16\beta_2 - 128) + 16(3\beta_2 - 4)(5\beta_2 + 4)^{\frac{1}{2}} \}.$$

* Note $\beta_4 > \beta_2^2$ and therefore $7\beta_4 > 5\beta_2^2$ for all values of β_2 and β_4 .

The numerators of the corresponding expressions $(\phi_r - \phi_s)$ each contain a square root and in obtaining the results above we may have introduced inadmissible values by squaring; furthermore the values of β_4 and β_6 must satisfy the inequalities

$$(i) \beta_6 > \beta_2 \beta_4 > \beta_2^3, \quad \beta_4 > \beta_2^2,$$

$$(ii) L > 0 \text{ and } Q > 0.$$

After some very lengthy algebra, I found for which values of β_2 the above solutions are admissible and interpreted the + and - signs of the square roots above. This discussion is given in the original manuscript*.

(g) *On the Division of the $\beta_2, \beta_4, \beta_6$ Space into Zones corresponding to the different Types.*

In the previous Section the value of the expressions $(\phi_r - \phi_s)$ was found and by discussing their sign we are able to find for which values of β_2 and β_4 , $\phi_r > \text{or} < \phi_s$. This enables us to determine the relative positions of $F_r = 0$ and $F_s = 0$ for any value of β_2, β_4 and β_6 and also to discuss theoretically the curves of section of the transition surfaces corresponding to any given value of β_2 . The introduction of the surfaces $F_6 = 0$ and $F_8 = 0$ makes the classification of the different systems of curves of section very complicated and I have confined myself to the surfaces F_0, F_2 and $F_4 = 0$. Their curves of section can be divided into three main classes, according as $\beta_2 < 1.8$, $1.8 < \beta_2 < 3$, $\beta_2 > 3$, as shown in Diagram I, Figs. I, II, and III. The types of curve in the various zones are found by discussing the signs of F_0, F_2 and F_4 for the corresponding values of β_2 and β_4 . These figures were of great use in the construction of Diagrams XII to XX, serving not only as a check, but also enabling us to determine very rapidly the types of curve corresponding to the different zones.

Note. The curve $F_2 = 0$ in Diagram I, Fig. III, has now no meaning because points on it do not satisfy the condition $c_a = 0$. I have reproduced it for the sake of uniformity.

(h) *The Constants for the main Types.*

The constants for the main types of frequency curves are expressed in terms of β_2, β_4 and β_6 , viz.

$$m_1 = \frac{9}{4K} (5\beta_2 - 9) (\beta_6 - \phi_4),$$

$$q_1 p_1 = -\frac{9\beta_2}{K} (\beta_6 - \phi_0),$$

$$q_1 + p_1 = \frac{9(3 - \beta_2)}{2K} (\beta_6 - \phi_2),$$

$$\frac{m_1}{q_1 - p_1} = \pm \sqrt{1 + \frac{m_1^2 - (q_1 - p_1)^2}{(q_1 - p_1)^2}}$$

$$= \pm \sqrt{1 + \frac{3(\beta_2 - 1)(7\beta_2 - 15)(\beta_6 - \phi_6')(\beta_6 - \phi_6'')}{4(\beta_2 - 3)^2(\beta_6 - \phi_6')(\beta_6 - \phi_6'')}}.$$

* Copy in the Library, University of London.

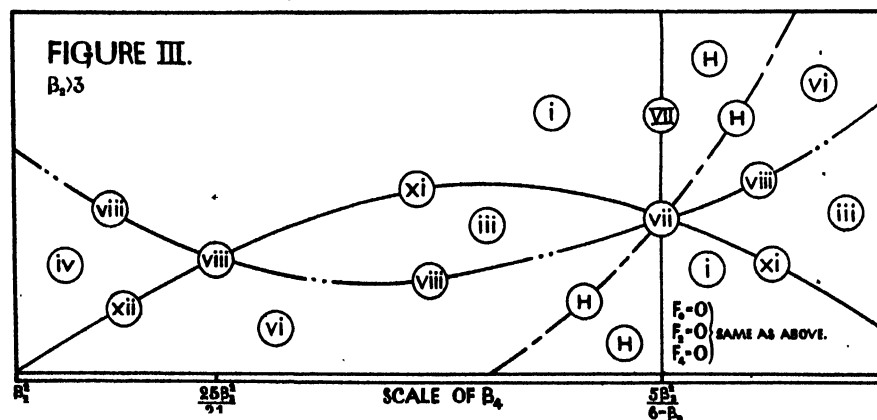
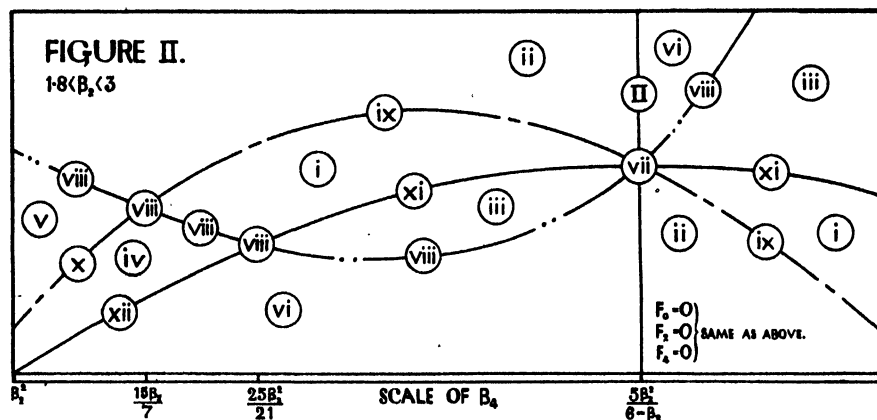
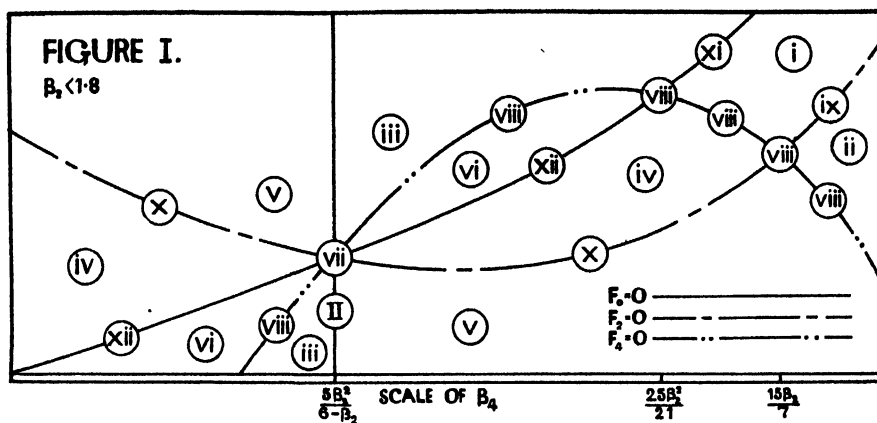


Diagram I. Figs. I, II and III.

The last formula which gives $k = \frac{m}{q-p}$ and also $k' = \frac{m}{q+p}$ is too complicated to be of any use in practice. It is far simpler to find m_1 and $(q_1 - p_1)$ from the formulae (6) and (7) and then $\frac{m_1}{q_1 - p_1}$ by division.

Let us now find the constants of integration of each curve. Let the total frequency be N , $f(x)$ the frequency curve which is symmetrical, $-b$ and $+b$ the limits of frequency, then

$$N = \int_{-b}^{+b} f(x) dx = 2 \int_0^{+b} f(x) dx.$$

By expanding $f(x)$, we easily find the following expressions for y_0 :

TABLE IV. *Values of y_0 .*

Type	Integral for N	Expression for y_0
(i)	$N = 2y_0 \int_0^{\sqrt{q}\sigma} \left(\frac{1 - \frac{x^2}{q\sigma^2}}{1 + \frac{x^2}{p\sigma^2}} \right)^{k'} dx$	$y_0 = \frac{N}{\sigma\sqrt{p}} / \left\{ \sum_{r=0}^{\infty} (-1)^r \left(\frac{p}{q} \right)^r \frac{k'!}{(k'-r)! r!} I_r \right\}^*$
(ii)	$N = 2y_0 \int_0^{\sigma\sqrt{p}} \left(\frac{1 - \frac{x^2}{p\sigma^2}}{1 + \frac{x^2}{q\sigma^2}} \right)^{k'} dx$	$y_0 = \frac{N}{\sigma\sqrt{p}} \frac{\Gamma(\frac{3}{2} + k')}{\sqrt{\pi} \Gamma(1 + k') F(k', \frac{1}{2}, k' + \frac{3}{2}, -p/q)}$
(iii)	$N = 2y_0 \int_0^{\sigma\sqrt{p}} \left(\frac{1 - \frac{x^2}{q\sigma^2}}{1 - \frac{x^2}{p\sigma^2}} \right)^k dx$	$y_0 = \frac{N}{\sigma\sqrt{p}} \frac{\Gamma(\frac{3}{2} - k)}{\sqrt{\pi} \Gamma(1 - k) F(-k, \frac{1}{2}, \frac{3}{2} - k, p/q)}$
(iv)	$N = 2y_0 \int_0^{\sigma\sqrt{q}} \left(\frac{1 + \frac{x^2}{p\sigma^2}}{1 - \frac{x^2}{q\sigma^2}} \right)^{k'} dx$	$y_0 = \frac{N\sqrt{q} \left(\frac{p}{q} \right)^{k'}}{\sigma(p+q)} \frac{1}{I'}^{\dagger}$
(v)	$N = 2y_0 \int_0^{\sigma\sqrt{p}} \left(\frac{1 + \frac{x^2}{q\sigma^2}}{1 - \frac{x^2}{p\sigma^2}} \right)^{k'} dx$	$y_0 = \frac{N}{\sigma\sqrt{p}} \frac{\Gamma(\frac{3}{2} - k')}{\sqrt{\pi} \Gamma(1 - k') F(-k', \frac{1}{2}, \frac{3}{2} - k', -p/q)}$
(vi)	$N = 2y_0 \int_0^{\sigma\sqrt{p}} \left(\frac{1 - \frac{x^2}{p\sigma^2}}{1 - \frac{x^2}{q\sigma^2}} \right)^k dx$	$y_0 = \frac{N}{\sigma\sqrt{p}} \frac{\Gamma(\frac{3}{2} + k)}{\sqrt{\pi} \Gamma(1 + k) F(k, \frac{1}{2}, \frac{3}{2} + k, p/q)}$

* Reduction formula is

$$I_r = -\frac{l^{r-\frac{1}{2}}(1-l)^{k'-r-\frac{1}{2}}}{k'-r-\frac{1}{2}} + \frac{r-\frac{1}{2}}{k'-r-\frac{1}{2}} I_{r-1}, \quad I_0 = \beta_1(\frac{1}{2}, k' - \frac{1}{2}) \text{ and } l = \frac{q}{p+q}.$$

†

$$I' = I'_0 + \frac{1}{2} \frac{p}{q} I'_1 + \frac{1}{2} \cdot \frac{8}{2} \cdot \frac{1}{2} \frac{p^2}{q^2} I'_2 + \dots,$$

where

$$I'_0 = B_{q/q+p}(1-k', k' + \frac{1}{2}), \text{ an incomplete B-function, } I'_r = -\frac{l^{r-k'}(1-l)^{k'+\frac{1}{2}-r}}{k'+\frac{1}{2}-r} + \frac{r-k'}{k'+\frac{1}{2}-r} I'_{r-1}.$$

(For continuation, see Table VI, p. 150.)

The series and the hypergeometrical functions converge rapidly when p/q is small. In other cases it is quicker to use quadrature formula.

The Transition Curves.

There now exists a relation between β_2 , β_4 and β_6 ; by substituting for β_6 in terms of β_2 and β_4 we are able to express the constants of the curve in terms of β_2 and β_4 and thus explain the ambiguity of sign which we find when working from the difference equations. This was necessary

(1) to discuss whether a given point on any of the transition surfaces corresponds to a fourth order Pearson curve consistent with our hypothesis;

(2) to determine the type of curve at all points on the different branches of $F_6 = 0$.

The discussion is long and summarised to some extent in Table V on p. 147. I have excluded from the table the Type (viii) or rectangular distribution, the second order Pearson Curves and the Normal Curve which for convenience I denoted by Types (xvi), (xvii), (xviii) and (xix).

It is interesting to find the values of the constants, expressed in terms of β_2 and β_4 , directly from the corresponding difference equation which can be found from first principles or more simply from the general equation (1) on p. 130 by making the appropriate change.

Let us now find the type of curve corresponding to the point I whose ordinates are $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$, $\beta_6 = \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}$. Let us approach the point I along either

$$F_6' = 0 \text{ or } F_6'' = 0,$$

i.e.

$$\begin{aligned} p &= \frac{-8\beta_2 \cdot K \pm 4|K| \cdot \sqrt{L}}{4(\beta_2 - 3) \cdot K} \\ &= \frac{-8\beta_2 \cdot K + 4\sqrt{K^2} \cdot L}{4K(\beta_2 - 3)} \\ &= \frac{-2\beta_2 + \sqrt{L(\beta_4)}}{4(\beta_2 - 3)}, \end{aligned}$$

and hence putting $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ and simplifying,

$$\frac{-2\beta_2 + \beta_2 \sqrt{\frac{(\beta_2 - 1)(5\beta_2 - 9)}{6 - \beta_2}}}{\beta_2 - 3}$$

Similarly the value of m when $\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}$ is ...(11)

$$M = \frac{(5\beta_2 - 9)\beta_2 \sqrt{\beta_2 - 1}}{(6 - \beta_2)(\beta_2 - 3)^2} [3\sqrt{\beta_2 - 1} - \sqrt{(5\beta_2 - 9)(6 - \beta_2)}],$$

on simplifying.

TABLE V. Values of Constants in terms of β_2 and β_4 .

Type of Curve	Values of the Constants in terms of β_2 and β_4	Remarks
(vii)	$p = \frac{1}{3-\beta_2} (\mp \sqrt{L} + 2\beta_2)$ $m = \frac{1}{2(3-\beta_2)^2} \{ \mp \sqrt{L} (5\beta_2 - 9) - [7\beta_4(3-\beta_2) - \beta_2(5\beta_2 + 9)] \}$	<p>The form of the equation is $y = y_0 e^{-\frac{m}{p-u^2}}$. β_2, β_4 and β_6 satisfy $F_6 = 0$, i.e.</p> $\beta_6 = \frac{b \pm 4 K \sqrt{L}}{9(3-\beta_2)^2}.$ <p>We must take the + sign if (i) $F_6' = 0$ and $K < 0$ or (ii) $F_6'' = 0$ and $K > 0$, the - sign if (i) $F_6' = 0$ and $K > 0$ or (ii) $F_6'' = 0$ and $K < 0$.</p>
(ix)	$p^2 = \frac{1}{3-\beta_2} (7\beta_4 - 5\beta_2)$ $m = \frac{1}{2(3-\beta_2)} (7\beta_4 - 15\beta_2)$	<p>The form of the equation is $y = y_0 \left(\frac{p-u^2}{p+u^2} \right)^{\frac{m}{2p}}$. β_2, β_4 and β_6 satisfy $F_2 = 0$, i.e.</p> $\beta_6 = \frac{1}{9(\beta_2 - 3)} (7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3).$ <p>Since p^2 and m must be positive we must also introduce the conditions $\beta_2 < 3$ and $\beta_4 > \frac{15\beta_2}{7}$.</p>
(x)	$p^2 = \frac{1}{3-\beta_2} (7\beta_4 - 5\beta_2)$ $m = \frac{1}{2(3-\beta_2)} (7\beta_4 - 15\beta_2)$	<p>The form of the equation is $y = y_0 \left(\frac{p+u^2}{p-u^2} \right)^{\frac{m}{2p}}$. β_2, β_4 and β_6 satisfy $F_2 = 0$. We also have the condition $\beta_2 < 3$ and $\beta_4 < \frac{15\beta_2}{7}$.</p>
(xi)	$q = \frac{1}{2\beta_2} (7\beta_4 - 5\beta_2^2)$ $m = \frac{1}{4\beta_2} (21\beta_4 - 25\beta_2^2)$	<p>The form of the curve is $y = y_0 \left(\frac{q-u^2}{u^2} \right)^{\frac{m}{q}}$. β_2, β_4 and β_6 satisfy $F_0 = 0$, i.e.</p> $\beta_6 = \frac{1}{9\beta_2} (14\beta_2^2 - 5\beta_2^3\beta_4).$ <p>We must also have $\beta_4 > \frac{25\beta_2^2}{21}$ but $< \frac{10\beta_2^2}{7}$.</p>
(xii)	$q = \frac{7\beta_4 - 5\beta_2^2}{2\beta_2}$ $m = \frac{1}{4\beta_2} (25\beta_2^2 - 21\beta_4)$	<p>The form of the curve is $y = y_0 \left(\frac{u^2}{q-u^2} \right)^{\frac{m}{q}}$. β_2, β_4 and β_6 satisfy $F_0 = 0$. The Type (xii) satisfies $c_a = 0$ provided $\beta_4 < \frac{1}{4}\beta_2^2$.</p>
(xiii)	$p = \frac{1}{2(7\beta_2 - 15)} \{ \pm \sqrt{Q} - (7\beta_4 - 23\beta_2) \}$ $q = \frac{1}{6(\beta_2 - 1)} \{ \pm \sqrt{Q} + 7(\beta_4 - \beta_2) \}$	<p>The form of the curve is $y = y_0 \left(\frac{q-u^2}{p+u^2} \right)^{\frac{m}{q}}$. β_2, β_4 and β_6 satisfy $F_8 = 0$, i.e.</p> $\beta_6 = \frac{B \pm 4 K \sqrt{Q}}{7(\beta_2 - 1)(7\beta_2 - 15)}.$ <p>We must take the + sign if (i) $F_8' = 0$ and $K > 0$ or (ii) $F_8'' = 0$ and $K < 0$, and the - sign if (i) $F_8' = 0$ and $K < 0$ or (ii) $F_8'' = 0$ and $K > 0$.</p>

$< \frac{1}{4}$ for finite frequency, which easily reduces to $\beta_4 < \frac{10\beta_2^2}{7}$.

TABLE V (continued).

Type of Curve	Values of the Constants in terms of β_2 and β_4	Remarks
(xiv)	$p = \frac{1}{6(\beta_2 - 1)} \{ \pm \sqrt{Q} + (7\beta_4 - 7\beta_2) \}$ $q = \frac{1}{2(7\beta_2 - 15)} \{ \pm \sqrt{Q} - (7\beta_4 - 23\beta_2) \}$	<p>The form of the curve is $y = y_0 \left(\frac{p - u^2}{q + u^2} \right)$.</p> <p>$\beta_2, \beta_4$ and β_6 satisfy $F_8 = 0$. We must take the + sign if (i) $F_8' = 0$ and $K > 0$ or (ii) $F_8'' = 0$ and $K < 0$; otherwise the - sign.</p>
(xv)	$p = \frac{1}{2(7\beta_2 - 15)} \{ \pm \sqrt{Q} - (7\beta_4 - 23\beta_2) \}$ $q = \frac{1}{6(\beta_2 - 1)} \{ \pm \sqrt{Q} + (7\beta_4 - 7\beta_2) \}$	<p>The form of the curve is $y = y_0 \left(\frac{p - u^2}{q - u^2} \right)$.</p> <p>$\beta_2, \beta_4, \beta_6$ satisfy $F_8 = 0$. We must take the + sign if (i) $F_8' = 0$ and $K > 0$ or (ii) $F_8'' = 0$ and $K < 0$; otherwise the - sign.</p>
(xx)	$p = 2 + (5\beta_2 + 4)^{\frac{1}{2}}$	<p>The form of the equation is $y = y_0 \left(\frac{p - u^2}{p + u^2} \right)$.</p> <p>$\beta_2, \beta_4, \beta_6$ satisfy</p> $\beta_4 = \frac{1}{7} [7\beta_2 + 24 + 4(3 - \beta_2) \sqrt{5\beta_2 + 4}]$ $\text{and } \beta_6 = \frac{1}{21} [(85\beta_2^2 + 16\beta_2 - 128) + 16(3\beta_2 - 4) \sqrt{5\beta_2 + 4}].$ <p>We find</p> $p = \{2(3 - \beta_2) \pm 3 - \beta_2 \sqrt{5\beta_2 + 4}\} / (3 - \beta_2),$ <p>but the only admissible value is</p> $p = 2 + (5\beta_2 + 4)^{\frac{1}{2}}.$

At the point I we accordingly find the Type (viii) curve, viz.

$$y = y_0 e^{-\frac{M}{P - u^2}}, \text{ where } u = \frac{x}{\sigma}.$$

We have managed to obtain limits to the expressions

$$\frac{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3}{4[5\beta_2^3 - (6 - \beta_2) \cdot \beta_4]},$$

$$\frac{9(3 - \beta_2)\beta_6 + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3}{2[5\beta_2^3 - (6 - \beta_2) \cdot \beta_4]},$$

$$\frac{14\beta_4^2 - 9\beta_2\beta_6 - 5\beta_2^3\beta_4}{5\beta_2^3 - (6 - \beta_2) \cdot \beta_4},$$

when

$$\beta_4 \rightarrow \frac{5\beta_2^2}{6 - \beta_2} \text{ and } \beta_6 \rightarrow \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}.$$

Notes a. (1) P and M are imaginary for $\beta_2 < 1.8$ and for such values of β_2 we do not get a fourth order Pearson curve at the point I .

(2) If $\beta_2 = 1.8$, $M = 0$ and we get $y = \text{const.}$

(3) Let us see for what values of β_2 , P and M are positive.

Now

$$\sqrt{\frac{(\beta_2 - 1)(5\beta_2 - 9)}{6 - \beta_2}} \geq 2,$$

according as

$$(\beta_2 - 1)(5\beta_2 - 9) \geq 4(6 - \beta_2),$$

" "

$$5(\beta_2 - 1)(\beta_2 - 3) \geq 0,$$

" "

$$\beta_2 \geq 3.$$

Hence P is always positive.

Again,

$$3\sqrt{\beta_2 - 1} \geq \sqrt{(5\beta_2 - 9)(6 - \beta_2)},$$

according as

$$9(\beta_2 - 1) \geq (5\beta_2 - 9)(6 - \beta_2),$$

" "

$$5(\beta_2 - 3)^2 \geq 0.$$

Hence $3\sqrt{\beta_2 - 1} > \sqrt{(5\beta_2 - 9)(6 - \beta_2)}$ for all values of β_2 . Accordingly $M > 0$ for all values of $\beta_2 > 1.8$.

(4) When $\beta_2 = 3$, we must find P and M from the difference equation, which is easily found to be

$$(2s + 1)P^2\mu_{2s} - 2P(2s + 3)\frac{\mu_{2s+2}}{\mu_2} + (2s + 5)\frac{\mu_{2s+4}}{\mu_2^2} = -2M\frac{\mu_{2s+2}}{\mu_2}.$$

Hence for

$$s = 0,$$

$$P^2 - 6P + 5\beta_2 = -2M,$$

$$s = 1,$$

$$3P^2 - 10P\beta_2 + 7\beta_4 = -2M\beta_2.$$

Put $\begin{cases} \beta_2 = 3 \\ \beta_4 = 15 \end{cases}$ in the equations above and solve for P and M ; therefore $P = 5$ and $M = 5$.

When $\beta_2 = 3$, $\beta_4 = 15$ and $\beta_6 = 91\frac{1}{2}$ we get the curve

$$y = y_0 e^{\frac{-5}{5 - u^2}}.$$

The ordinates of this curve and the corresponding normal curve are plotted in Diagram II, showing how a change in β_6 changes the shape of the resulting frequency curve.

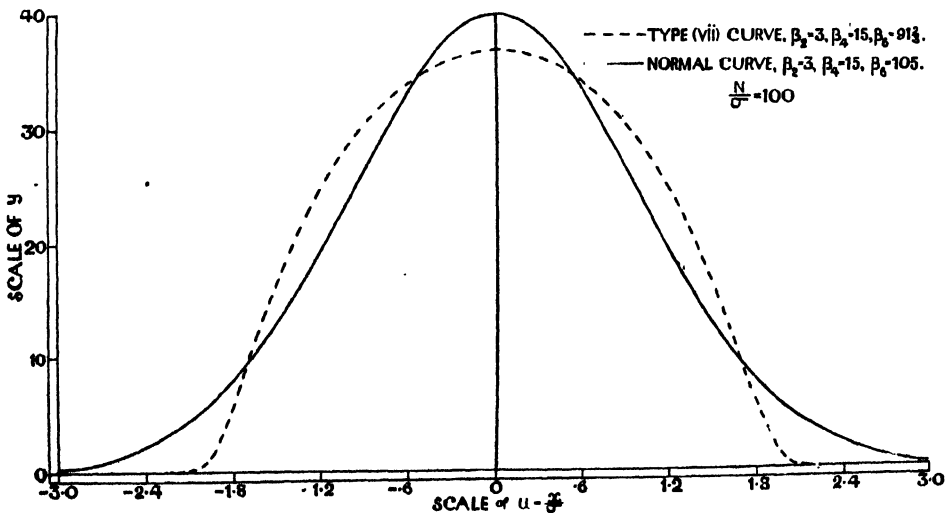


Diagram II.

150 On certain Non-Normal Symmetrical Frequency Distributions

The constants of integration for the transition types found by considering the areas under the curves are as follows:

TABLE VI. Values of y_0 (continuation of Table IV).

Type	Value for N	Expression for y_0
(vii)	$N\bar{\xi} = Sn_x x $	$\frac{N\bar{\xi}^*}{m\sigma^2 E_1\left(-\frac{m}{p}\right) + e^{-m/p} \left\{ p\sigma^2 - \frac{h^2}{12} - \frac{7h^4}{480} \frac{m}{p^2\sigma^2} \dots \right\}}$
(ix)	$N = 2y_0 \sqrt{p} \cdot \sigma \int_0^1 \left(\frac{1-x^2}{1+x^2} \right)^n dx$	$\frac{N}{2\sigma\sqrt{p}} \frac{1}{I_n}$
(x)	$N = 2y_0 \sigma \sqrt{p} \int_0^1 \left(\frac{1+x^2}{1-x^2} \right)^n dx$	$\frac{N}{2\sigma\sqrt{p}} \frac{1}{I_n}$
(xi)	$N = 2y_0 \int_0^{\sigma/q} \left(\frac{q\sigma^2 - x^2}{x^2} \right)^{\frac{m}{q}} dx$	$\frac{N\sqrt{\pi}}{2\sigma\sqrt{q}} \frac{1}{\Gamma(\frac{1}{2}-l)\Gamma(1+l)}$ where $l = m/q$
(xii)	$N = 2y_0 \int_0^{\sigma/q} \left(\frac{x^2}{q\sigma^2 - x^2} \right)^{\frac{m}{q}} dx$	$\frac{N\sqrt{\pi}}{2\sigma\sqrt{q}} \frac{1}{\Gamma(\frac{1}{2}+l)\Gamma(1-l)}$
(xiii)	$N = 2y_0 \int_0^{\sigma/q} \left(\frac{q\sigma^2 - x^2}{p\sigma^2 + x^2} \right) dx$	$\frac{N\sqrt{p}}{2\sigma [(q+p) \tan^{-1} \sqrt{q/p} - \sqrt{pq}]}$
(xiv)	$N = 2y_0 \int_0^{\sigma/q} \left(\frac{p\sigma^2 - x^2}{p\sigma^2 + x^2} \right) dx$	$\frac{N\sqrt{q}}{2\sigma [(p+q) \tan^{-1} \sqrt{p/q} - \sqrt{pq}]}$
(xv)	$N = 2y_0 \int_0^{\sigma/p} \left(\frac{p\sigma^2 - x^2}{q\sigma^2 - x^2} \right) dx$	$\frac{N\sqrt{q}}{\sigma [2\sqrt{pq} - q - p \{ \log_e (\sqrt{q} + \sqrt{p}) - \log_e (\sqrt{q} - \sqrt{p}) \}]}$
(xx)	$N = 2y_0 \int_0^{\sigma/p} \left(\frac{p\sigma^2 - x^2}{p\sigma^2 + x^2} \right) dx$	$\frac{N}{2\sigma\sqrt{p} \left[\frac{\pi}{2} - 1 \right]}$

$E_1(-a) = \int_a^\infty \frac{e^{-z}}{z} dz$ and is tabled in *Mathematical Tables* of the British Association for the Advancement of Science, Vol. 1. The expression is easily found by replacing the summation by one of integration. Unfortunately I have not had time to try it out sufficiently, but in the trials I have made it gave satisfactory results. $\bar{\xi}$ denotes the mean absolute derivation.

† The reduction formula is $I_{n+1} = I_{n-1} - \frac{I_n}{n}$, $I_0 = 1$, $I_1 = \frac{\pi}{2} - 1$,

$$I_{\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \left\{ \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} - 4 \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \right\} = .71196, \quad I_{-\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \left\{ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} + 4 \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \right\} = 1.01010$$

and from these values I constructed Table VII (p. 151). I also proved that

$$I_{r+\theta} = I_r + \theta \Delta I_r - \frac{\theta(1-\theta)}{2!} \Delta^2 I_r + \dots,$$

where

$$\Delta I_r = I_{r+1} - I_r \text{ etc.,}$$

and hence we can find $I_{r+\theta}$ from I_r , I_{r+1} etc. by ordinary interpolation formulae.

Note β. The Types (xi) and (xii) are rather interesting. Consider Type (xi) which for convenience I shall write in the form

$$y = y_0 \left(\frac{b^2 - x^2}{x^2} \right)^l.$$

We have

$$N\mu_{2s} = 2 \int_0^b y_0 x^{2s} \left(\frac{b^2 - x^2}{x^2} \right)^l dx$$

$$= y_0 b^{2s+1} B(s-l+\frac{1}{2}, l+1) = y_0 b^{2s+1} \frac{\Gamma(s+\frac{1}{2}-l) \Gamma(l+1)}{\Gamma(s+\frac{3}{2})}.$$

TABLE VII. *Table of I_n and its differences.*

I_n	I	$\Delta^2 I_n$	$\Delta^4 I_n$	$\Delta^6 I_n$
		+	+	+
I_2	·429 2037	97229	13063	3070
$I_{2.5}$	·387 8376	64951	7028	1389
I_3	·356 1945	45737	4063	673
$I_{3.5}$	·331 0465	33551	2487	358
I_4	·310 4722	25428	1594	207
$I_{4.5}$	·293 2529	19792	1059	116
I_5	·278 5765	15750	730	70
$I_{5.5}$	·265 8792	12767	518	42
I_6	·254 7569	10513	375	36
$I_{6.5}$	·244 9113	8778	274	19
I_7	·236 1170	7418	209	—
$I_{7.5}$	·228 2005	6833	163	—
I_8	·221 0259	5456	125	—
$I_{8.5}$	·214 4845	4741	95	—
I_9	·208 4887	4151	78	—
$I_{9.5}$	·202 9670	3657	65	—
I_{10}	·197 8605	3240	53	—
I_{10}	·197 86048	116081	5687	593
I_{11}	·188 70268	93518	3970	365
I_{12}	·180 70569	76641	2847	233
I_{13}	·173 64387	63735	2088	153
I_{14}	·167 34846	53675	1563	103
I_{15}	·161 69041	45704	1190	72
I_{16}	·156 56911	39295	920	50
I_{17}	·151 90484	34077	722	32
I_{18}	·147 63353	29778	574	34
I_{19}	·143 70298	26202	458	10
I_{20}	·140 07021	23200	376	21

$$I_{-.5} = \cdot 4310\ 632, \quad I_0 = 1.0, \quad I_{+.5} = \cdot 711\ 9587,$$

$$I_{1.0} = \cdot 570\ 7963, \quad I_{1.5} = \cdot 486\ 1816.$$

Put

$$s = 0, \quad N = by_0 \frac{\Gamma(\frac{1}{2} - l) \Gamma(1 + l)}{\Gamma(\frac{3}{2})},$$

$$s = 1, \quad \sigma^2 = b^2 (\frac{1}{2} - l) / \frac{3}{2},$$

$$s = 2, \quad \beta_2 = \frac{3(3b^2 - 2m)}{5(b^2 - 2m)},$$

$$s = 3, \quad \beta_4 = \frac{1}{4} \beta_2 (10\beta_2 - 3),$$

$$s = 4, \quad \beta_6 = \frac{1}{8} \beta_2 (10\beta_2 - 3) (15\beta_2 - 6),$$

giving
$$m = \frac{\sigma^2(5\beta_2 - 9)}{4}, \quad b^2 = \frac{\sigma^2(5\beta_2 - 3)}{2}, \quad l = \frac{5\beta_2 - 9}{2(5\beta_2 - 3)}.$$

Hence in every plane of section by a plane $\beta_2 = \text{constant}$ we get at one point of the curve of section of $F_0 = 0$, viz. when β_4 and β_6 have the values above a special Type (xi) curve with the constants expressible in terms of β_2 only. The difference equation is now

$$(2s + 1)a^2\mu_{2s} - (2s + 3)\frac{\mu_{2s+2}}{\mu_2} = 2m\mu_{2s}, \quad (b^2 = a^2\sigma^2).$$

Put

$$\begin{aligned} s = 0, & \quad a^2 - 3 = 2m, \\ s = 1, & \quad 3a^2 - 5\beta_2 = 2m, \\ s = 2, & \quad 5a^2\beta_2 - 7\beta_4 = 2m\beta_2. \end{aligned}$$

The first two of these equations give us the values found above, while the last two equations give us the general values.

II. THE TRANSITION SURFACES.

We have seen in Section I (g) (p. 143) how a knowledge of the curves of section of the transition surfaces for different values of β_2 will enable us to determine what type of frequency curve corresponds to any given values of β_2 , β_4 and β_6 .

In this Section we suppose β_2 known and then deduce general formulae in terms of β_2 , to simplify the calculation of the β_4 - and β_6 -ordinates of the transition curves.

(a) The auxiliary co-ordinates r and Φ .

There is no way of finding the approximate range of values which β_2 , β_4 and β_6 can assume. The value Professor Pearson finds for β_4 for his symmetrical curves is

$$\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}.$$

The lowest possible value for β_4 is β_2^2 , and I suggest taking as the range for β_4 the expression

$$2\left(\frac{5\beta_2^2}{6 - \beta_2} - \beta_2^2\right) = \frac{2\beta_2^2(\beta_2 - 1)}{6 - \beta_2}.$$

in other words, I assume that

$$\beta_2^2 \leq \beta_4 \leq \beta_2^2 + \frac{2\beta_2^2(\beta_2 - 1)}{6 - \beta_2}$$

The only justification that I have for this assumption is that in all the symmetrical curves I have fitted I found β_4 well within the limits

$$\beta_2^2 \text{ and } \frac{\beta_2^2(\beta_2 + 4)}{(6 - \beta_2)}.$$

To make the model, I shall give β_2 different values and find the corresponding curves of section in the different β_2 -planes. The curves of section depend then on β_4 and β_6 and it is an advantage to calculate the same number of β_6 -ordinates for each curve, and at equal intervals of β_4 , for each β_2 . But the suggested range of possible values for β_4 varies as β_2 varies, e.g. the range is from 4 to 6 when $\beta_2 = 2$ and from 25 to 225 when $\beta_2 = 5$, and hence the interval between the successive values of β_4 must vary as β_2 varies. Similarly we must also have a proportionate unit for β_6 .

All the surfaces pass through the point I , whose co-ordinates are

$$\begin{aligned}\beta_2 &= \beta_2, \\ \beta_4 &= \frac{5\beta_2^2}{6 - \beta_2}, \\ \beta_6 &= \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}\end{aligned}$$

I propose calculating the β_6 -ordinates at intervals of

$$\beta_4 = \frac{1}{7} \frac{\beta_2^2(\beta_2 - 1)}{6 - \beta_2} = \frac{1}{7} \left(\frac{5\beta_2^2}{6 - \beta_2} - \beta_2^2 \right)$$

and taking the unit for β_6 as

$$\frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} - \beta_2^3 = \frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2}.$$

In this way I introduce two auxiliary co-ordinates r and Φ , which I shall measure from the point I as origin. The relations between β_4 and r , β_6 and Φ , r and Φ_x then become

$$\begin{aligned}\beta_4 &= \frac{5\beta_2^2}{6 - \beta_2} + \frac{r}{7} \frac{\beta_2^2(\beta_2 - 1)}{6 - \beta_2}, \\ \Phi &= \beta_6 - \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} / \frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2}, \\ \Phi_x &= \frac{\phi_x \left(\frac{5\beta_2^2}{6 - \beta_2} + \frac{r}{7} \frac{\beta_2^2(\beta_2 - 1)}{6 - \beta_2} \right) - \frac{25\beta_2^3(\beta_2 + 8)}{(6 - \beta_2)^2}}{\frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2}},\end{aligned}$$

where ϕ_x is the β_6 -ordinate on the surface $F_x = 0$.

Let us represent graphically these relations.

Let OQ, OR represent the axis of β_4 and β_6 ,
 IQ', IR' " " " r and Φ ,
 O be the point (β_2^2, β_2^3) ,
 I " " $\left(\frac{5\beta_2^2}{6 - \beta_2}, \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} \right)$.

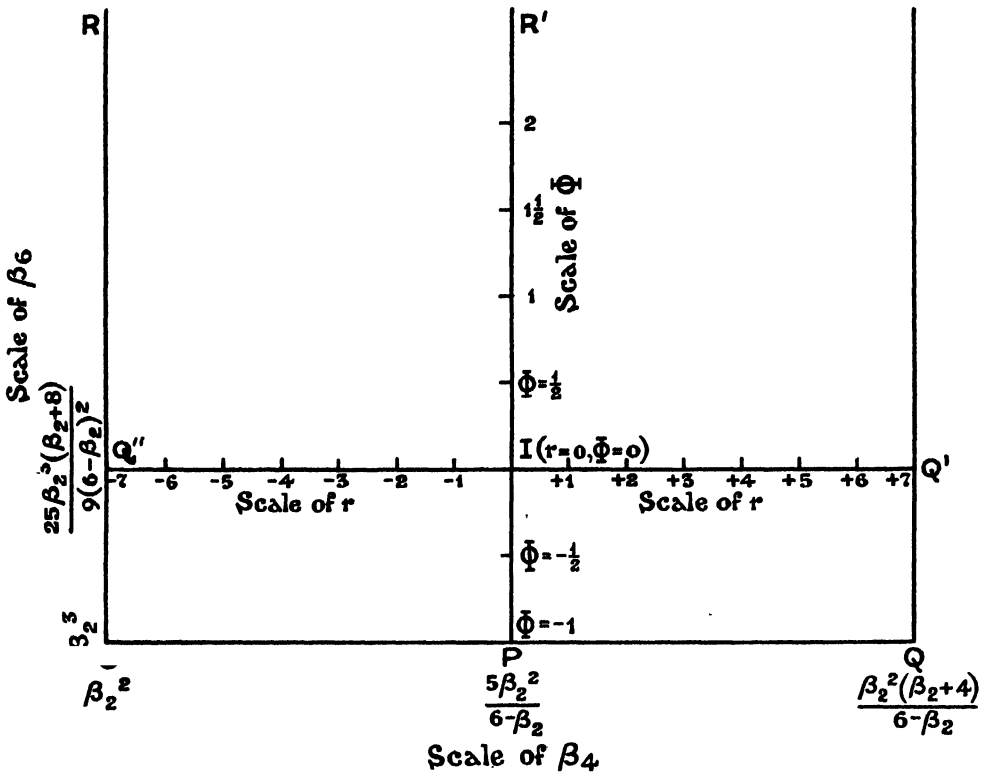


Diagram III.

Then
$$OP = \frac{\beta_2^2(\beta_2 - 1)}{6 - \beta_2},$$

$$PI = \beta_2^3(124 - 9\beta_2)(\beta_2 - 1)/9(6 - \beta_2)^2.$$

I calculate the β_6 -ordinates at intervals of $\frac{1}{4}OP$, transfer to the point I as origin and divide by PI , to give me the corresponding value of Φ .

The following table gives values of PI and $\frac{1}{4}OP$ for different values of β_2 .

TABLE VIII.

β_2	$\frac{1}{4}OP$	PI	β_2	$\frac{1}{4}OP$	PI
1.25	.01175	.27117	3.5	1.75	176.2639
1.5	.03571	1.02315	4	3.4286	469.3333
1.75	.07721	2.67660	4.5	6.75	1315.125
2	.14286	5.88889	5	14.2857	4388.889
2.5	.38265	21.57738	5.5	38.8922	24789.88
3	.85714	64.66667	6	∞	∞

Note that since $\beta_4 = \frac{5\beta_2^2}{6-\beta_2}$ is negative for $\beta_2 > 6$ we cannot use the r and Φ co-ordinates for $\beta_2 > 6$. I shall not make any diagrams for such values of β_2 .

(b) Section I $f(\alpha)$ (p. 142) gives the relations

$$\phi_0 = \frac{14\beta_4^2 - 5\beta_2^2\beta_4}{9\beta_2} = \frac{f_0}{9\beta_2},$$

$$\phi_2 = \frac{-\{7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3\}}{9(3-\beta_2)} = \frac{f_2}{9(3-\beta_2)} \quad (\beta_2 \neq 3),$$

$$\phi_4 = \frac{49\beta_4^2 - 210\beta_2\beta_4 + 125\beta_2^3}{9(5\beta_2-9)} = \frac{f_4}{9(5\beta_2-9)} \quad (\beta_2 \neq 1.8),$$

$$\phi_6'''' = \frac{b \pm 4|K|\sqrt{L}}{9(3-\beta_2)^2} \quad (\beta_2 \neq 3),$$

$$\phi_8'''' = \frac{-\beta \pm 4|K|\sqrt{Q}}{27(\beta_2-1)(7\beta_2-15)} \quad (\beta_2 \neq 2\frac{1}{7}),$$

where

$$b = (3-\beta_2)f_2 - 8\beta_2 \cdot K,$$

$$B = (5\beta_2-9)f_4 - 4b,$$

$$K = 5\beta_2^2 - 6 - \beta_2 \cdot \beta_4,$$

$$L = \beta_2^2(19-5\beta_2) - 7(3-\beta_2)\beta_4,$$

$$Q = 49\beta_4^2 - 14\beta_4(37\beta_2-30) + 7\beta_2^2(60\beta_2-53).$$

I assume for the present $\beta_2 \neq 1.8$, $\beta_2 \neq 2\frac{1}{7}$, $\beta_2 \neq 3$.

Write

$$l = \frac{5\beta_2^2}{6-\beta_2},$$

$$m = \frac{r\beta_2^2(\beta_2-1)}{7(6-\beta_2)},$$

$$n = \frac{25\beta_2^3(\beta_2+8)}{9(6-\beta_2)^2}.$$

We want the values of $\phi_x(l+m)$, or $f_x(l+m)$. Using Taylor's theorem, we have

$$f_x(l+m) = f_x(l) + mf'_x(l) + \frac{m^2}{2!}f''_x(l) + \dots$$

for $x=0, 2, 4, 6, 8$. The expressions f_0, f_2, f_4, L , and Q are quadratic expressions in β_4 and hence all their third and higher order differentials with respect to β_4 vanish. Again, we simplify the algebra by using the result

$$\phi_x(l) = \frac{25\beta_2^3(\beta_2+8)}{9(6-\beta_2)^2} = n$$

for $x=0, 2, 4, 6, 8$ since all the surfaces pass through the point I .

Now

$$f_0'(\beta_4) = 28\beta_4 - 5\beta_2^2,$$

$$f_0''(\beta_4) = 28,$$

$$f_0'(l) = \frac{5\beta_2^2}{6 - \beta_2}(\beta_2 + 22),$$

$$f_0(l) = 9\beta_2 \phi_0(l) = 9\beta_2 \cdot n.$$

Hence
$$f_0(l+m) = f_0(l) + m f_0'(l) + \frac{m^2}{2!} f_0''(l)$$

$$= 9\beta_2 \cdot n + \frac{r \beta_2^2 (\beta_2 - 1)}{7} \frac{5\beta_2^2 (\beta_2 + 22)}{6 - \beta_2} + \frac{1}{2} \frac{r^2}{49} \frac{\beta_2^4 (\beta_2 - 1)^2}{(6 - \beta_2)^2}.$$

But

$$\begin{aligned} \Phi_0 &= \frac{\phi_0(l+m) - n}{\beta_2^3 (124 - 9\beta_2)(\beta_2 - 1)} \\ &= \frac{r \{2r(\beta_2 - 1) + 5(\beta_2 + 22)\}}{7(124 - 9\beta_2)}. \end{aligned}$$

Similarly, after lengthy algebra,

$$\Phi_2 = \frac{r \{r\beta_2(\beta_2 - 1) + 60(2\beta_2 - 5)\}}{7(124 - 9\beta_2)(\beta_2 - 3)}, \quad \dots(12)$$

$$\Phi_4 = \frac{r \{r\beta_2(\beta_2 - 1) + 20(5\beta_2 - 9)\}}{(5\beta_2 - 9)(124 - 9\beta_2)},$$

$$\Phi_6''' = \frac{r \{r\beta_2(\beta_2 - 1)(\beta_2 - 3) + 4\psi\}}{7(\beta_2 - 3)^2(124 - 9\beta_2)},$$

where

$$\psi = 32\beta_2^3 - 189\beta_2 + 297 \mp \{(6 - \beta_2)^3(\beta_2 - 1)[(5\beta_2 - 9) + r(\beta_2 - 3)]\}^{\frac{1}{2}},$$

$$\Phi_8''' = \frac{r \{r\beta_2(31\beta_2 - 51) + 4\chi\}}{21(124 - 9\beta_2)(7\beta_2 - 15)},$$

where

$$\chi = 9(83\beta_2 - 183) \mp (6 - \beta_2)[84(5\beta_2 - 9) + 72r(2\beta_2 - 5) + \beta_2^2 r^2]^{\frac{1}{2}}.$$

Note $\Phi_6' > \Phi_6''$, $\Phi_8' > \Phi_8''$ according as $\beta_2 \geq \frac{1}{2}$. By giving β_2 its constant value and putting $r = -7$ to $r = +7$ we can easily calculate the ordinates.

Not only have I solved the difficulty of choosing a suitable scale for β_4 and β_6 and so made the construction of a model practicable, but I have simplified the calculation of the ordinates of the transition surfaces considerably.

(c) *Calculation of the Ordinates.*

(i) Substituting for β_2 in Equations (12) we find the following relations between Φ_0 , Φ_2 , Φ_4 , and r for different values of β_2 :

TABLE IX. *Values of Φ_0, \dots, Φ_4 for values of β_2 .*

β_2	Φ_0	Φ_2	Φ_4
1	$\frac{r}{7}$	$\frac{18r}{161}$	$\frac{4r}{23}$
1.25	$\frac{r[465+2r]}{3157}$	$\frac{r[2400-5r]}{22099}$	$\frac{r[880-5r]}{4961}$
1.5	$\frac{r[235+2r]}{1547}$	$\frac{r[160-r]}{1547}$	$\frac{r[40-r]}{221}$
1.75	$\frac{r[475+6r]}{3031}$	$\frac{r[1440-21r]}{15155}$	$\frac{r[80-21r]}{433}$
2	$\frac{r[60+r]}{371}$	$\frac{r[30-r]}{371}$	$\frac{r[10+r]}{53}$
2.5	$\frac{r[245+6r]}{1421}$	$\frac{-15r^2}{1421}$	$\frac{r[280+15r]}{1421}$
3	$\frac{r[125+4r]}{679}$	—	$\frac{r[20+r]}{97}$
3.5	$\frac{r[51+2r]}{259}$	$\frac{r[96+7r]}{259}$	$\frac{r[136+7r]}{629}$
4	$\frac{r[45+3r]}{154}$	$\frac{r[65+3r]}{308}$	$\frac{r[55+3r]}{242}$
4.5	$\frac{r[265+14r]}{1169}$	$\frac{r[320+21r]}{1169}$	$\frac{r[120+7r]}{501}$
5	$\frac{r[135+8r]}{553}$	$\frac{r[150+10r]}{553}$	$\frac{r[80+5r]}{316}$
5.5	$\frac{r[275+18r]}{1043}$	$\frac{r[1440+99r]}{5215}$	$\frac{r[1480+99r]}{5513}$

158 *On certain Non-Normal Symmetrical Frequency Distributions*

In the same way we find the following table, which gives the relations between Φ_6 , Φ_8 and r for different values of β_2 :

TABLE X. *Values of Φ_6 and Φ_8 for Values of β_2 .*

β_2	Φ_6	Φ_8
1	$\frac{4r}{23}$	$\frac{r \{3600 + 20r \pm 20 [r^2 - 216r - 336]^{\frac{1}{2}}\}}{19320}$
1.25	$\frac{r [28352 - 35r \mp 152 (-19.7r + 11)^{\frac{1}{2}}]}{154693}$	$\frac{r [45648 + 245r \pm 76 \{25r^2 - 2880r - 3696\}^{\frac{1}{2}}]}{236775}$
1.5	$\frac{r [304 - r \mp 12 (-6.7r + 1)^{\frac{1}{2}}]}{1547}$	$\frac{r [312 + r \pm 4 (r.7r - 64 - 56)^{\frac{1}{2}}]}{1547}$
1.75	$\frac{r [16448 - 105r \mp 136 \{51 (-5r - 1)\}^{\frac{1}{2}}]}{75775}$	$\frac{r [21744 - 91r \pm 68 \{49r^2 - 1728r - 336\}^{\frac{1}{2}}]}{100023}$
2	$\frac{r [94 - r \mp (256.1 - r)^{\frac{1}{2}}]}{371}$	$\frac{r [306 - 11r \pm 8 (84 - r.72 - 4r)^{\frac{1}{2}}]}{1113}$
2.5	$\frac{r [784 - 15r \mp (42.7 - r)^{\frac{1}{2}}]}{1421}$	$\frac{r [3528 + 265r \mp 28 (1176 + 25r)^{\frac{1}{2}}]}{21315}$
3	$\frac{r [600 + r (36 + r)]}{2716}$	$\frac{r [132 + 7r \mp 2 (56 + r.8 + r)^{\frac{1}{2}}]}{2716}$
3.5	$\frac{r [176 + 7r \mp 20 (2.17 + r)^{\frac{1}{2}}]}{259}$	$\frac{r [3096 + 161r \mp 4 (2856 + r.576 + 49r)^{\frac{1}{2}}]}{14763}$
4	$\frac{r [53 + 3r \mp (24.11 + r)^{\frac{1}{2}}]}{154}$	$\frac{r [1341 + 73r \mp 2 (924 + r.216 + 16r)^{\frac{1}{2}}]}{6006}$
4.5	$\frac{r [336 + 21r \mp 4 (14.9 + r)^{\frac{1}{2}}]}{1169}$	$\frac{r [3048 + 177r \mp 4 (504 + r.128 + 9r)^{\frac{1}{2}}]}{12859}$
5	$\frac{r [152 + 10r \mp (64 + 8r)^{\frac{1}{2}}]}{553}$	$\frac{r [2088 + 130r \mp (1344 + r.360 + 25r)^{\frac{1}{2}}]}{8295}$
5.5	$\frac{r [7216 + 495r \mp 12 (74 + 10r)^{\frac{1}{2}}]}{26075}$	$\frac{r [39384 + 2629r \mp 4 (6216 + r.1728 + 121r)^{\frac{1}{2}}]}{147063}$

(d) From the previous tables it is now an easy matter to calculate the Φ -ordinates of each of the transition surfaces for integral values of r from -7 to $+7$ and β_2 from $1.25 - (.25) - 2.0$ and $2.0 - (.5) - 5.5$. The results are given in Tables XI—XXII and the ordinates are plotted in Diagrams VI—XIX.

TABLE XI. *The Ordinates of the Transition Curves for $\beta_2 = 1.25$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.00000	-.77130	-1.29107	-1.10922	-1.47886	-1.59766	-1.00000
-6	-.86094	-.65976	-1.10058	-.96474	-1.25090	-1.35127	-.88771
-5	-.72062	-.54867	-.91211	-.81714	-1.02696	-1.10891	-.76726
-4	-.57903	-.43803	-.72566	-.66610	-.80737	-.87104	-.63817
-3	-.43617	-.32784	-.54122	-.51124	-.59250	-.63829	-.49983
-2	-.29205	-.21811	-.35880	-.35263	-.38230	-.41131	-.35158
-1	-.14666	-.10883	-.17839	—	—	—	—
+1	+ .14793	+ .10838	+ .17638	—	—	—	—
+2	+ .29712	+ .21630	+ .35074	—	—	—	—
+3	+ .44758	+ .32377	+ .52308	—	—	—	—
+4	+ .59930	+ .43079	+ .69341	—	—	—	—
+5	+ .75230	+ .53735	+ .86172	—	—	—	—
+6	+ .90656	+ .64347	+ 1.02802	—	—	—	—
+7	+ 1.06208	+ .74913	+ 1.19230	—	—	—	—

TABLE XII. *The Ordinates of the Transition Curves for $\beta_2 = 1.5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.00000	-.75566	-1.48869	-1.08145	-1.00000
-6	-.86490	-.64383	-1.24887	-.94741	-.89083
-5	-.72721	-.53329	-1.01810	-.80870	-1.18871	-1.21202	-.77246
-4	-.58694	-.42405	-.79638	-.66474	-.92802	-.94839	-.64438
-3	-.44086	-.31610	-.58371	-.51473	-.67596	-.69263	-.50582
-2	-.29864	-.20944	-.38009	-.35760	-.43361	-.44586	-.35569
-1	-.15061	-.10407	-.18552	-.19716	-.19716	-.20879	-.19328
+1	+ .15320	+ .10278	+ .17647	—	—	—	—
+2	+ .30899	+ .20427	+ .34389	—	—	—	—
+3	+ .46736	+ .30446	+ .50226	—	—	—	—
+4	+ .62831	+ .40336	+ .65158	—	—	—	—
+5	+ .79186	+ .50097	+ .79186	—	—	—	—
+6	+ .95798	+ .59729	+ .92308	—	—	—	—
+7	+ 1.12670	+ .69231	+ 1.04525	—	—	—	—

— Denotes that the value of Φ is imaginary.

... Denotes that Φ is now a large negative quantity.

TABLE XIII. *The Ordinates of the Transition Curves for $\beta_2 = 1.75$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.00000	-.73303	...	-1.06418	-1.00000
-6	-.86902	-.61999	...	-.93813	-1.78641	-1.78012	-.89407
-5	-.73408	-.50973	-2.13626	-.80600	-1.43392	-1.44152	-.77788
-4	-.59518	-.40224	-1.51501	-.66695	-1.11390	-1.11741	-.65082
-3	-.45233	-.29753	-.99076	-.51979	-.80754	-.80869	-.51203
-2	-.30551	-.19558	-.56351	-.36277	-.51657	-.51672	-.36012
-1	-.15473	-.09640	-.23326	-.19281	-.24408	-.24405	-.19255
+1	+ .15869	+ .09363	+ .13626	—	—	—	—
+2	+ .32135	+ .18449	+ .17552	—	—	—	—
+3	+ .48796	+ .27258	+ .11778	—	—	—	—
+4	+ .65853	+ .35790	-.03695	—	—	—	—
+5	+ .83306	+ .44045	-.28868	—	—	—	—
+6	+1.01155	+ .52022	-.63741	—	—	—	—
+7	+1.19400	+ .59723	-1.08314	—	—	—	—

TABLE XIV. *The Ordinates of the Transition Curves for $\beta_2 = 2$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.00000	-.69811	-.39623	-1.07066	-1.00000
-6	-.87332	-.58221	-.45283	-.94881	-.89745
-5	-.74124	-.47170	-.47170	-.81952	-.78351
-4	-.60377	-.36658	-.45283	-.67087	-1.44234	-1.85820	-.65752
-3	-.46092	-.26685	-.39623	-.52561	-1.04313	-1.30901	-.51848
-2	-.31267	-.17251	-.30189	-.36812	-.66692	-.81395	-.36484
-1	-.15903	-.08356	-.16981	-.19507	-.31706	-.37574	-.19390
+1	+ .16442	+ .07817	+ .20755	+ .25067	+ .25067	+ .23630	+ .29380
+2	+ .33423	+ .15094	+ .45283	—	—	—	—
+3	+ .50943	+ .21833	+ .73585	—	—	—	—
+4	+ .69003	+ .28032	+1.05660	—	—	—	—
+5	+ .87601	+ .33693	+1.41509	—	—	—	—
+6	+1.06739	+ .38814	+1.81132	—	—	—	—
+7	+1.26415	+ .43396	+2.24528	—	—	—	—

TABLE XV. *The Ordinates of the Transition Curves for $\beta_2 = 2.5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.00000	-.51724	-.86207	-1.03466	...	-.09885	-1.00000
-6	-.88248	-.38001	-.80225	-.92780	...	-.18641	-.90465
-5	-.75651	-.26390	-.72132	-.81070	...	-.23803	-.79551
-4	-.62210	-.16890	-.61928	-.68167	...	-.25455	-.67175
-3	-.47924	-.09500	-.49613	-.53871	...	-.23715	-.53217
-2	-.32794	-.04222	-.35186	-.37948	-1.91187	-.18746	-.37515
-1	-.16819	-.01066	-.18649	-.20109	-.92347	-.10756	-.19861
+1	+ .17664	-.01066	+ .20760	+ .85397	+ .22837	+ .22347	+ .13243
+2	+ .36172	-.04222	+ .43631	+1.63231	+ .49014	+ .47461	+ .28692
+3	+ .55524	-.09500	+ .68614	+2.32636	+ .79397	+ .75595	+ .46094
+4	+ .75721	-.16890	+ .95707	+2.92273	+1.15327	+1.06959	+ .65239
+5	+ .96763	-.26390	+1.24912	+3.39769	+1.59175	+1.41714	+ .85966
+6	+1.18649	-.38001	+1.56228	+3.69653	+2.16413	+1.79979	+1.08156
+7	+1.41379	-.51724	+1.89655	+3.34483	+3.34483	+2.21839	+1.31724

— Denotes that the value of Φ is imaginary.... Denotes that Φ is now a large negative quantity.

TABLE XVI. *The Ordinates of the Transition Curves for $\beta_2 = 3$.*

r	Φ_0	Φ_2^*	Φ_4	$\Phi_6'^*$	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	∞	- .938 144	- 1.023 196	∞	- .711 340	- 1.000 000
-6	- .892 489		- .865 979	- .927 835		- .678 057	- .912 517
-5	- .773 196		- .773 196	- .819 219		- .619 984	- .808 588
-4	- .642 121		- .659 794	- .695 140		- .538 150	- .687 182
-3	- .499 264		- .525 773	- .553 387		- .433 846	- .547 008
-2	- .344 624		- .371 134	- .391 753		- .308 493	- .386 647
-1	- .178 203		- .195 876	- .208 027		- .163 476	- .204 713
+1	+ .189 985	∞	+ .216 495	+ .234 536	∞	+ .228 460	+ .180 965
+2	+ .391 753		+ .453 608	+ .497 791		+ .481 401	+ .378 687
+3	+ .605 302		+ .711 340	+ .791 973		+ .759 358	+ .592 630
+4	+ .830 633		+ .989 691	+ 1.119 293		+ 1.062 716	+ .822 409
+5	+ 1.067 747		+ 1.288 660	+ 1.481 959		+ 1.391 753	+ 1.067 747
+6	+ 1.316 642		+ 1.608 247	+ 1.882 180		+ 1.746 666	+ 1.328 445
+7	+ 1.577 320		+ 1.948 454	+ 2.322 165		+ 2.127 600	+ 1.604 359

TABLE XVII. *The Ordinates of the Transition Curves for $\beta_2 = 3.5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	- 1.270 270	- .968 203	- 1.015 062	---	- .867 236	- 1.000 000
-6	- .903 475	- 1.250 975	- .896 661	- .931 081	---	- .811 839	- .919 516
-5	- .791 506	- 1.177 606	- .802 862	- .830 510	---	- .728 977	- .822 875
-4	- .664 093	- 1.050 193	- .686 804	- .710 728	---	- .624 750	- .703 978
-3	- .521 236	- .868 726	- .548 490	- .569 536	---	- .498 792	- .563 187
-2	- .362 934	- .633 205	- .387 917	- .405 062	---	- .352 184	- .399 425
-1	- .189 189	- .343 629	- .205 087	- .215 687	- 1.089 332	- .186 003	- .211 613
+1	+ .204 633	+ .397 683	+ .227 345	+ 1.169 884	+ .243 213	+ .236 605	+ .204 633
+2	+ .424 710	+ .849 421	+ .476 948	+ 2.419 215	+ .515 148	+ .498 185	+ .427 914
+3	+ .660 232	+ 1.355 212	+ .748 808	+ 3.747 001	+ .816 705	+ .784 911	+ .669 671
+4	+ .911 197	+ 1.915 058	+ 1.042 925	+ 5.152 352	+ 1.148 807	+ 1.095 818	+ .930 871
+5	+ 1.177 606	+ 2.528 958	+ 1.359 300	---	+ 1.512 259	+ 1.434 238	+ 1.177 606
+6	+ 1.459 459	+ 3.196 911	+ 1.697 933	---	+ 1.907 801	+ 1.796 979	+ 1.459 459
+7	+ 1.756 757	+ 3.918 919	+ 2.058 824	---	+ 2.336 106	+ 2.185 167	+ 1.756 757

TABLE XVIII. *The Ordinates of the Transition Curves for $\beta_2 = 4$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	...	- .983 471	- 1.009 184	...	- .934 732	- 1.000 000
-6	- .915 584	- 1.051 948	- .917 355	- .936 840	...	- .873 561	- .930 635
-5	- .811 688	- .974 026	- .826 446	- .844 156	...	- .786 513	- .838 529
-4	- .688 312	- .857 143	- .710 744	- .728 273	...	- .674 957	- .722 313
-3	- .545 454	- .701 299	- .570 248	- .587 213	- 1.127 073	- .539 906	- .580 913
-2	- .383 117	- .506 494	- .404 959	- .419 520	- .801 259	- .382 231	- .413 640
-1	- .201 299	- .272 727	- .214 876	- .224 079	- .425 272	- .202 162	- .220 082
+1	+ .220 779	+ .311 688	+ .239 669	+ .473 835	+ .253 438	+ .246 753	+ .224 109
+2	+ .461 039	+ .662 338	+ .504 132	+ .995 630	+ .536 838	+ .520 268	+ .470 075
+3	+ .720 779	+ 1.051 948	+ .793 388	+ 1.564 876	+ .850 708	+ .820 604	+ .737 838
+4	+ 1.000 000	+ 1.480 519	+ 1.107 438	+ 2.181 134	+ 1.195 489	+ 1.146 468	+ 1.028 691
+5	+ 1.298 701	+ 1.948 052	+ 1.446 281	+ 2.844 024	+ 1.571 561	+ 1.500 218	+ 1.340 276
+6	+ 1.616 883	+ 2.454 545	+ 1.809 917	+ 3.553 208	+ 1.979 259	+ 1.882 872	+ 1.671 574
+7	+ 1.954 545	+ 3.000 000	+ 2.198 847	+ 4.308 392	+ 2.418 881	+ 2.290 781	+ 2.026 236

* See (f) (iv) on p. 164.

— Denotes that the value of Φ is imaginary.- - - - Denotes that Φ is a large positive quantity.... .. Denotes that Φ is a large negative quantity.

TABLE XIX. *The Ordinates of the Transition Curves for $\beta_2 = 4.5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	-1.035 928	- .992 016	-1.004 994	-1.258 479	- .969 516	-1.000 000
-6	- .928 999	- .995 723	- .934 132	- .944 792	-1.210 896	- .912 209	- .941 123
-5	- .834 046	- .919 589	- .848 303	- .859 995	-1.116 053	- .826 372	- .855 718
-4	- .715 141	- .807 528	- .734 531	- .747 763	- .976 788	- .713 384	- .742 405
-3	- .572 284	- .659 538	- .592 814	- .606 517	- .794 681	- .573 985	- .600 446
-2	- .405 475	- .475 620	- .423 154	- .435 247	- .570 741	- .408 522	- .429 490
-1	- .214 713	- .255 774	- .225 549	- .233 249	- .305 673	- .217 164	- .229 371
+1	+ .238 666	+ .291 702	+ .253 493	+ .345 876	+ .264 903	+ .258 673	+ .242 922
+2	+ .501 283	+ .619 333	+ .534 930	+ .731 632	+ .561 782	+ .546 676	+ .511 571
+3	+ .787 853	+ .982 891	+ .844 311	+1.157 004	+ .890 900	+ .864 029	+ .805 930
+4	+1.098 375	+1.382 378	+1.181 637	+1.621 772	+1.252 479	+1.209 498	+1.127 231
+5	+1.432 849	+1.817 793	+1.546 906	+2.125 749	+1.646 707	+1.586 826	+1.471 732
+6	+1.791 275	+2.289 136	+1.940 120	+2.668 771	+2.073 744	+1.992 285	+1.843 161
+7	+2.173 653	+2.796 407	+2.361 277	+3.250 698	+2.533 733	+2.427 124	+2.240 268

TABLE XX. *The Ordinates of the Transition Curves for $\beta_2 = 5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	-1.012 658	- .996 835	-1.002 172	-1.073 778	- .988 186	-1.000 000
-6	- .943 942	- .976 492	- .949 367	- .954 792	-1.041 591	- .939 483	- .952 742
-5	- .858 951	- .904 159	- .870 253	- .877 948	- .966 537	- .858 951	- .874 623
-4	- .745 027	- .795 660	- .759 494	- .769 209	- .851 044	- .747 710	- .764 526
-3	- .602 170	- .650 995	- .617 089	- .627 534	- .696 155	- .606 107	- .622 102
-2	- .430 380	- .470 163	- .443 038	- .452 339	- .502 453	- .434 260	- .447 235
-1	- .229 656	- .253 165	- .237 342	- .243 249	- .270 313	- .232 216	- .239 875
+1	+ .258 590	+ .289 331	+ .268 987	+ .308 292	+ .277 603	+ .272 403	+ .262 378
+2	+ .546 112	+ .614 828	+ .569 620	+ .654 410	+ .589 713	+ .577 340	+ .554 908
+3	+ .862 568	+ .976 492	+ .901 899	+1.038 232	+ .936 451	+ .914 455	+ .877 950
+4	+1.207 957	+1.374 322	+1.265 823	+1.459 660	+1.317 917	+1.284 835	+1.230 414
+5	+1.582 278	+1.808 318	+1.661 392	+1.918 608	+1.734 195	+1.687 397	+1.613 386
+6	+1.985 533	+2.278 481	+2.088 608	+2.415 005	+2.185 356	+2.122 505	+2.026 501
+7	+2.417 722	+2.784 810	+2.547 468	+2.948 791	+2.671 463	+2.590 158	+2.469 758

TABLE XXI. *The Ordinates of the Transition Curves for $\beta_2 = 5.5$.*

r	Φ_0	Φ_2	Φ_4	Φ_6'	Φ_6''	Φ_8'	Φ_8''
-7	-1.000 000	-1.002 685	- .999 274	-1.000 537	-1.013 423	- .997 335	-1.000 000
-6	- .960 690	- .973 346	- .964 266	- .966 696	- .987 360	- .960 930	- .965 592
-5	- .886 865	- .906 040	- .893 343	- .897 838	- .920 381	- .888 767	- .895 434
-4	- .778 523	- .800 767	- .786 505	- .792 488	- .813 955	- .781 465	- .788 910
-3	- .635 666	- .657 526	- .643 751	- .650 209	- .668 525	- .638 844	- .646 197
-2	- .458 293	- .476 318	- .465 033	- .470 782	- .484 309	- .461 002	- .467 199
-1	- .246 405	- .257 143	- .250 499	- .254 075	- .261 438	- .248 080	- .251 774
+1	+ .280 920	+ .295 110	+ .286 414	+ .299 942	+ .291 506	+ .288 123	+ .283 238
+2	+ .596 357	+ .628 188	+ .608 743	+ .638 339	+ .620 491	+ .612 596	+ .601 632
+3	+ .946 309	+ .999 233	+ .966 987	+1.015 154	+ .986 994	+ .973 420	+ .955 182
+4	+1.330 777	+1.408 245	+1.361 146	+1.430 355	+1.391 045	+1.370 595	+1.343 889
+5	+1.749 760	+1.855 225	+1.791 221	+1.883 917	+1.832 670	+1.804 121	+1.767 750
+6	+2.203 260	+2.340 173	+2.257 210	+2.375 818	+2.311 890	+2.273 998	+2.226 767
+7	+2.691 275	+2.863 087	+2.759 115	+2.906 040	+2.828 725	+2.780 227	+2.720 940

(e) The limiting curve $\beta_6 = \beta_2\beta_4$; the Φ co-ordinate of the second order Pearson curve.

(i) I have calculated ordinates at values of r from -7 to $+7$ and found the corresponding Φ -ordinates for different values of β_2 . All these values of Φ will not be possible because β_2 , β_4 , and β_6 must satisfy the inequalities $\beta_4 > \beta_2^2$, $\beta_6 > \beta_2\beta_4 > \beta_2^3$. For all values of r from -7 to $+7$, $\beta_4 > \beta_2^2$ and $\beta_6 > \beta_2^3$. Let us consider now the surface $\beta_6 = \beta_2\beta_4$ and express this relation as an equation in r and Φ .

$$\Phi = \left\{ \beta_6 - \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} \right\} / \left\{ \frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2} \right\}$$

$$= \frac{\beta_2 \left\{ \frac{5\beta_2^2}{6 - \beta_2} + \frac{r\beta_2^2(\beta_2 - 1)}{6 - \beta_2} \right\} - \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2}}{\frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2}},$$

since

$$\beta_4 = \frac{5\beta_2^2}{6 - \beta_2} + \frac{r\beta_2^2(\beta_2 - 1)}{6 - \beta_2}.$$

Hence, on simplifying we find

$$\Phi = \frac{9r(6 - \beta_2) - 490}{7(124 - 9\beta_2)} \dots\dots\dots(13).$$

Let us denote this limiting surface $\beta_6 = \beta_2\beta_4$ by $F_1 = 0$ and this limiting value of Φ by Φ_1 . For given β_2 , Φ lies along a straight line which passes through the point ($r = -7$, $\Phi = -1$). I shall accordingly calculate Φ_1 for $r = 0$ and 7 , and for different values of β_2 .

TABLE XXII a.

β_2	$\Phi_1(0)$	$\Phi_1(7)$	β_2	$\Phi_1(0)$	$\Phi_1(7)$
1.25	-.621	-.242	3.5	-.757	-.514
1.5	-.634	-.267	4	-.796	-.591
1.75	-.647	-.293	4.5	-.838	-.677
2	-.660	-.321	5	-.886	-.772
2.5	-.690	-.379	5.5	-.940	-.879
3	-.722	-.443			

(ii) The values of β_4 and β_6 corresponding to the second order symmetrical Pearson curves are

$$\beta_4 = \frac{5\beta_2^2}{6 - \beta_2}, \text{ i.e. } r = 0 \text{ and } \beta_6 = \frac{35\beta_2^3}{(9 - 2\beta_2)(6 - \beta_2)}.$$

$$\text{Hence } \Phi = \left\{ \frac{35\beta_2^3}{(9 - 2\beta_2)(6 - \beta_2)} - \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} \right\} / \left\{ \frac{\beta_2^3(124 - 9\beta_2)(\beta_2 - 1)}{9(6 - \beta_2)^2} \right\} \quad (14).$$

$$= \frac{10(5\beta_2 - 9)}{(124 - 9\beta_2)(9 - 2\beta_2)}$$

Note that Φ is negative for $\beta_2 < 1.8$ and for $\beta_2 > 4.5$.

164 *On certain Non-Normal Symmetrical Frequency Distributions*

If $\beta_2 = 4.5$, Φ becomes infinite. By substituting for β_2 we can find the Φ ordinate corresponding to that value of β_2 . We find

TABLE XXII *b*.

β_2	Φ	β_2	Φ
1.25	-0.038	2.5	+0.086
1.5	-0.023	3	+0.206
1.75	-0.002	3.5	+0.483
2	+0.019	4	+1.189

(*f*) I will now discuss the curves of section corresponding to critical values of β_2 , viz. $\beta_2 = 1, 1.8, 2\frac{1}{2}, 3$.

(i) $\beta_2 = 1$. We now have the "two-lumps" distribution for which $\beta_2 = \beta_4 = \beta_6 = 1$. Now the points *O* and *I* coincide, in other words, for all values of r , β_4 always remains unity. I have added the curves of section in the case $\beta_2 = 1$ to each of the contour diagrams for the sake of completeness.

(ii) $\beta_2 = 1.8$. The curve of section of the surface $F_4 = 0$ reduces to the two coincident straight lines $\beta_4 = 2\frac{1}{2}$. The discussion is similar to that for $\beta_2 = 3$.

(iii) $\beta_2 = 2\frac{1}{2}$. The equation for the surface $F_6 = 0$ simplifies. Here also the discussion must unfortunately be omitted owing to lack of space.

(iv) $\beta_2 = 3$. The section of the surface $F_2 = 0$ by the plane $\beta_2 = 3$ reduces to the two parallel straight lines $\beta_4 = 15$ and $7\beta_4 = 45$, i.e. $r = 0$ and $r = -10$, whereas the curves of section of the surface $F_6 = 0$ reduce to the straight line $\beta_4 = 15$ and the parabola $3\beta_6 - 275 = 49\beta_4^2 + 42\beta_4 + 9945$, i.e.

$$\Phi_6 = \frac{3\beta_6 - 275}{194} = \frac{r[600 + 36r + r^2]}{2716}, \text{ since } \beta_4 = 15 + \frac{6r}{7}.$$

Now
$$m_1 = 9 \frac{(\beta_6 - \phi_4)}{(15 - \beta_4)},$$

$$q_1 p_1 = - \frac{9(\beta_6 - \phi_0)}{(15 - \beta_4)},$$

$$q_1 + p_1 = -\frac{1}{3}(7\beta_4 - 45),$$

$$(q_1 - p_1)^2 = \frac{1296\beta_6 - [49\beta_4^3 - 693\beta_4^2 + 9315\beta_4 - 30375]}{36(15 - \beta_4)},$$

$$m_1^2 - (q_1 - p_1)^2 = \frac{81}{4(15 - \beta_4)^2} (\beta_6 - \phi_6') (\beta_6 - \phi_6'').$$

The values of the ordinates $\Phi_0, \Phi_4, \Phi_6, \Phi_8$ are shown in Table XVI, p. 161, and the transition curves drawn in Diagrams XVI and XVII (right-hand figures).

We find that $F_6 = 0$ lies below both $F_0 = 0$ and $F_4 = 0$ for $\beta_4 < 15$ and above for $\beta_4 > 15$. Hence for points on $F_6 = 0$, $m_1 < 0$, $p_1 q_1 > 0$ and $q_1 + p_1 < 0$, and so we have the Type (vii) curve. Between the line $\beta_4 = 15$ and the curve $F_6 = 0$, $(q_1 - p_1)^2 < 0$ and we find a heterotypic area.

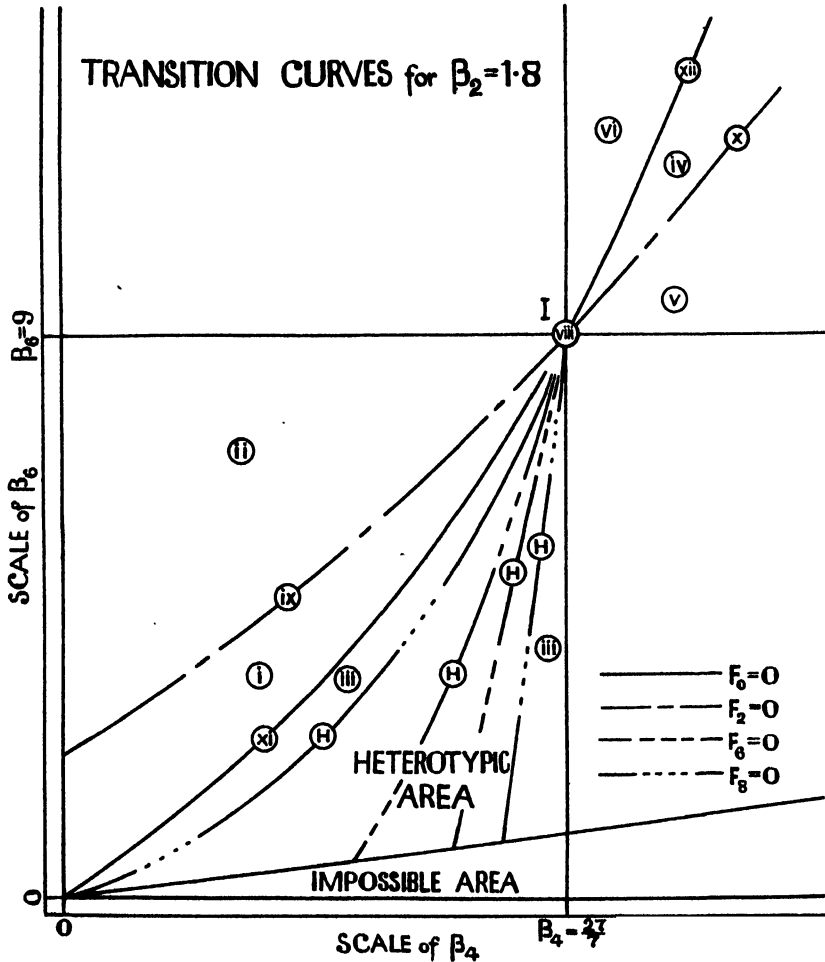


Diagram IV.

Again, the area which corresponds to the Type (iii) curve, viz. that between $F_0 = 0$ and $F_4 = 0$, always lies between the two branches of $F_8 = 0$ and hence the condition $\frac{m}{q_1 - p_1} < 1$ is always satisfied. Furthermore, points on $F_8' = 0$ correspond to the Type (xiii) curve when $\beta_4 < 15$ and to the Type (xv) when $\beta_4 > 15$; on the other hand, for $F_8'' = 0$ we first get the Type (xv) for $\beta_4 < 15$ and then the Type (xiii) for $\beta_4 > 15$.

TABLE XXIII*a*.(1) $\beta_4 < 10\frac{1}{2}$.

Zone	Signs of			Type
	m_1	$q_1 p_1$	$q_1 + p_1$	
Above $F_4 = 0$	+	—	—	(i)
On $F_4 = 0$	0	—	—	(viii)
Between $F_4 = 0$ and $F_0 = 0$	—	—	—	(iv)
On $F_0 = 0$	—	0	—	(xii)
Below $F_0 = 0$	—	+	—	(vi)

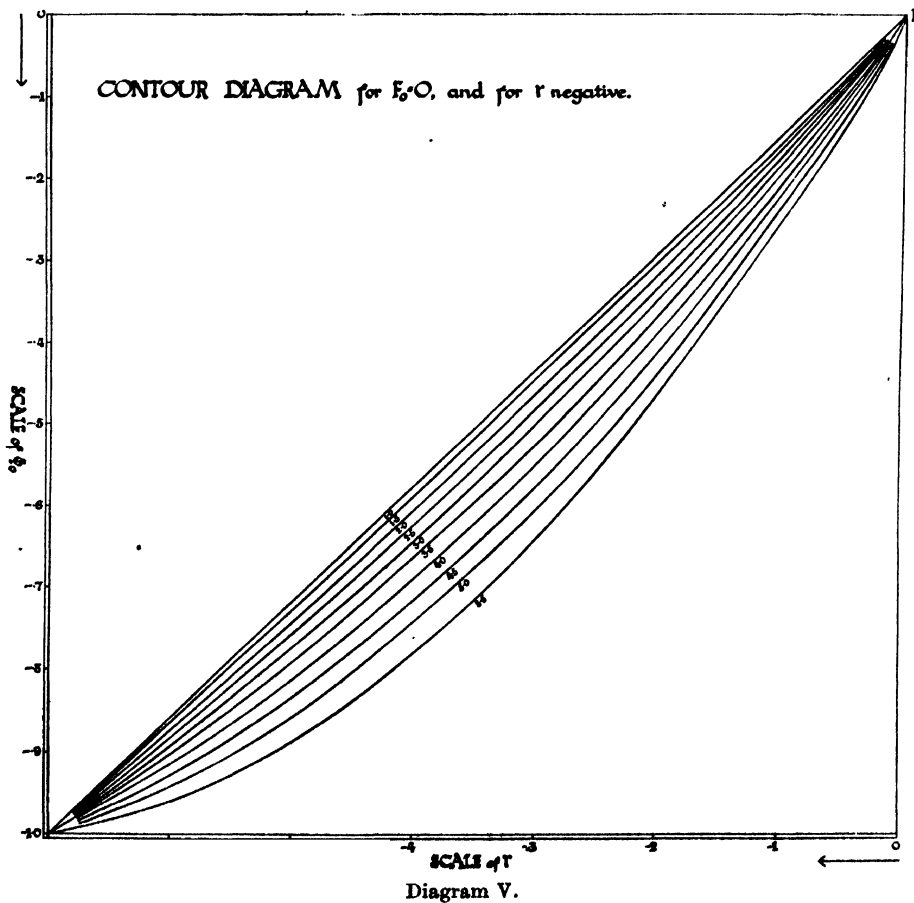
TABLE XXIII*b*.(2) $10\frac{1}{2} < \beta_4 < 15$.

Zone	Signs of			Type
	m_1	$q_1 p_1$	$q_1 + p_1$	
Above $F_0 = 0$	+	—	—	(i)
On $F_0 = 0$	+	0	—	(xi)
Between $F_0 = 0$ and $F_4 = 0$	+	+	—	(iii)
On $F_4 = 0$	0	+	—	(viii)
Below $F_4 = 0$	—	+	—	(vi)

TABLE XXIIIc.

(3) $\beta_2 > 15$.

Zone	Signs of			Type
	m_1	$q_1 p_1$	$q_1 + p_1$	
Above $F_4=0$	-	+	-	(vi)
On $F_4=0$	0	+	-	(viii)
Between $F_4=0$ and $F_0=0$	+	+	-	(iii)
On $F_0=0$	+	0	-	(xi)
Below $F_0=0$	+	-	-	(i)



(g) The contours of the transition surfaces.

In plotting the Φ -ordinate for different values of r , we find it convenient to make separate diagrams for $r > 0$ and $r < 0$.

(i) The contour diagrams for the surface $F_0 = 0$ are straightforward. Note that each curve of section must pass through the points $(-7, -1)$ and $(0, 0)$ and that Φ_0 decreases as β_2 increases. See Diagrams V and VI.

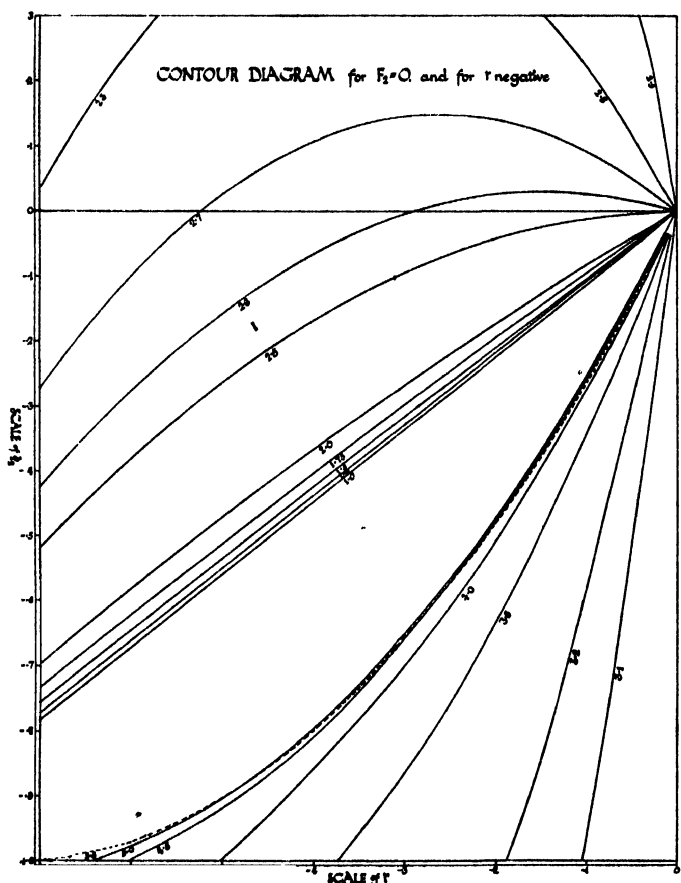


Diagram VI.

(ii) The contour diagrams for $F_2 = 0$ and $F_4 = 0$, shown in Diagrams VII, VIII and IX, present greater difficulty and we need more ordinates. As before we easily find the Φ -ordinates for

$r = -7 - (1.0) - +7$ and $\beta_2 = 1.7 - (1.5) - 1.9$ and $\beta_2 = 2.6(-1) - 2.9, 3.1, \text{ and } 3.2$.

Let us now consider the contours of the surfaces of $F_2 = 0$ and $F_4 = 0$. When $r < 0$ increases as β_2 increases till it becomes infinite at $\beta_2 = 3$. It now changes sign and as β_2 is further increased, decreases. On the other hand, for $r < 0$, ϕ_4 increases as β_2 increases till it becomes infinite at $\beta_2 = 3$. It now changes sign and as β_2 is further increased, decreases.

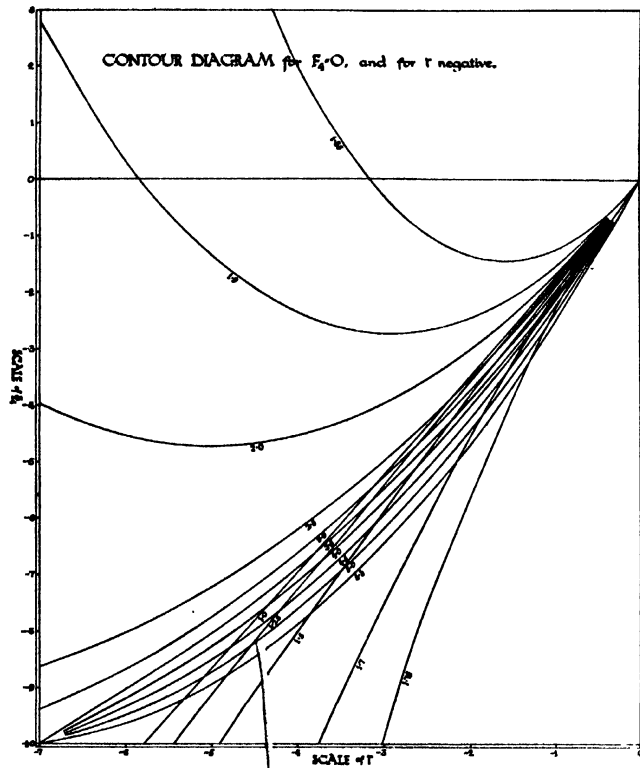


Diagram VII.

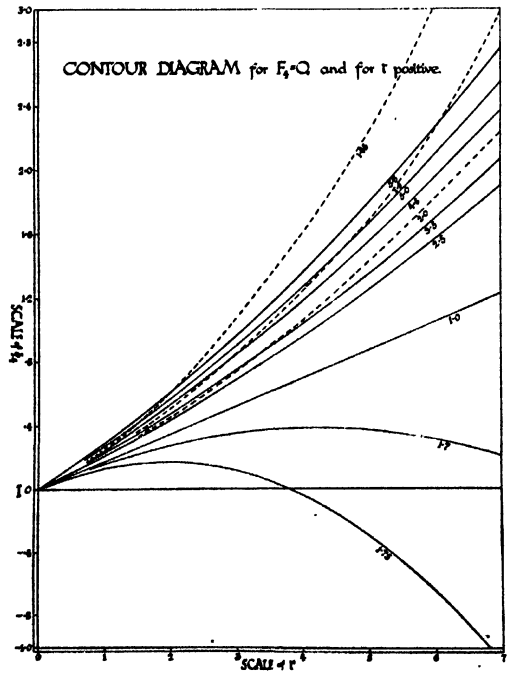
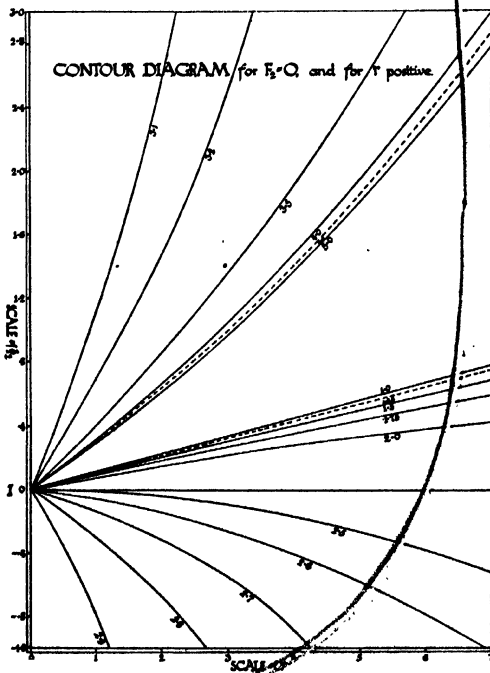


Diagram VIII.

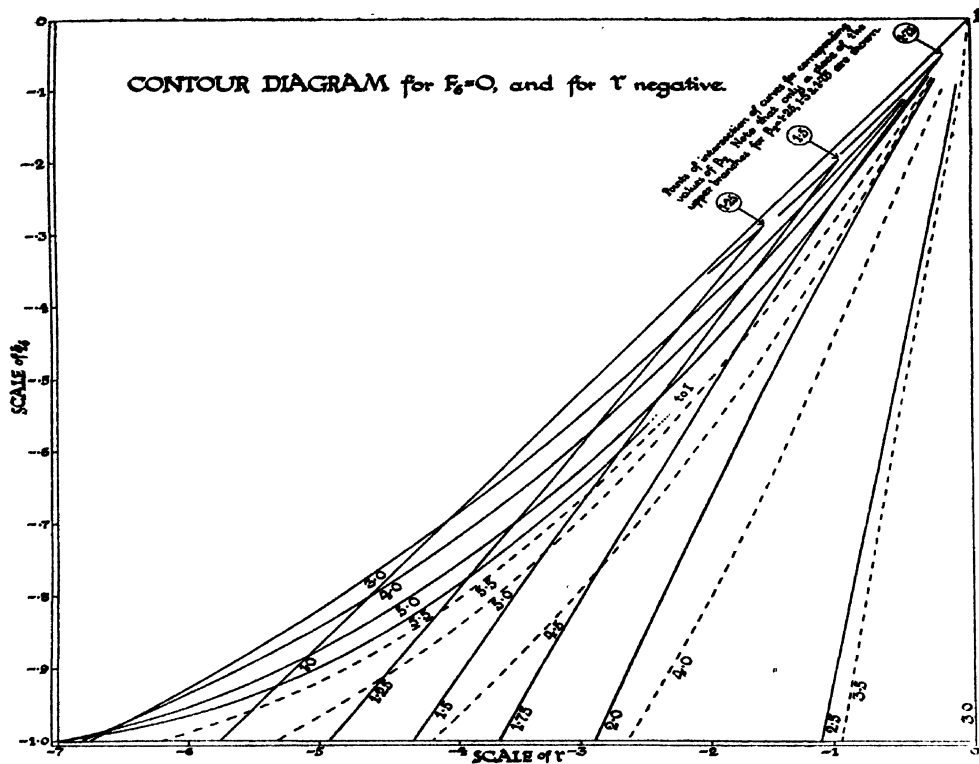


Diagram IX.

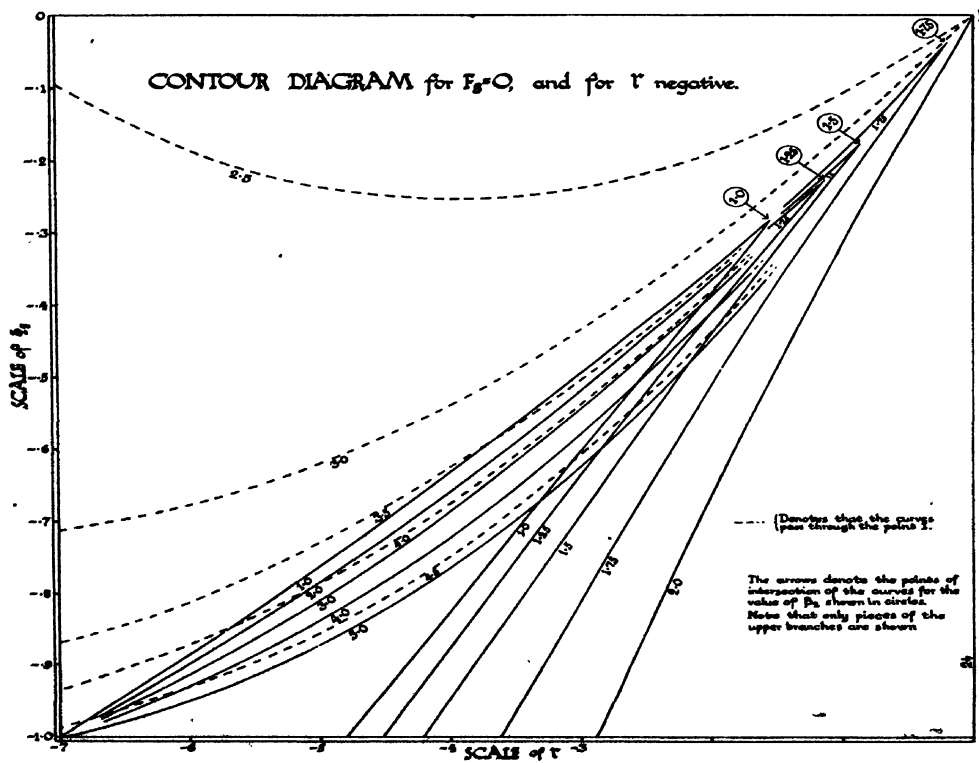


Diagram X.

decreases, becomes infinite at $\beta_2 = 1.8$, where it changes sign and then decreases again; but if $r > 0$, ϕ_4 decreases, changes sign at $\beta_2 = 1.8$, decreases till $\beta_2 = 2.5$ and then increases again.

The contours of the surfaces $F_6 = 0$ and $F_8 = 0$ are more complicated and I have omitted several of the lines to make their reproduction possible. Note that since the surfaces consist of two parts, each section gives two contour lines. If $r < 0$, the contour lines of $F_6' = 0$ are such that each member of the system cuts all the preceding members, e.g. the contour line for $\beta_2 = 2$ cuts the contour lines for $\beta_2 = 1.75$, $\beta_1 = 1.5$, and $\beta_2 = 1.25$. Furthermore, ϕ_6'' decreases as β_2 increases, becomes infinite at $\beta_2 = 3$, but does not change sign and then increases. Again, the lower of the contour lines for the surface $F_8 = 0$ always passes through the points $(0, 0)$ and $(-7, -1)$, whereas corresponding ordinates on the upper branch decrease until $\beta_2 = 2\frac{1}{2}$, where they become infinite, change sign and then decrease again, as β_2 is further increased.

The contour diagrams for $F_6 = 0$ and $F_8 = 0$ simplify for $r > 0$ because there are now fewer lines. ϕ_6' increases as β_2 increases, becomes infinite at $\beta_2 = 3$ and now decreases as β_2 increases. $F_6'' = 0$ represents a system of curves, such that each member cuts the preceding members. Both ϕ_6' and ϕ_6'' increase as β_2 increases.

(h) The Transition Curves.

Diagrams XIII to XXI show the sections of the transition surfaces by planes corresponding to different values of β_2 and the frequency types corresponding to the different areas*. These areas are determined as in Section I (g), p. 143, but I have now introduced the conditions

$$(q_1 - p_1)^2 = \frac{81(\beta_2 - 3)^2}{4K^2} (\beta_6 - \phi_6') (\beta_6 - \phi_6'') \geq 0$$

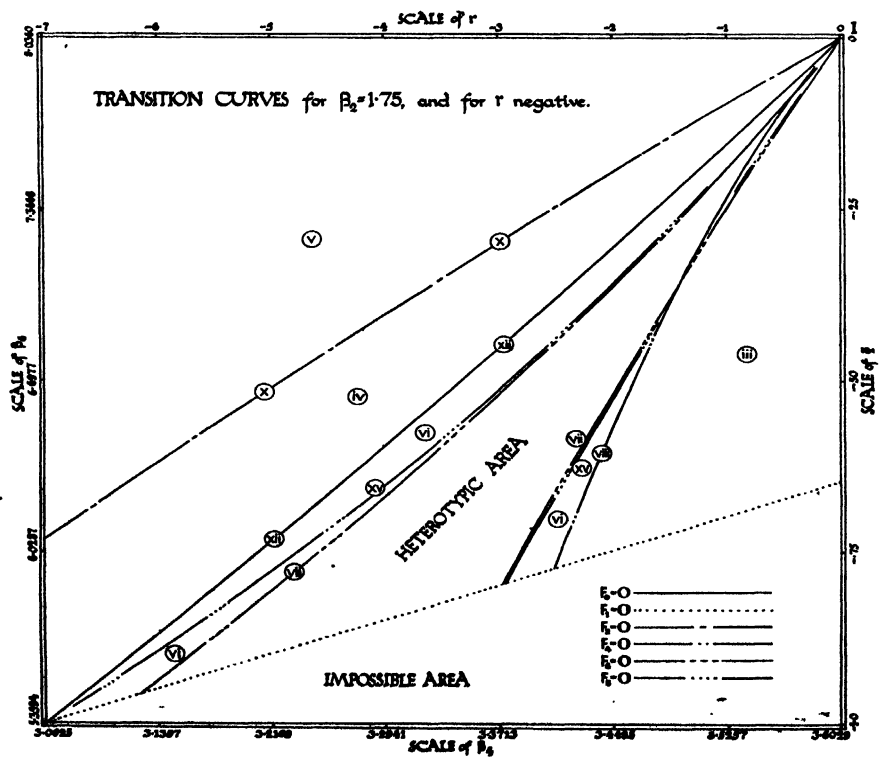
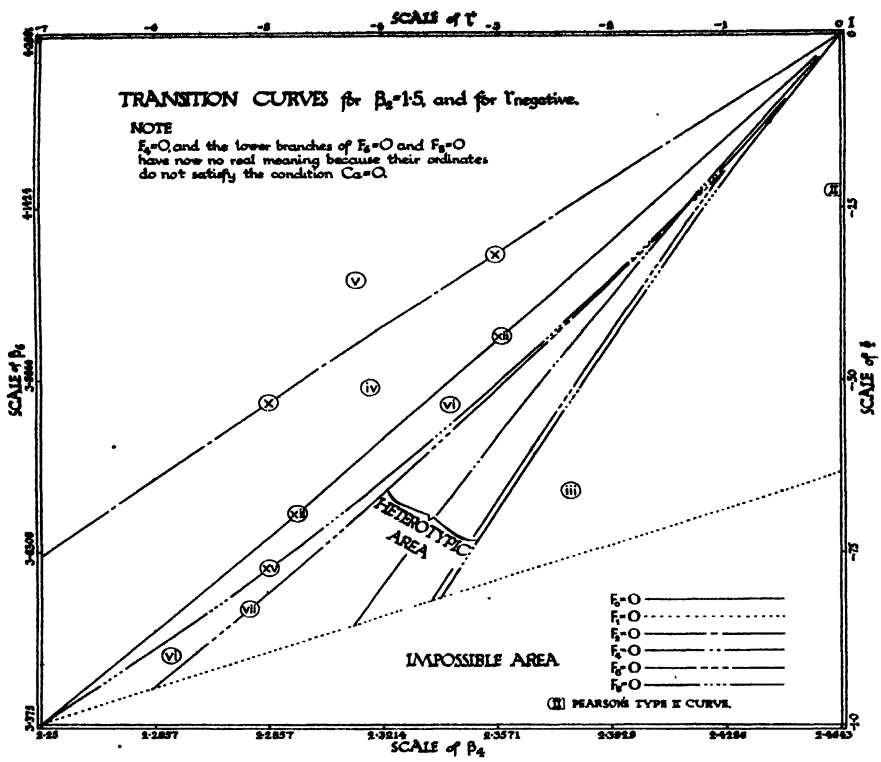
$$\text{and} \quad m_1^2 - (q_1 - p_1)^2 = \frac{243(\beta_2 - 1)(7\beta_2 - 15)}{16K^2} (\beta_6 - \phi_6') (\beta_6 - \phi_6'') \leq 0.$$

Accordingly the zones between the two branches of $F_6 = 0$ and also those parts of the zones for the main Types (iii), (iv), and (v) and the transition Types (x) where $m_1^2 - (q_1 - p_1)^2$ is not negative, are now heterotypic areas, points of which do not satisfy the condition $c_4 = 0$.

Furthermore, note that the Type (vii) can only occur when $F_6 = 0$ borders an area corresponding to the Type (vii) curve. Again, the Types (xiii), (xiv), (xv) must occur in the areas of the Type (i), (ii), and (vi) curves respectively. At all other points of $F_6 = 0$ and $F_8 = 0$ we get a different type of curve.

Clearly $m_1^2 - (q_1 - p_1)^2 < 0$ if we are outside the two branches of F_8 when $\beta_2 < 1\frac{1}{2}$ or if we are inside these two branches when $\beta_2 > 1\frac{1}{2}$.

* The heterotypic areas are the zones where we may get other types of curves, but not those considered above, and the impossible areas are those wherein the necessary inequalities $\beta_4 > \beta_2^2$, $\beta_4 > \beta_2\beta_4 > \beta_4^2$ do not hold.



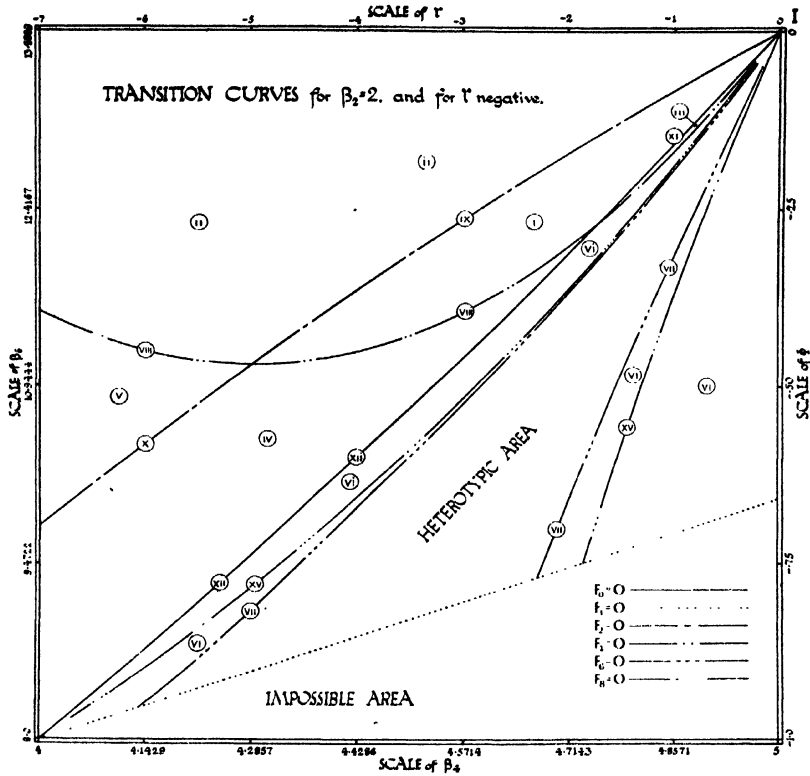


Diagram XIII.

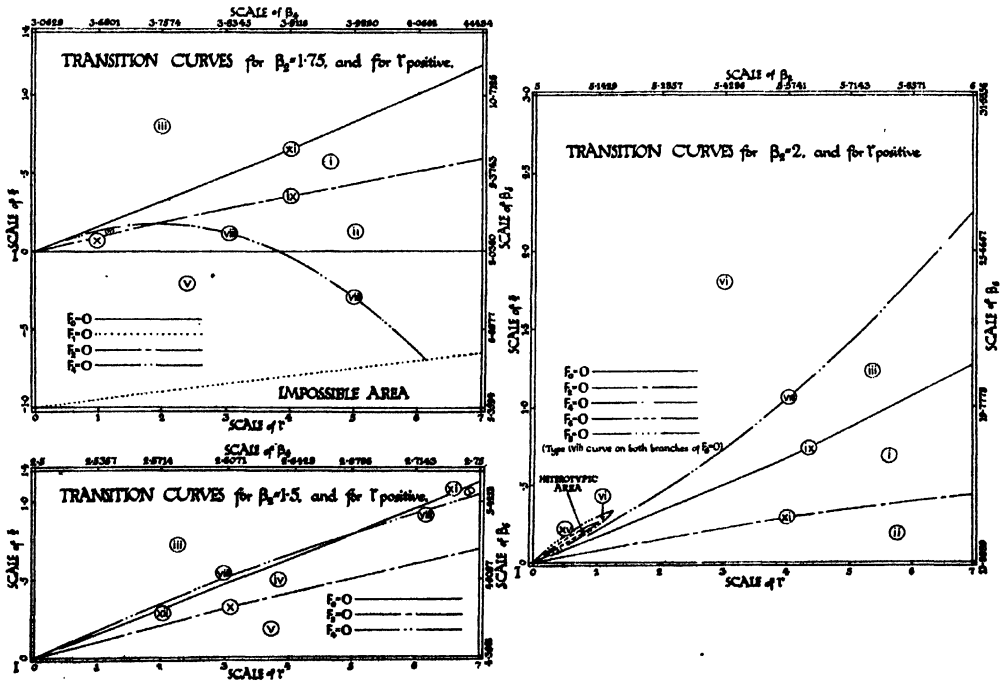
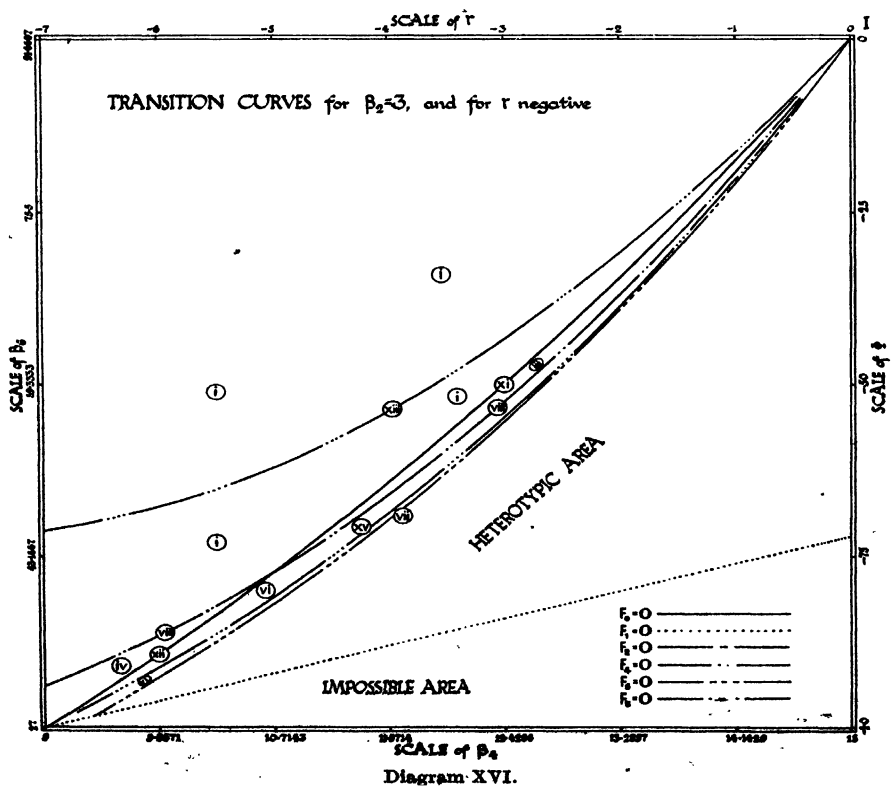
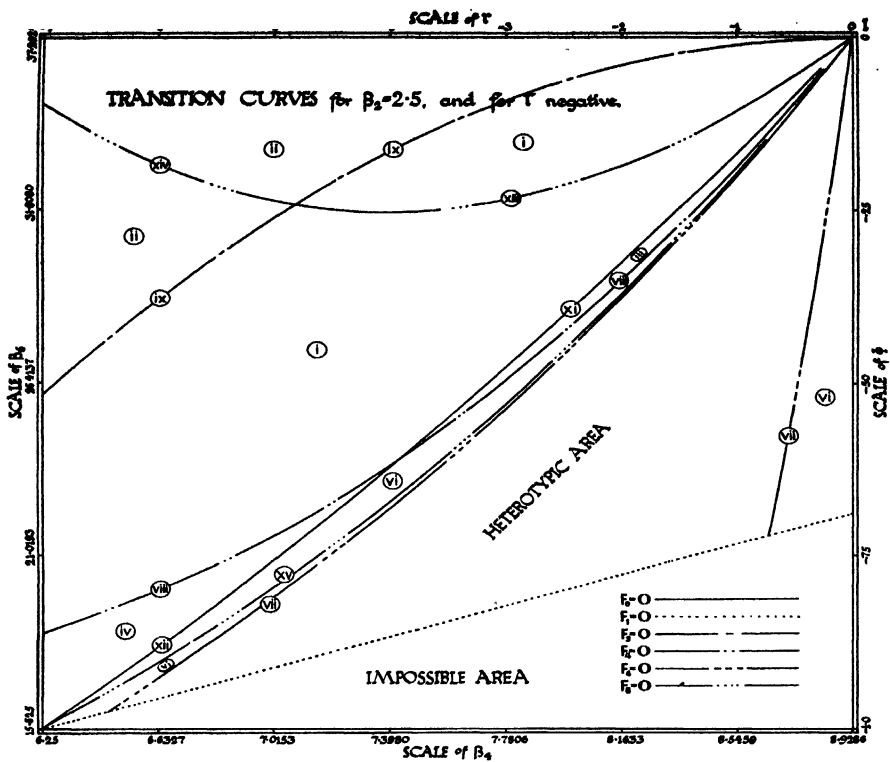


Diagram XIV.



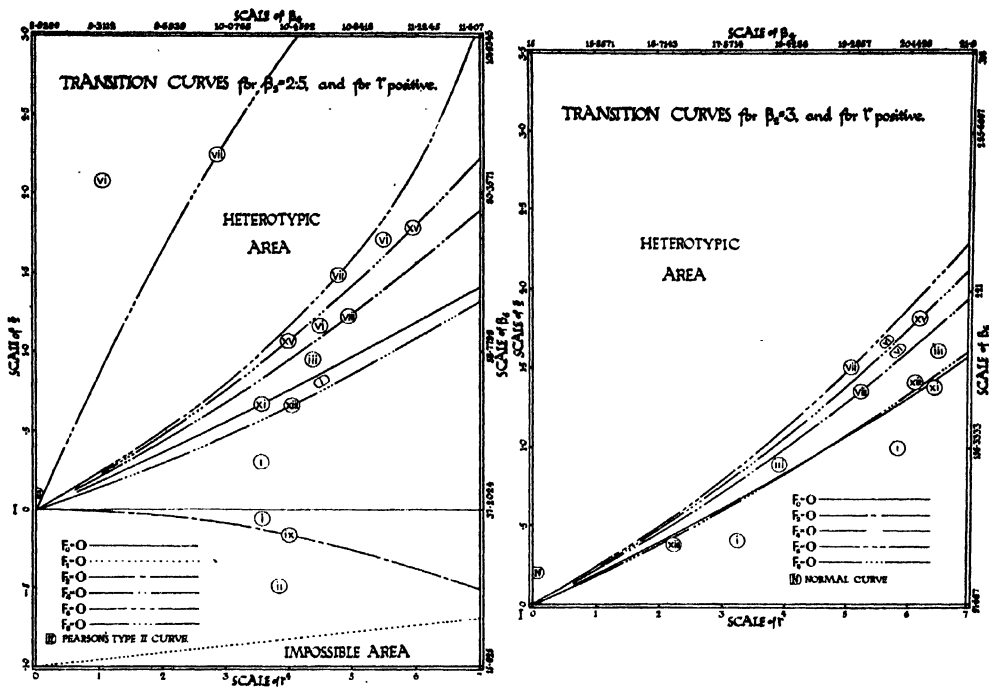


Diagram XVII.

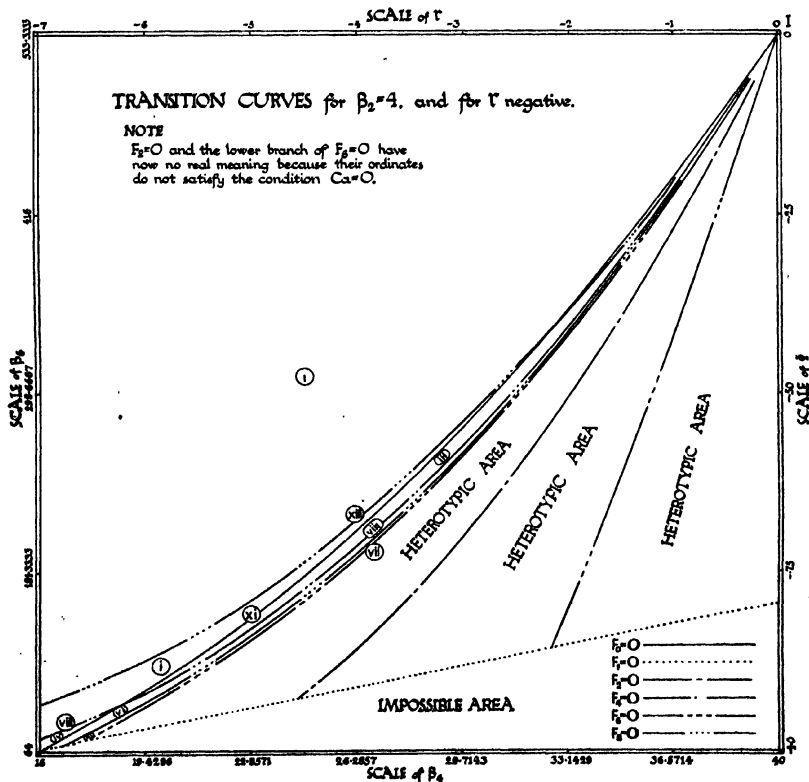
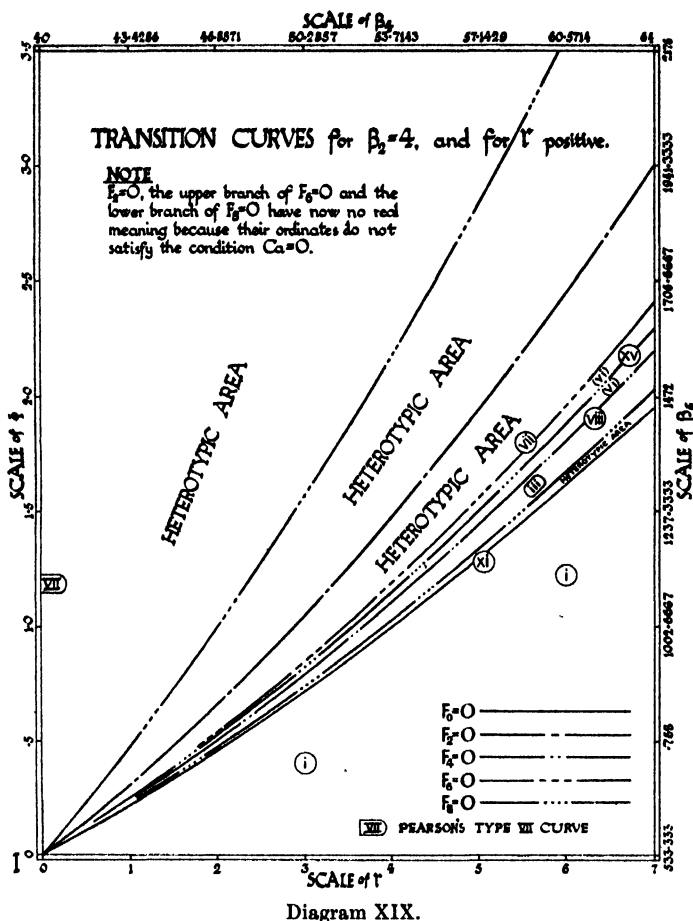


Diagram XVIII.



Note (1) that the sign of K does not trouble us.

(2) When $F_6' = 0$ and $F_6'' = 0$ become imaginary, $(q_1 - p_1)^2$ is always positive; when $F_3' = 0$ and $F_3'' = 0$ become imaginary, $m_1^2 - (q_1 - p_1)^2$ is positive or negative according as $\beta_2 >$ or $< 1\frac{1}{2}$.

In (f) of the first Section (see p. 141) I gave the values of β_4 and β_6 at the intersections of the transition surfaces with each other, but all these values do not come into the diagrams of the transition surfaces because I have given an upper limit to β_4 . Table XXIV gives the values of r and Φ at the intersections of the transition curves for different values of β_2 . These values check the Diagrams XI to XIX.

By painting the transition curves on glass and fixing the sheets corresponding to each value of β_2 in their right order, one above the other, we can make a model in the β_2, β_4 and β_6 -space showing the zones corresponding to each of the fourth order Pearson curves. But unfortunately the curves are so close to each other that the model becomes too complicated and its reproduction in print would be impossible.

TABLE XXIV.

β_2	Surfaces which intersect	Co-ordinates of points of intersection	
		r	ϕ
1.5	$F_8''=0, F_6'=0, \text{ and } F_4=0$ $F_0=0 \text{ and } F_4=0$	-1.302 5.0	- .2437 + .7919
1.75	$F_8''=0, F_6'=0 \text{ and } F_4=0$ $F_6''=0 \text{ and } F_4=0$ $F_0=0 \text{ and } F_4=0$ $F_2=0 \text{ and } F_4=0$	-1.333 - .409 + .556 +1.905	- .3326 - .0835 + .0877 + .1759
2.0	$F_0=0 \text{ and } F_4=0$ $F_2=0 \text{ and } F_4=0$	-1.667 -5.0	- .2621 - .4716
2.5	$F_8''=0 \text{ and } F_2=0$ $F_0=0 \text{ and } F_4=0$	-4.807 -3.889	- .2439 - .6066
3.0	$F_8''=0 \text{ and } F_0=0$ $F_0=0 \text{ and } F_4=0$	+5.0 -5.0	+1.0677 - .7732
4.0	$F_8'=0 \text{ and } F_0=0$ $F_0=0 \text{ and } F_4=0$	-1.667 -6.111	- .3247 - .9259

III. STATISTICAL APPLICATIONS.

(a) *General Remarks on Curve Fitting.*

A model would enable us to determine by inspection what type of curve must be used given the values of β_2 , β_4 and β_6 . This method is not practicable, because the model cannot readily be produced with sufficient accuracy for this purpose.

In the illustrations dealt with in this Section, I have shown how to determine the curve to be fitted:

(1) from first principles as in Example (d) on p. 188;

(2) by interpolating into the Tables XI—XXI and using Diagram I, Figs. I, II and III, to determine the type of curve, then finding if the point falls into a heterotypic area or corresponds to a transition curve on the surfaces $F_6=0$ or $F_8=0$ by considering the contour lines of these surfaces as in Example (e), p. 191;

(3) by inspection of the two neighbouring transition curve diagrams, etc., as in Example (f), p. 193.

I have discussed the transition curves at great length for $\beta_2 < 6$. For higher values of β_2 we must calculate the type of curve to be used from first principles, following the method used in Example (d), p. 188.

(b) *Distribution of Product Moment Coefficients, p_{11} .*

The distribution of the first product moment coefficient in samples drawn from an indefinitely large normal population is a Bessel-function* curve. When the sample is larger than 25 an excellent fit is obtained by using a Pearson curve with the same mean, standard deviation, and constants β_1 and β_2 . For smaller samples a Pearson curve cannot be used to replace the Bessel-function curve. The distribution of the first product moment coefficient is symmetrical when the variates of the population sampled are uncorrelated. The problem arises whether one of my curves with the same mean, the same standard deviation and the same constants β_2 , β_4 and β_6 can describe the corresponding symmetrical Bessel-function curve.

The distribution of $v = p_{11} \sqrt{\frac{n}{1-\rho^2}} \frac{1}{\sigma_1 \sigma_2}$ is

$$y = \frac{N(1-\rho^2)^{\frac{n-1}{2}}}{\sqrt{\pi} \cdot 2^{\frac{n-2}{2}} \Gamma\left(\frac{n-1}{2}\right)} e^{\rho v} \left\{ v^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(v) \right\}$$

$$= \frac{N(1-\rho^2)^{\frac{n-1}{2}}}{\sqrt{\pi} \cdot 2^{\frac{n-2}{2}} \Gamma\left(\frac{n-1}{2}\right)} e^{\rho v} T_m(v), \quad m = \frac{n-2}{2},$$

where n is the size of the sample, σ_1 , σ_2 and ρ refer to the parent population. Log $T_m(v)$ is tabled* for $m=0$ to $m=11.5$ and for different values of v .

The s th moment coefficient* of p_{11} about zero is

$$\mu_s' = \frac{I_s}{I_0},$$

where

$$I_s = A_s \int_0^\infty [e^{2\rho u} + (-1)^s e^{-2\rho u}] u^s F(u) du,$$

and

$$F(u) = \frac{1}{2} \Gamma\left(\frac{n-2}{2}\right) u^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(2u).$$

A_s is found to be

$$A_s = \frac{N 2^{n-2} (1-\rho^2)^{\frac{n-1}{2}}}{\pi \Gamma(n-2)} [2\sigma_1 \sigma_2 (1-\rho^2)]^s.$$

On evaluating I_s and dividing by I_0 we find

$$\mu_s' = 2 \left(\frac{\sigma_1 \sigma_2}{n} \right)^s \frac{\Gamma(n+s-1) \Gamma(s+1)}{\Gamma\left(\frac{n+2s+1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}.$$

* *Biometrika*, Vol. xxi. p. 164, sq.

The reduction formula for $f_s(\rho)$ is

$$f_s(\rho) = \frac{1}{2} \left(\frac{n-1}{2} + s \right) \int f_{s-1}(\rho) d\rho + \text{constant}.$$

The constant of integration is found from the two reductions

$$f_s(0) = 0 \text{ for odd } s,$$

and
$$f_s(1) = \frac{\Gamma\left(\frac{n+1}{2} + s\right) \Gamma\left(\frac{n-1}{2} + s\right)}{\Gamma(n+s-1) \Gamma(s+1)}.$$

$f_{-1}(\rho) = 0$, and hence we can find $f_r(\rho)$ ($r = 0, 1, \dots, 8$)*.

We find

$$f_5(\rho) = \frac{1}{768} \frac{\Gamma\left(\frac{n+11}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \left[\frac{\rho^5}{5} + \frac{2\rho^3}{n} + \frac{3\rho}{n(n+2)} \right],$$

$$f_6(\rho) = \frac{1}{46080} \frac{\Gamma\left(\frac{n+13}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \left[\rho^6 + \frac{15\rho^4}{n} + \frac{45\rho^2}{n(n+2)} + \frac{15}{n(n+2)(n+4)} \right],$$

$$f_7(\rho) = \frac{1}{92160} \frac{\Gamma\left(\frac{n+15}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \left[\frac{\rho^7}{7} + \frac{3\rho^5}{n} + \frac{15\rho^3}{n(n+2)} + \frac{15\rho}{n(n+2)(n+4)} \right],$$

$$f_8(\rho) = \frac{1}{10321920} \frac{\Gamma\left(\frac{n+17}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \left[\rho^8 + \frac{28\rho^6}{n} + \frac{210\rho^4}{n(n+2)} + \frac{420\rho^2}{n(n+2)(n+4)} + \frac{105}{n(n+2)(n+4)(n+6)} \right].$$

Put $\rho = 0$, then the s th moment about the mean is

$$\mu_s = \left(\frac{2\sigma_1\sigma_2}{n} \right)^s \frac{\Gamma(n+s-1) \Gamma(s+1)}{\Gamma\left(\frac{n+2s+1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} f_s(0).$$

Putting $s = 2, 4, 6, 8$ and substituting for $f_s(0)$, we can determine μ_s , and therefore the corresponding β 's. We easily find

$$\beta_2 = \frac{3(n+1)}{n-1},$$

$$\beta_4 = \frac{15(n+1)(n+3)}{(n-1)^2},$$

$$\beta_6 = \frac{105(n+1)(n+3)(n+5)}{(n-1)^3}.$$

* $f_0(\rho)$, $f_1(\rho)$, $f_2(\rho)$, $f_3(\rho)$ and $f_4(\rho)$ are worked out *loc. cit.*—the higher functions follow after some lengthy algebra.

Hence from the equations of (b), p. 136, we find

$$\begin{aligned} p_1 q_1 &= -\frac{(n+1)(n+3)(n+15)}{n-1} = -qp, \\ p_1 + q_1 &= -\frac{n^2 + 28n + 90}{n-1} = -(q-p), \\ m_1 &= \frac{n^3 + 22n^2 + 132n + 300}{2(n-1)} = m, \end{aligned}$$

where $q = |q_1|$, $p = |p_1|$, $m = |m_1|$, $q > p$. Therefore $q = -q_1$, $p = p_1$, $m = m_1$.

Similarly

$$k' = \frac{m}{q+p} = \frac{1}{2} \frac{n^3 + 22n^2 + 132n + 300}{\sqrt{(5n^4 + 128n^3 + 1140n^2 + 4968n + 7920)}}$$

on simplification. The size of the sample, n , is > 0 , and therefore my Type (i) curve must always be used. The equation is

$$y = y_0 \left[\frac{1 - \frac{u^2}{q}}{1 + \frac{u^2}{p}} \right]^{k'}$$

where

$$u = \frac{v}{\sigma_v} = \frac{v}{\sqrt{(n-1)}}.$$

Taking the total frequency to be 1000 in each case, I have calculated the ordinates and the frequencies for samples of size $n = 3, 5, 10, 25$ and 50 , for the Type VII curve, the normal curve, and my Type (i) curve. I compared the values found with the corresponding values of $y = 1000 T_m$, as found from the tables, the ordinates by plotting, and the frequencies by using the χ^2 -test for goodness of fit. Strictly speaking, we cannot use the χ^2 -test in this case, because the test assumes an observed distribution, found in sampling. I used it because I know of no better test for comparing the frequencies.

For the Type (i) curve, I found y_0 by quadrature. I took

$$y_0'' = c_{\frac{n-2}{2}}(0) = 1000 T_{\frac{n-2}{2}}(0)$$

to find the ordinates, and then found the correcting factor (y_0') which would reduce the area under the curve to 1000. Thus the true value for

$$y_0 = (y_0') \times (y_0'') = (y_0') \times 1000 T_{\frac{n-2}{2}}(0).$$

The equations at the top of this page give the following values for the constants of the curves:

TABLE XXV.

n	p	q	k'	y_0'	$c_{\frac{n-2}{2}}$
3	2.3027	93.8027	2.3958	0.6995	500.00
5	3.5653	67.3153	2.8834	0.9495	250.00
10	6.7372	58.9594	4.0760	0.9864	145.51
25	16.1536	75.1120	7.5273	0.9985	84.09
50	31.6961	113.1247	13.1690	1.0004	57.83

Professor Pearson's Type VII curve, viz.

$$y = y_0 \frac{1}{\left(1 + \frac{v^2}{a^2}\right)^l},$$

where $y_0 = \frac{N}{\sqrt{\pi(n^2 - 1)}} \frac{\Gamma\left(\frac{n+4}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}, \quad a^2 = n^2 - 1, \quad l = \frac{1}{2}(n+4),$

gives the following constants:

TABLE XXVI.

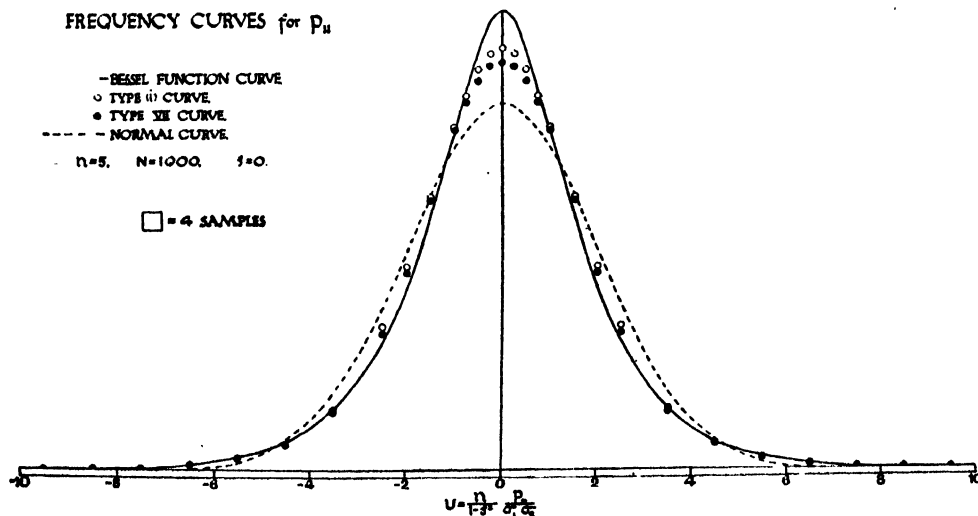
n	a^2	l	y_0
3	8	3.5	331.45
5	24	4.5	223.26
10	99	7.0	141.81
25	624	14.5	83.76
50	2499	27.0	57.83

The normal curve is

$$y = \frac{1000}{\sigma \sqrt{(2\pi)}} e^{-\frac{1}{2} \frac{v^2}{n-1}}.$$

I calculated the ordinates and the frequencies for each of the three types when $n = 3, 5, 10, 25, 50$. The ordinates when $n = 5$ and $n = 10$ are plotted in Diagrams XX and XXI. The values of P are approximately equal for the Type (i) and the Type VII curves, but the normal curve fits badly. The values of χ^2 however show that the Type (i) can better describe the Bessel-function curve than the Type VII, and this is also shown by the diagrams. For samples of 10 the Type (i) curve effectively replaces the Bessel-function curve, for values of $n > 25$ the Type VII and the Type (i) curves become identical.

FREQUENCY CURVES for p_u



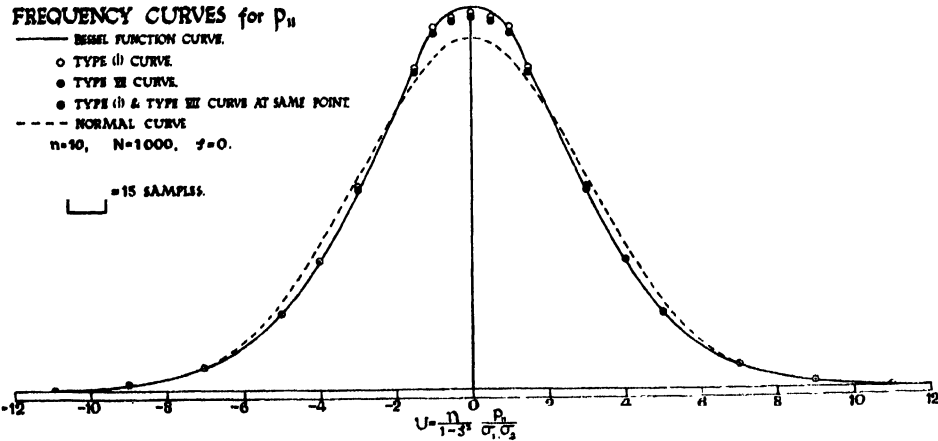


Diagram XXI.

(c) *Distribution of μ_3 in samples.*

Dr J. Pepper* drew samples of 10 from a normal paper with variance 100 and calculated the 3rd moment coefficient of each sample. He found the following distribution:

TABLE XXVII†.

Group	Frequency		Group	Frequency		Group	Frequency		Group	Frequency		Group	Frequency	
	+	-		+	-		+	-		+	-		+	-
0—1	42	49	15—	7	4	30—	—	—	45—	—	1	60—	—	—
1—2	42	31	16—	7	15	31—	—	—	46—	—	—	61—	—	—
2—3	42	36	17—	5	3	32—	1	1	47—	—	—	62—	—	—
3—4	27	21	18—	4	2	33—	1	1	48—	—	—	63—	—	—
4—5	26	28	19—	3	2	34—	2	1	49—	—	—	64—	—	—
5—6	28	17	20—	4	3	35—	2	—	50—	—	—	65—	—	—
6—7	15	17	21—	5	2	36—	—	—	51—	—	1	66—	—	—
7—8	19	12	22—	4	2	37—	2	—	52—	—	—	67—	—	—
8—9	15	15	23—	1	7	38—	1	—	53—	—	—	68—	—	—
9—10	11	2	24—	2	4	39—	—	—	54—	—	—	69—	—	—
10—11	9	16	25—	1	1	40—	—	—	55—	—	—	70—71	1	1
11—12	7	8	26—	1	4	41—	—	—	56—	—	—	Total	Frequency	
12—13	4	3	27—	1	4	42—	—	3	57—	—	—		+	-
13—14	7	8	28—	1	—	43—	1	—	58—	—	—		366	334
14—15	11	8	29—	3	1	44—	1	—	59—	—	—		700	

* I am greatly indebted to Dr Pepper for placing his sampling data at my disposal.

† I denote by Frequency + and Frequency - the frequency in groups to the right and to the left of the origin respectively.

We want Sheppard's correction formulae for the higher μ 's. Now the general formula* is

$$\nu_p = \mu_p + \frac{h^2}{3!2^2} p(p-1) \mu_{p-2} + \frac{h^4}{5!2^4} p(p-1)(p-2)(p-3) \mu_{p-4} + \dots,$$

where $\nu_p = p$ th raw moment coefficient about the mean and $\mu_p =$ corrected p th moment coefficient.

Hence

$$\left. \begin{aligned} \mu_5 &= \nu_5 - h^2 \cdot \frac{5}{8} \mu_3 \\ \mu_6 &= \nu_6 - h^2 \cdot \frac{5}{4} \mu_4 - h^4 \cdot \frac{3}{16} \mu_2 - \frac{1}{4} \frac{1}{8} h^6 \\ \mu_7 &= \nu_7 - h^2 \cdot \frac{7}{4} \mu_5 - h^4 \cdot \frac{7}{16} \mu_3 \\ \mu_8 &= \nu_8 - h^2 \cdot \frac{7}{3} \mu_6 - h^4 \cdot \frac{7}{8} \mu_4 - h^6 \cdot \frac{1}{16} \mu_2 - \frac{h^8}{2304} \end{aligned} \right\} \quad (15).$$

Again, if ν_p' denotes the uncorrected moment coefficient about the origin, then

$$\nu_p = \nu_p' - p\nu_{p-1}'\nu_1' + \frac{p(p-1)}{2!} \nu_{p-2}'\nu_1'^2 - \dots$$

Hence

$$\begin{aligned} \nu_5 &= \nu_5' - 5\nu_4'\nu_1' + 10\nu_3'\nu_1'^2 - 10\nu_2'\nu_1'^3 + 4\nu_1'^5, \\ \nu_6 &= \nu_6' - 6\nu_5'\nu_1' + 15\nu_4'\nu_1'^2 - 20\nu_3'\nu_1'^3 + 15\nu_2'\nu_1'^4 - 5\nu_1'^6, \\ \nu_7 &= \nu_7' - 7\nu_6'\nu_1' + 21\nu_5'\nu_1'^2 - 35\nu_4'\nu_1'^3 + 35\nu_3'\nu_1'^4 - 21\nu_2'\nu_1'^5 + 6\nu_1'^7, \\ \nu_8 &= \nu_8' - 8\nu_7'\nu_1' + 28\nu_6'\nu_1'^2 - 56\nu_5'\nu_1'^3 + 70\nu_4'\nu_1'^4 - 56\nu_3'\nu_1'^5 + 28\nu_2'\nu_1'^6 - 7\nu_1'^8. \end{aligned}$$

The following constants were found:

$$\begin{aligned} \mu_1 &= .1957, & \sigma &= 12.334, & \beta_1 &= .0102, \\ \beta_2 &= 7.538, & \beta_4 &= 139.04, & \beta_6 &= 3702.24, \end{aligned}$$

$$p_1 + q_1 = -57.815 = -(q-p), \quad p_1 q_1 = -40.208 = -qp, \quad m_1 = 87.980 = m;$$

therefore
$$p = .6873, \quad q = 58.50, \quad k' = 1.4864.$$

The range is from $x = -94.3$ to $x = +94.3$.

The Type (i) curve accordingly is

$$y = y_0 \cdot \left\{ \frac{1 - \frac{x^2}{8899.45}}{1 + \frac{x^2}{104.55}} \right\}^{1.4864}.$$

I calculated ordinates at $x = 0 - (.125) - .5$, $x = 1.5 - (.2) - .95$ and found the frequencies in the corresponding groups. The fit is very bad. The frequencies in the first twelve groups are:

* Whittaker and Robinson, *Calculus of Observations*, p. 196.

Group	Theoretical	Observed +	Observed -
0-1	35.1	42	49
1-2	34.2	42	31
2-3	32.3	42	36
3-4	29.8	27	21
4-5	27.0	26	28
5-6	24.1	28	17
6-7	21.2	15	17
7-8	18.5	19	12
8-9	16.0	15	15
9-10	13.8	11	2
10-11	11.9	9	16
11-12	10.3	7	8

Pepper also obtained the values for higher β 's of the distribution of the 3rd moment in samples*, viz.

$$\beta_2 = \frac{3(n^2 + 27n - 70)}{(n-1)(n-2)}$$

$$\beta_4 = \frac{15(n^4 + 84n^3 + 2695n^2 - 15,168n + 20,020)}{(n-1)^2(n-2)^2} \quad \dots(16).$$

$$\beta_6 = \frac{105 \left\{ \begin{array}{l} n^6 + 171n^5 + 13,893n^4 + 580,401n^3 - 5,131,014n^2 \\ + 14,132,268n - 12,932,920 \end{array} \right\}}{(n-1)^3(n-2)^3}.$$

Hence substituting for $\beta_2, \beta_4, \beta_6$ in the equations giving $p_1q_1, p_1 + q_1$ and m_1 we obtain, after some very lengthy algebra,

$$p_1q_1 = \frac{14\beta_4^2 - 9\beta_2\beta_6 - 5\beta_2^2\beta_4}{5\beta_2^2 - \beta_4(6 - \beta_2)}$$

$$= -\frac{1}{18} \frac{[A]}{[B]} \dots\dots\dots(17 a),$$

where $A = n^8 + 348n^7 + 50,975n^6 + 4,064,292n^5 + 34,629,558n^4$

$$- 646,758,090n^3 + 2,928,581,096n^2$$

$$- 5,507,413,800n + 3,806,202,400,$$

and $B = (n-1)(n-2)(3n^4 + 642n^3 - 4879n^2 + 12,228n - 10,220),$

$$p_1 + q_1 = \frac{9\beta_6(3 - \beta_2) + 7\beta_4^2 - 50\beta_2\beta_4 + 25\beta_2^3}{2\{5\beta_2^2 - \beta_4(6 - \beta_2)\}}$$

$$= -\frac{1}{12} \frac{[C]}{[B]} \dots\dots\dots(17 b),$$

* *Biometrika*, Vol. xxiv. p. 60.

where
$$C = 10n^7 + 3546n^6 + 521,736n^5 + 34,738,968n^4 \\ - 392,792,064n^3 + 1,575,584,616n^2 \\ - 2,789,485,552n + 1,854,410,880.$$

Further
$$m_1 = \frac{9\beta_6(5\beta_2 - 9) - 49\beta_4^2 + 210\beta_2\beta_4 - 125\beta_2^3}{4(6\beta_2^2 - 6 - \beta_2\beta_4)} \\ = \frac{1}{36} \frac{[D]}{[B]} \dots\dots\dots(17 c),$$

where
$$D = n^8 + 393n^7 + 66,122n^6 + 6,216,894n^5 + 187,648,764n^4 \\ - 2,369,922,228n^3 + 9,840,116,048n^2 \\ - 17,754,485,784n + 11,957,893,360.$$

For all positive integral values of n the signs of p_1q_1 , $q_1 + p_1$ and m_1 are negative, negative, and positive respectively. Hence we find in all cases that the Type (i) curve has the same first eight moment coefficients as the actual curve of distribution of μ_3 .

Let us regroup the data, in groups of 5 times Pepper's grouping unit, and compare the distribution so obtained with the curves fitted from the theoretical values and observed values of β_2 , β_4 and β_6 .

The regrouped distribution is

TABLE XXVIII.

Group	Frequency		Group	Frequency		Group	Frequency		Group	Frequency		Group	Frequency	
	+	-		+	-		+	-		+	-		+	-
0-5	179	165	15-20	26	26	30-35	4	3	45-50	—	1	60-65	—	—
5-10	88	63	20-25	16	18	35-40	5	—	50-55	—	1	65-70	—	—
10-15	38	43	25-30	7	10	40-45	2	3	55-60	—	—	70-75	1	1

We want to have the standard deviation of the curve calculated from the theoretical betas expressed in the same units as the standard deviation of the distribution above.

Now
$$\sigma_{\mu_3}^2 = \frac{6(n-1)(n-2)}{n^3} \mu_2^3,$$

and μ_3 was taken = 100 and the size of the sample is $n = 10$. Hence

$$\sigma_{\mu_3}^2 = \frac{6 \cdot 9 \cdot 8}{10^3} \cdot 10^6 = 432,000$$

in actual units. Now the interval between successive groups in the distribution is $5\sigma = 50$ and therefore

$$\sigma_{\mu_3} = \frac{432,000}{2500} = 172.8.$$

Put $n = 10$ in the Equations (17 a), (17 b) and (17 c) and we obtain

$$\begin{array}{ll} q_1 + p_1 = -554.69, & \text{therefore } p = 1.8823, \\ q_1 p_1 = -1047.65, & \text{,, } q = 556.572, \\ m_1 = 2076.89, & \text{,, } k' = 2.372. \end{array}$$

Hence the curve required is

$$y = 28.183 \left(\frac{1 - x^2 \times .00001040}{1 + x^2 \times .003074} \right)^{2.372}.$$

The regrouped data corresponds to the Type (i) curve, viz.

$$y = 182.37 \left(\frac{1 - x^2 \times .00010442}{1 + x^2 \times .010480} \right)^{1.4708}.$$

The following frequencies were found:

TABLE XXIX.

Group	Theoretical*		Observed		Group	Theoretical		Observed	
	(a)	(b)	+	-		(a)	(b)	+	-
0—5	132.9	162.5	179	165	50—55	.64	.70	—	1
5—10	96.4	92.9	88	63	55—60	.43	.51	—	—
10—15	56.0	43.9	38	43	60—65	.29	.35	—	—
15—20	29.5	21.5	26	26	65—70	.20	.23	—	—
20—25	14.7	11.4	16	18	70—75	.15	.15	1	1
25—30	8.1	6.6	7	10	75—80	.11	.09	—	—
30—35	4.5	4.0		3	80—85	.08	.05	—	—
35—40	2.6	2.5	5	—	85—90	.06	.03	—	—
40—45	1.6	1.7	2	3	χ^2 P	46.28	20.68	$n = 14$	
45—50	1.0	1.1	—	1		.00006	.08		

* The brackets show the grouping for the χ^2 test. (a) and (b) refer respectively to the theoretical curve and the curve from the data.

The poor fit when we fit the theoretical curve to the observed data is due to the large difference in the β 's which seems to suggest that the total number of samples, viz. 700, is far too small, e.g.

β_2	β_4	β_6	Calculated from
7.538	139.04	3702	Data in Table XXVII
7.640	150.23	4257	Data in Table XXVIII
12.5	670.8	99225	Formulae (17 a-c)

Although $P = \cdot 08$ for the curve fitted from the data, this is not a bad fit, because the contribution to χ^2 of the frequency in the group -5 to -10 , viz. 63 against a theoretical value of 92.9, is 9.6. Diagram XXII gives the regrouped distribution with the ordinates of the corresponding Type (i) and Type II curves.

DISTRIBUTION OF μ_3 in SAMPLES of SIZE 10 N^o PEPPER'S DATA

— TYPE (i) CURVE FROM ACTUAL OBSERVED VALUES
OF β_2, β_4 & β_6

- - - - - TYPE (ii) CURVE CALCULATED FROM THEORETICAL
VALUES OF β_2, β_4 & β_6

□ = 5 SAMPLES

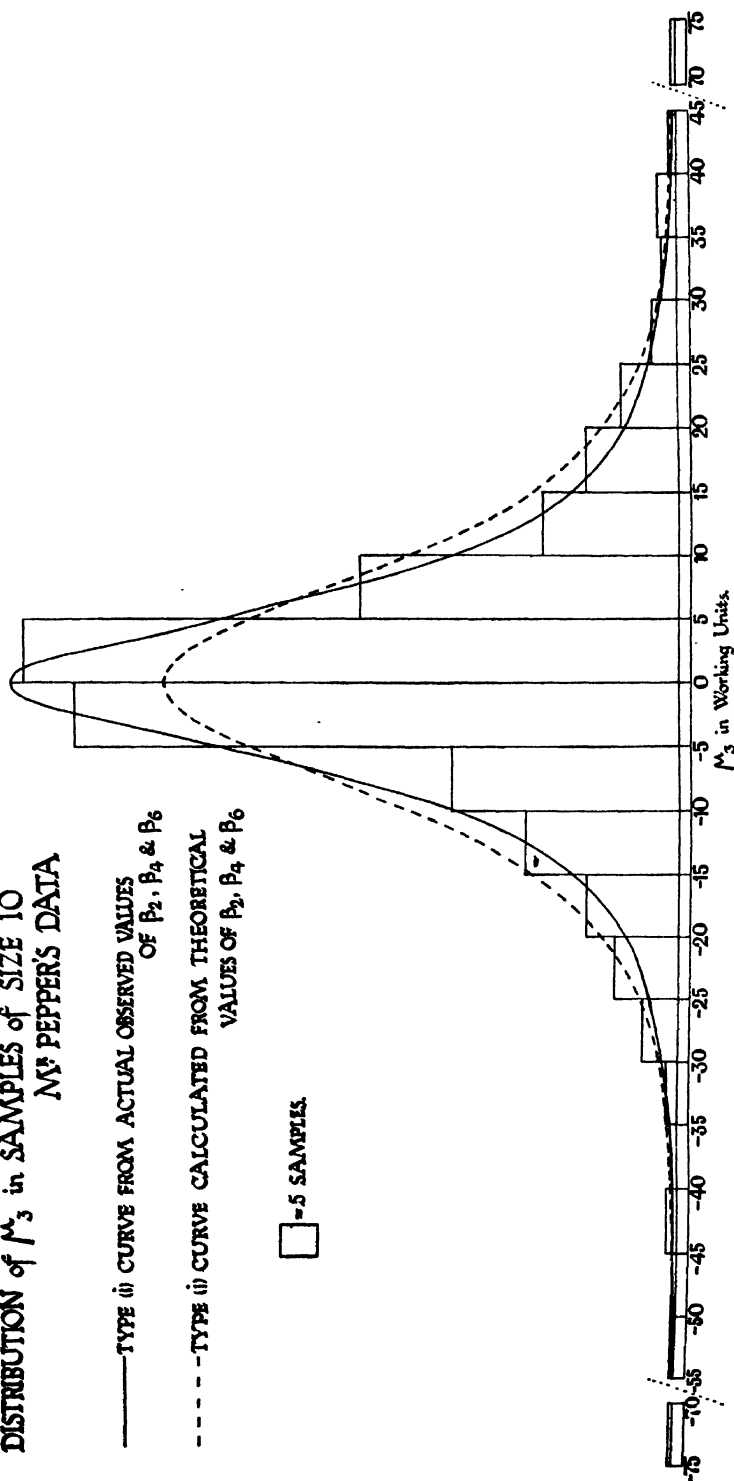


Diagram XXII.

(d) *American Artillery Experience.*

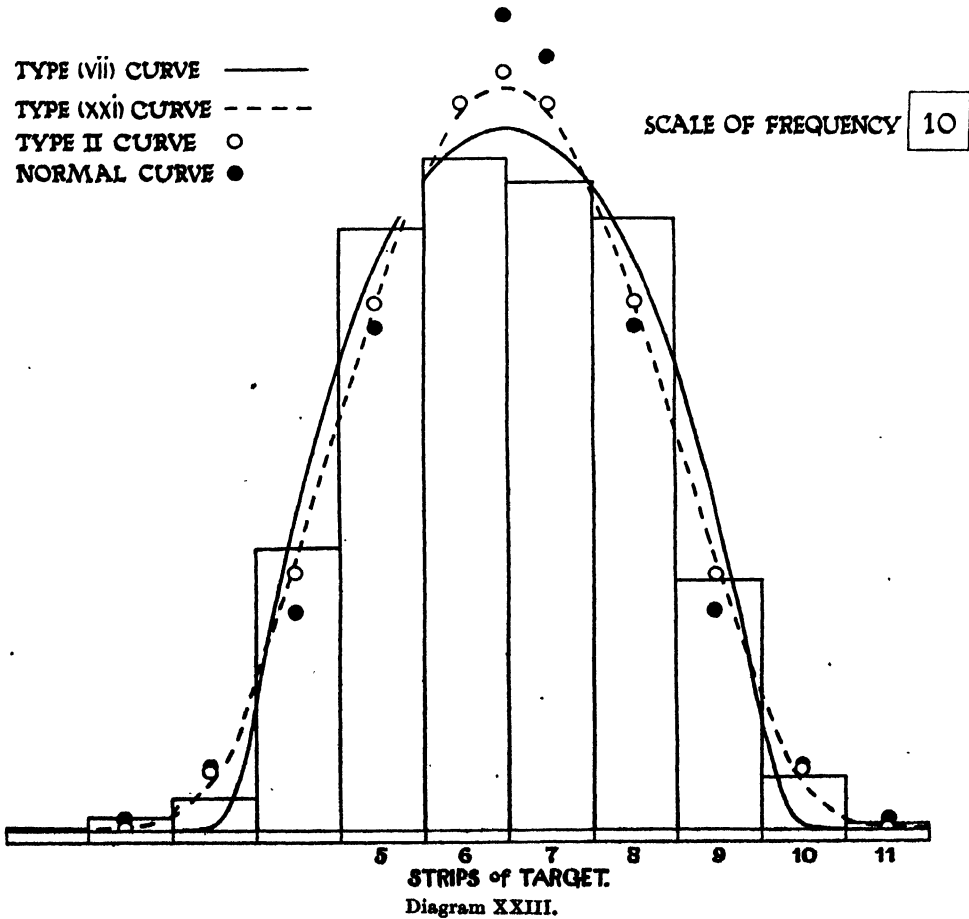
1000 shots were fired from a battery gun in the United States at a target 52 feet long and 11 feet high. All the shots hit the target, and their distribution in the eleven horizontal strips of one foot each, the point of aim being the central horizontal line of the target, was found to be:

No of strip	1	2	3	4	5	6	7	8	9	10	11
Frequency	1	4	10	89	190	212	204	193	79	16	2

I found the mean to be a distance of .482 from the centre of the sixth strip. The constants referred to the mean and corrected by Sheppard's formulae gave

$$\begin{aligned}\sigma &= 1.5499, & \beta_4 &= 10.9184, \\ \beta_1 &= .0008, & \beta_6 &= 69.2137. \\ \beta_2 &= 2.5261,\end{aligned}$$

DISTRIBUTION OF 1000 SHOTS FIRED FROM A BATTERY GUN.



(i) The ordinates of the transition curves for $\beta_2 = 2.5261$ and $\beta_4 = 10.9184$ are

$$\begin{aligned}\phi_0 &= 58.089, & \phi_6'' &= 68.352, \\ \phi_2 &= 33.191, & \phi_8' &= 66.051, \\ \phi_4 &= 63.1822, & \phi_8'' &= 55.889. \\ \phi_6' &= 118.490,\end{aligned}$$

Since $\phi_6' > \text{observed } \phi_6 > \phi_6''$ the point corresponding to the observed β 's falls into a heterotypic zone.

I found the ordinates of the Type (vii) curve for

$$\beta_2 = 2.526, \quad \beta_4 = 10.918, \quad \beta_6 = 61.352,$$

but although a better fit is obtained than that given by Pearson Type II curve, the fit is not good. I also tried to fit a sixth order Pearson curve, viz.

$$y = y_0 e^{-(p-x/\sigma)^2},$$

which corresponds to the different equation

$$\frac{1}{y} \frac{dy}{dx} = \frac{-4m(x/\sigma^2)}{(p-x^2/\sigma^2)^3}.$$

p is given by the cubic*

$$p^3(\beta_2 - 3) + 6p^2\beta_2 - 3p(7\beta_4 - 5\beta_2^2) + 9\beta_6 - 7\beta_2\beta_4 = 0$$

and m by the equation $p^3 - 9p^2 + 15p\beta_2 - 7\beta_4 = 4m$.

Here again the fit is not good. Diagram XXIII shows the above distribution and the ordinates of the various curves plotted.

Note. As a matter of interest I now considered the different types along the line $\beta_4 = 10.92$ in the plane $\beta_2 = 2.53$. The Table XXX below gives all the types discussed. Diagram XXIV (p. 190) shows some of the various shapes obtained.

TABLE XXX.

β_6	Zone†	Signs of					Type
		m_1	$q_1 p_1$	$q_1 + p_1$	$(q_1 - p_1)^2$	$m_1^2 - (q_1 - p_1)^2$	
30	below $F_2=0$	+	-	+	+	+	(ii)
33.19	$F_2=0$	+	-	0	+	+	(ix)
50	(F_2, F_8'')	+	-	-	+	+	(i)
55.89	$F_2''=0$	+	-	-	+	0	(xiii)
57	(F_8'', F_0)	+	-	-	+	-	(j)
58.09	$F_0=0$	+	0	-	+	-	(xi)
60	(F_0, F_4)	+	+	-	+	-	(iii)
63.18	$F_4=0$	0	+	-	+	-	(viii)
64	(F_4, F_8')	-	+	-	+	-	(vi)
66.05	$F_8'=0$	-	+	-	+	0	(xv)
67	(F_8', F_6'')	-	+	-	+	+	(vi)
68.36	$F_6''=0$	-	+	-	0	+	(vii)
70	(F_6'', F_6')	-	+	-	-	+	(H)
118.49	$F_6'=0$	-	+	-	0	+	(vii)
140	above $F_6'=0$	-	+	-	+	+	(vi)

* In our case the cubic has only one real root.

† I denote for shortness by $F_x=0$, on the curve $F_x=0$ and by $(F_x, F_y)=0$, between the curves $F_x=0$ and $F_y=0$. H marks that the curve lies in a heterotypic region.

FREQUENCY CURVES for $\beta_2 = 2.53$ and $\beta_4 = 10.92$, and varying β_3

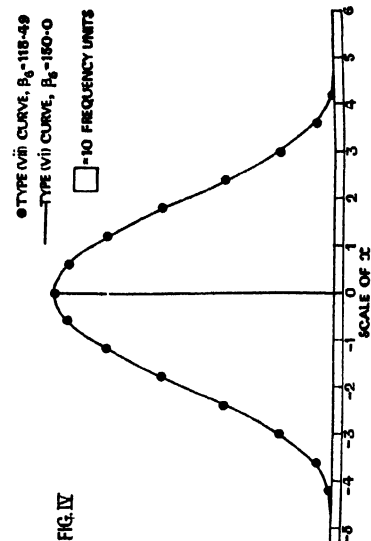
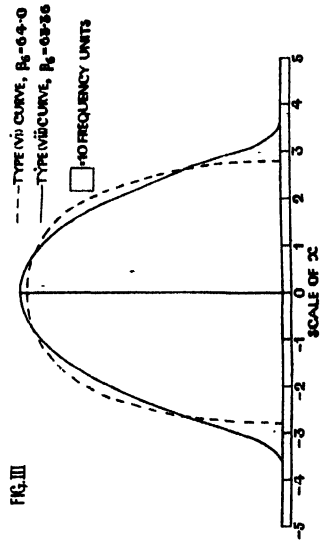
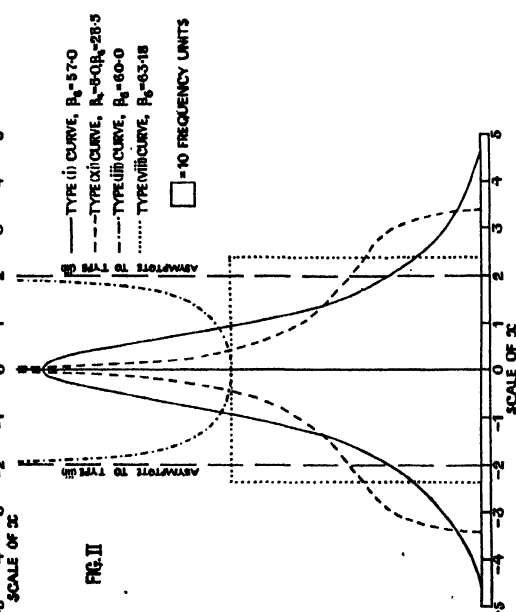
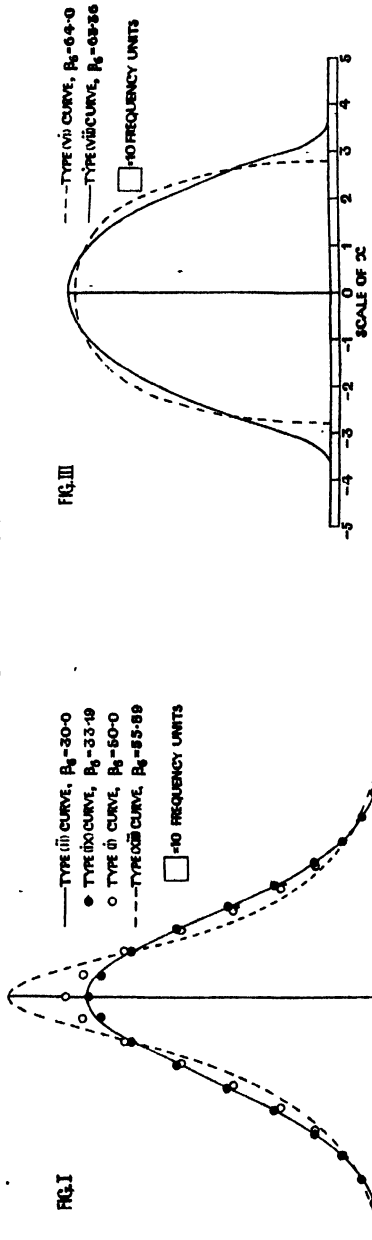


Diagram XXIV.

(e) *Errors of the Right Ascensions of Stars**.

Bessel provides the following data :

Limits	Actual Errors	Limits	Actual Errors
0—0·1	114	0·5—0·6	6
·1—	84	·6—	3
·2—	53	·7—	1
·3—	24	·8—	1
·4—	14	·9—	0

The table appears to add the frequencies in the positive and negative groups and to give their sums only. We find the constants

$$\mu_2 = \cdot 0513, \quad \mu_4 = \cdot 0101, \quad \mu_6 = \cdot 00361, \quad \mu_8 = \cdot 00174,$$

$$\beta_2 = 3\cdot 843, \quad \beta_4 = 26\cdot 81, \quad \beta_6 = 252\cdot 2.$$

$$\text{Hence} \quad r = \left(\beta_4 - \frac{5\beta_2^2}{6 - \beta_2} \right) / \frac{1}{7} \left(\frac{5\beta_2^2}{6 - \beta_2} - \beta_2^2 \right) = -2\cdot 672,$$

$$\Phi = \left(\beta_6 - \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} \right) / \left\{ \frac{25\beta_2^3(\beta_2 + 8)}{9(6 - \beta_2)^2} - \beta_2^3 \right\} = -\cdot 433.$$

Let us now interpolate linearly into the tables of the Φ'_8 ordinates for $\beta_2 = 3\cdot 843$ and $r = -2\cdot 672$.

Values of Φ'_8 .

r	$\beta_2 = 3\cdot 5$	$\beta_2 = 3\cdot 843$	$\beta_2 = 4$
—2·0	—·352	—	—·382
—2·672	—·451	—·482	—·488
—3·0	—·499	—	—·540

We find $\Phi'_8 = -\cdot 482$, approximately. But the observed value of Φ'_8 is $-\cdot 433$ and therefore $\beta'_8 > \Phi'_8 > \phi_0$, since by inspection of the tables for $\beta_2 = 3\cdot 5$ and $\beta_2 = 4$, $\phi'_8 > \phi_0$. Accordingly, referring to Diagram I, Fig. III, we see that we must fit the Type (i) curve. Note that we could have reached this result by an inspection of the contour diagrams.

$$\text{Now} \quad (q - p) = -(q_1 + p_1) = 19\cdot 206, \quad \text{therefore} \quad q = 21\cdot 107,$$

$$qp = -q_1 p_1 = 40\cdot 125, \quad \text{,,} \quad p = 1\cdot 901,$$

$$m = m_1 = 39\cdot 264, \quad \text{,,} \quad k' = 1\cdot 7065.$$

* *Astronomische Nachrichten*, Bd. 15, Nr. 358—359.

192 *On certain Non-Normal Symmetrical Frequency Distributions*

The Type (i) curve accordingly is $y = y_0 \left(\frac{1 - u^2/21.107}{1 + u^2/1.901} \right)^{1.7065}$, $y_0 = 124.0$ and the range is from -1.040 to $+1.040$.

Pearson's Type VII and the normal curves are respectively

$$y = 114.145 (1 + 2.140 x^2)^{-0.068},$$

and $y = 264.99 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(4.4165 x^2)} \right).$

The ordinates are plotted in Diagram XXV.

Using six groups the following values of χ^2 were found and hence P by the goodness of fit test:

Type of curve	(i)	VII	Normal
χ^2	.669	1.348	5.517
P	.975	.923	.359

Both the Type (i) and the Type VII give a better fit than the normal curve. There is not much to choose between the Type (i) and the Type VII curves.

**THE DISTRIBUTION OF THE RIGHT ASCENSION OF A STAR.
300 OBSERVATIONS MADE BY BESSEL.**

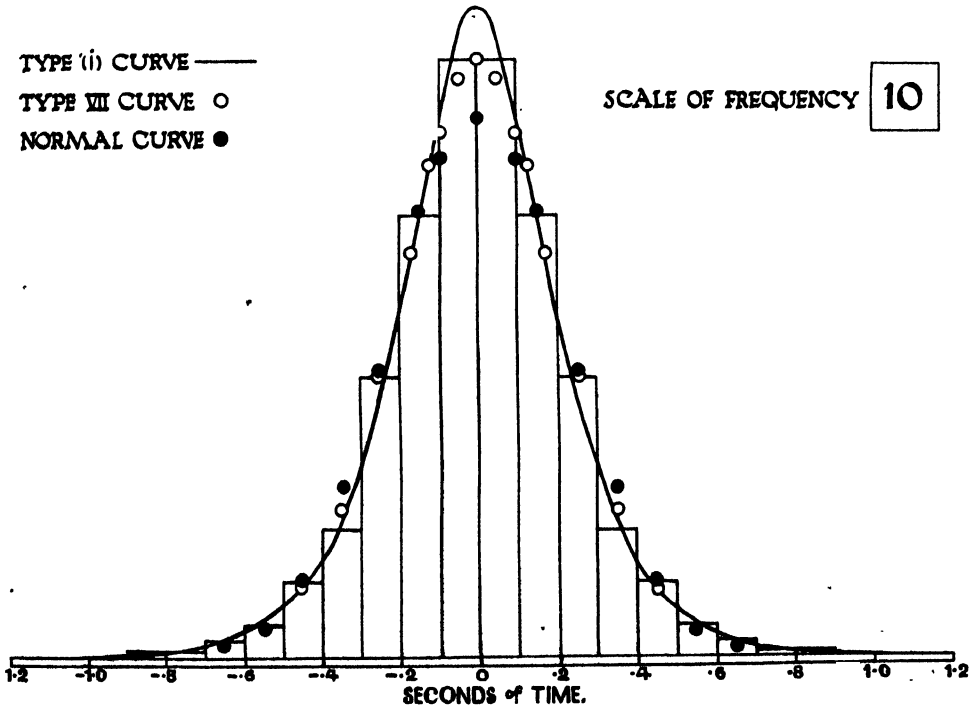


Diagram XXV.

(f) *W. Palin Elderton's Data.*

The following example is taken from Elderton's "Frequency Curves and Correlation":

Central Value	Observations	Type II	Type A	Edgeworth	Normal
-2.0	11	14	15	16	20
-1.5	116	109	106	106	95
-1.0	274	286	284	285	270
- .5	451	433	437	436	456
+ .5	432	433	437	436	456
+1.0	267	285	283	284	270
+1.5	116	109	106	106	95
+2.0	16	14	15	16	20

There is no statement of what this distribution describes, and I have taken the groups as shown above. The constants are

$$\mu_1' = -.00149, \mu_2 = 1.829, \beta_1 = .00237, \beta_2 = 2.548, \beta_3 = 9.122, \text{ and } \beta_6 = 39.50.$$

The corresponding auxiliary co-ordinates are $r = -.682$ and $\Phi = -.049$.

By inspection of the Diagram XV we see that the observed values of r and Φ fall into the zone between the curves of section of $F_0 = 0$ and $F_2 = 0$. Hence, from Diagram I, Type (i) must be used. We find

$$p = 4.926, \quad q = 6.321, \quad k' = 1.330, \quad y_0 = 472.14.$$

$$\text{The required curve is } y = y_0 \left(\frac{1 - \frac{x^2}{14.1123}}{1 + \frac{x^2}{9.0103}} \right)^{1.3302}$$

Note the range is from -3.76 to $+3.76$. We find the following frequencies:

Central Value	Observed Frequency	Frequency from Type (i)
-2.0	11	14
-1.5	116	111
-1.0	274	278
- .5	451	438
+ .5	432	438
+1.0	267	278
+1.5	116	111
+2.0	16	14

Although we have few groups the Type (i) curve gives a very good fit. All the other types, except the normal curve, fit equally well. Note that in this case the corresponding r and Φ co-ordinates are $r = -.68$ and $\Phi = -.04$, as compared to $r = 0$ and $\phi = .09$ for Pearson's Type II curve.

(g) *Distribution in Samples of the Mean of an Array as determined by the Regression Line* (K. Pearson, *Proc. Roy. Soc., A*, Vol. 112, 1926).

Let $\check{y}_x, m_1, m_2, \sigma_1, \sigma_2, \rho$ refer to parent populations,
and $\bar{y}_x, \bar{x}, \bar{y}, \Sigma_1, \Sigma_2, r$ refer to the sample of size M .

Hence
$$\check{y}_x = m_2 + \rho \frac{\sigma_2}{\sigma_1} (x - m_1),$$

and
$$\bar{y}_x = \bar{y} + r \frac{\Sigma_2}{\Sigma_1} (x - \bar{x}).$$

The moments of the distribution for $\bar{y}_x' = \bar{y}_x - \check{y}_x$ are

$$\mu_2 = \frac{\sigma_2^2 (1 - \rho^2) \phi^2}{M - 3}, \text{ where } \phi^2 = 1 - \frac{2}{M} + \frac{(x - m_1)^2}{\sigma_1^2},$$

$$\mu_4 = \frac{3\sigma_2^4 (1 - \rho^2)^2}{(M - 3)(M - 5)} \left\{ \phi^4 - \frac{2}{M} \left(1 - \frac{2}{M} \right) \right\},$$

$$\mu_6 = \frac{15\sigma_2^6 (1 - \rho^2)^3}{(M - 3)(M - 5)(M - 7)} \left\{ \phi^6 - \frac{6}{M} \left(1 - \frac{2}{M} \right) \phi^2 + \frac{8}{M^2} \left(1 - \frac{2}{M} \right) \right\},$$

$$\begin{aligned} \mu_8 = & \frac{105\sigma_2^8 (1 - \rho^2)^4}{(M - 3)(M - 5)(M - 7)(M - 9)} \\ & \times \left\{ \phi^8 - \phi^4 \left(1 - \frac{2}{M} \right) \frac{12}{M} + \phi^2 \left(1 - \frac{2}{M} \right) \frac{32}{M^2} + \left(1 - \frac{2}{M} \right) \left(1 - \frac{6}{M} \right) \frac{12}{M^2} \right\}. \end{aligned}$$

Hence

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(M - 3)}{(M - 5)} \left\{ 1 - \frac{2}{M} \left(1 - \frac{2}{M} \right) \frac{1}{\phi^4} \right\},$$

$$\beta_4 = \frac{15(M - 3)^2}{(M - 5)(M - 7)} \left\{ 1 - \frac{1}{\phi^4} \frac{6}{M} \left(1 - \frac{2}{M} \right) + \frac{8}{M^2} \left(1 - \frac{2}{M} \right) \frac{1}{\phi^6} \right\},$$

$$\begin{aligned} \beta_6 = & \frac{105(M - 3)^3}{(M - 5)(M - 7)(M - 9)} \\ & \times \left\{ 1 - \frac{12}{M} \left(1 - \frac{2}{M} \right) \frac{1}{\phi^4} + \frac{32}{M^2} \left(1 - \frac{2}{M} \right) \frac{1}{\phi^6} + \frac{12}{M^2} \left(1 - \frac{2}{M} \right) \left(1 - \frac{6}{M} \right) \frac{1}{\phi^8} \right\}. \end{aligned}$$

(1) Take $M = 50$. From the equations above we find the following table*:

$x - m_1$	β_2	β_4	β_6	r	r_{VII}	Φ	Φ_{VII}
$\frac{1}{2}\sigma_1$	3.051	15.806	119.339	.024	0	.3768	.1874
σ_1	3.102	16.618	129.431	.016	0	.2486	.2051
$2\sigma_1$	3.128	17.045	136.792	.003	0	.2538	.2146

* r_{VII} and Φ_{VII} are the r and Φ co-ordinates of the point corresponding to the Type VII curve.

(2) Take $M = 25$.

$x - m_1$	β_2	β_4	β_6	r	r_{VII}	Φ	Φ_{VII}
$\frac{1}{3}\sigma_1$	3.072	16.195	144.875	.091	0	.5460	.1946
$3\sigma_1$	3.298	20.208	193.408	.068	0	.3326	.2964

(3) For larger samples we are still closer to the point corresponding to Pearson's Type VII curve. For very large samples the curve of distribution of the mean of the arrays follows a normal curve.

We are so close to the point $(0, \Phi_{VII})$ that in all cases the Type VII curve can excellently describe the distribution of the mean of the arrays. We expect this since

$$\beta_2 = \frac{3(M-3)}{M-5},$$

$$\beta_4 = \frac{15(M-3)^2}{(M-5)(M-7)},$$

$$\beta_6 = \frac{105(M-3)^3}{(M-5)(M-7)(M-9)},$$

satisfy $\beta_4 = \frac{5\beta_2^2}{6-\beta_2}$ and $\beta_6 = \frac{35\beta_2^3}{(6-\beta_2)(9-2\beta_2)}$.

Conclusions.

I have shown that the fourth order symmetrical Pearson curves give a better graduation than the second order Pearson curves or the normal curve, and that the improvement in fit, as judged by the χ^2 test, justifies the extra amount of labour involved in finding the higher betas.

I have also succeeded in finding frequency curves whose higher beta-coefficients are positive, in the cases $\beta_2 > 6$ and $4.5 < \beta_2 < 6$.

Finally, I have discussed the $\beta_2, \beta_4, \beta_6$ -space and the types of curve at different points, but the question of finding the type of curve corresponding to the heterotypic area must be left unanswered. This problem demands an analysis of the sixth order Pearson curves and a discussion of β_8 and β_{10} will be necessary. We cannot expect any advantage from such a system of curves.

In conclusion I would like to thank Professor Karl Pearson for suggesting this problem to me and to express my appreciation of his continual advice and criticism. I wish also to thank Miss M. Kirby for the very capable way in which she has prepared my diagrams.

A BIOMETRIC STUDY OF THE "FLATNESS" OF THE FACIAL SKELETON IN MAN.

By T. L. WOO, PH.D., IN CONJUNCTION WITH G. M. MORANT, D.Sc.

(1) *Introduction.* It has long been recognised that differences in the degree of "flatness" of the face, or facial skeleton, are sufficiently great to be of considerable importance in aiding the discrimination of the racial varieties of modern man. These differences have usually been taken for granted, and it is only in recent years that any attempt has been made to give them quantitative expression. The research described in the present paper was undertaken with the purpose of extending these quantitative methods and applying them to more adequate cranial material than any previously considered. No one appears to have made any serious attempt to define exactly what is meant by the "flatness" of the facial skeleton, but this feature of the cranium has been customarily referred to when discussing measurements of two different kinds. Those of the first are chiefly angular measurements taken in the "median sagittal" plane and the more important of them deal with prognathism and are modifications of Camper's facial angle. The greater the degree of prognathism the less "flat" the face is said to be. Measurements of the projection of the "median sagittal" section of the nasal bones have been dealt with in a similar way by A. Lüthy* and G. von Bonin†, the latter having obtained his data from sagittal type contours. We are not concerned in the present paper with such measurements which are confined to the "median sagittal" plane. Those of the second kind referred to are designed to give estimates, not of antero-posterior projection, but of transverse flattening. One may consider this feature at a number of different and convenient horizons and it is clear that a considerable number of measurements would be needed to give any appreciation of the transverse flattening of the whole facial skeleton. C. de Mérejkowsky seems to have been the first to make a useful contribution to this subject. In a paper which appeared in 1882‡ he described a new instrument—now known as a simometer—designed to measure the projection of the nasal bones. The means found for a number of racial series of crania suggested that his method would be a valuable one, but it was not adopted by craniometricians. Thirty years later a study of negro skulls was issued in *Biometrika*§ in which special attention was paid to the same character. The

* "Die vertikale Gesichtsprofilierung und das Problem der Schädelhorizontalen." *Archiv für Anthropologie*, N.F. Bd. xi. 1912, pp. 1—87.

† *Biometrika*, Vol. xxiii. 1931, pp. 260—261.

‡ "Sur un nouveau caractère anthropologique." *Bulletins de la Société d'Anthropologie de Paris*, 5ième série, t. v. pp. 293—304.

§ "A Study of the Negro Skull with special reference to the Congo and Gaboon Crania." By R. Crewdson Benington. Prepared for press by Karl Pearson. Vol. viii. 1912, pp. 292—339; see pp. 315—320.

new measurements taken for this purpose, in addition to those defined by de Mérejkowsky, were the chord from dacryon to dacryon and the minimum arc between the same points, so no special instrument was required other than a simometer. A paper published in the same journal in the following year* provided a detailed study in man and the anthropoid apes of the same measurements and also of the subtense from the dacryal chord, the simotic and dacryal subtenses being found with the simometer. Cranial series representing 16 races were dealt with, but several of these were admittedly too small to give reliable means. The dacryal and simotic measurements have been provided for the numerous cranial series described in later volumes of *Biometrika*, and the material available for them is now quite extensive. Their omission from all other recent craniometric studies may be largely attributed to the fact that the simometer is a somewhat complicated instrument which would need to be made to order. Other estimates of the transverse flattening of the facial skeleton—including in that the supraciliary region as Broca did—have been obtained from measurements of horizontal type contours. The section represented is one through the glabella parallel to the Frankfort horizontal plane and indices measuring its curvature anterior to the temporal lines have been given by G. M. Morant for several races†. More recently Davidson Black has defined certain angular measurements to measure the same feature‡. These are found with a goniometer or from the horizontal section on which the projections of certain facial points are marked: they are only available for Chinese series.

The writers' purpose was to make a fairly comprehensive survey of the degree of transverse flattening of the facial skeleton considered as a whole in a sample of races from all parts of the world. The use of any projective methods would have made it impossible to record a sufficiently large number of measurements in the time available, and accordingly they decided to use a simple pair of co-ordinate calipers only. Four pairs of measurements, defined in the next Section, were taken to give four indices, of which all but one are new. The simotic index, which was included, was first defined by de Mérejkowsky, and a considerable amount of material for it has accumulated since. Ryley, Bell and Pearson concluded that, for purposes of racial comparison, "a measure of the mesodacryal index is likely to be of as much value as the determination of the simotic index, possibly of greater value," but nevertheless we chose the latter rather than the former to measure the prominence of the nasal bridge because it can be determined on a far larger number of museum specimens. The lacrymal bones are frequently missing, and it is then impossible to locate the dacrya with precision.

(2) *Definitions of Measurements.* The object of the measurements used in this study was to provide accurate estimates of the transverse flattening of the facial skeleton at different horizons. This was effected by measuring directly, with the

* "A Study of the Nasal Bridge in the Anthropoid Apes and its Relationship to the Nasal Bridge in Man." By Kathleen V. Ryley and Julia Bell, assisted by Karl Pearson. Vol. ix. 1913, pp. 391—445.

† *Annals of Eugenics*, Vol. ix. 1927, pp. 355—356.

‡ *Palaeontologia Sinica*, Series D, Vol. vii. 1928, pp. 13—14.

aid of a special pair of calipers, the subtense of a particular “median” point from a corresponding chord, of which the terminals are the same point on the right and left sides, and also the length of the chord. The index obtained by expressing the subtense as a percentage of the chord provides a measure of the character with which we are concerned, and four different indices were found, whenever possible, for each skull. The readings required could be taken rapidly and accurately with a pair of co-ordinate calipers. This instrument consists of a bar, on which a scale is inscribed, with three arms which always remain mutually parallel to one another and perpendicular to the bar such that their tips, which are brought into contact with the object measured, are always co-planar with the scale. One of the arms is fixed to the zero end of the bar, and another of the same length can be moved along it so that these two are used in precisely the same way as the arms of an ordinary pair of rectangular calipers. The third is attached between the other two, and it can also be moved along the bar, the attachment permitting a movement so that the setting point of the arm can be shifted towards or away from the bar. This middle arm bears a scale on which the reading is zero when the tips of the three arms are in a straight line. When the middle arm is moved towards or away from the bar the divergence of its tip from that straight line—i.e. the subtense—can be read. In our case the tips of the outside arms were first brought into contact with the extremities of the transverse chord, and then the middle arm was moved until its tip was in contact with the appropriate “median” point, a screw enabling it to be fixed in that position. On removing the calipers both the length of the chord and the subtense of the “median” point from it were recorded. Another reading might have been taken at the same time. This is the distance of the foot of the perpendicular—i.e. the “working edge” of the middle arm—from one or other extremity of the chord which would have given an appreciation of asymmetry. The measurement in question was not recorded, but for two series the direct distance of the “median” point from the extremity of the chord was found on either side. The form of co-ordinate calipers made by P. Hermann, Richenbach und Sohn, of Zürich, was found to be unsuitable for taking the measurements devised for two reasons: the middle arm is too broad to make it possible to obtain the adjustment required in the case of narrow nasal bones, and the outside arms are not long enough to enable other subtenses as large as some needed to be found. Accordingly a similar instrument was made to order for Dr Woo by Messrs W. F. Stanley and Co. The breadth of the middle arm was reduced to 2.5 mm., and the outside arms were made 10 cm. long. All the measurements needed could be readily taken with these calipers on all the skulls examined.

The measurements are:

(1) *IOW*. The internal bi-orbital breadth between the points, right and left, where the fronto-malar sutures cross the outer margins of the orbits. The point used, which is Martin’s *fronto-malare orbitale*, can almost always be found precisely if the suture be open. If the orbital margin be blunt, it is advisable to mark it first as a pencil line. The measurement is No. 43 (1) in Martin’s list.

(1 *a*) Sub. *IOW*. The subtense of the nasion from the chord *IOW*.

(1 *b*) The frontal index of facial flatness is defined to be $100 \text{ Sub. } IOW/IOW$.

(2) *SC*. The simotic chord which is the minimum horizontal breadth of the two nasal bones, so that the extremities of this chord are points on the naso-maxillary sutures.

(2 *a*) *SS*. The simotic subtense found by first marking as a pencil line the "ridge" of the nasal bones, without necessarily following the inter-nasal suture, and then finding the minimum subtense from this line to the simotic chord. When found with co-ordinate calipers this may not correspond exactly with the simotic subtense found with the aid of de Mérejkowsky's simometer. A comparison between the two measurements found for the same series is made below.

(2 *b*) The simotic index is defined to be $100 SS/SC$.

(3) *MOW*. The mid-orbital breadth between the points, right and left, where the malar-maxillary sutures cross the lower margins of the orbits. It is generally advisable to mark the lower margins of the orbits as pencil lines before locating the points.

(3 *a*) Sub. *MOW*. The subtense from the "tip" of the nasal bones, which is accepted to be the lowest point on the inter-nasal suture (i.e. Martin's *rhinion*), to the chord *MOW*. This was found to be an unsatisfactory measurement owing to the fact that the lower parts of the nasal bones are often defective, even on skulls which have the remainder of the facial skeleton quite complete. It is sometimes difficult to decide whether the region in question is intact or not, and in all doubtful cases the measurement was omitted.

(3 *b*) The rhinal index is defined to be $100 \text{ Sub. } MOW/MOW$.

(4) *GB*. The chord between the points, right and left, on the zygomatico-maxillary sutures which are lowest with regard to the Frankfort horizontal plane. The positions of these points can be estimated with sufficient accuracy without orientating the cranium exactly. If the inferior extremity of the suture is a short length lying parallel to the horizontal plane, the anterior point on it is the one accepted. The point used is Martin's *Zygomaxillare* and the chord (No. 46 in his list) is one usually given in the descriptions of racial series.

(4 *a*) Sub. *GB*. The subtense of the alveolar point from *GB*. The true alveolar point was used, i.e. the inferior point on the process between the cavities of the central incisors, and not the prosthion.

(4 *b*) The premaxillary index of facial flatness is defined to be $100 \text{ Sub. } GB/GB$.

The first three of these indices are of a straightforward character, and in each case the three points involved usually lie in a plane which is approximately parallel to the Frankfort horizontal plane. For the first the points all lie on the inferior borders of processes of the frontal bone; the second measures the flatness of the bridge of the nose and it is confined to the nasal bones, while the third is of a less simple type since it involves the nasal and maxillary bones. The points used in determining the fourth index all lie on the premaxillary bones,

but the "median" point—the alveolar point—is always decidedly inferior to the extremities of the transverse chord. The subtense will obviously be affected by differences in the height of the premaxillary region, though the index may be supposed to give primarily a measurement of sub-nasal prognathism. We should anticipate that the latter would be highly correlated with a measurement of the total prognathism of the face.

Of the absolute measurements defined above, *IOW*, *SS*, *SC* and *GB* have been provided for a number of cranial series, and all except the first were previously available for several of the series measured for the purpose of this study. Hence, comparisons can be made to test the reliability of the same measurement taken on the same specimens by two different observers, and this is particularly necessary in the case of the simotic subtense as two different instruments, and slightly different definitions have been used to determine it. De Mérejkowsky describes how he measured the character in question with his simometer. The two arms of the instrument are placed so that their tips coincide with the extremities of the minimum breadth of the two nasal bones, and then, to bring the tip of the subtense arm into position, "on l'appuie sur le dos du nez là où celui-ci atteint son minimum d'élévation; les trois points de repaire (*sic!*) sont donc faciles à trouver et complètement fixes*." But the last statement does not seem to be true since "le dos du nez" is an inexact term and yet it forms an essential part of the definition, as it is only with reference to it that the *minimum d'élévation* can have any meaning. We must suppose that de Mérejkowsky was imagining a "median" line, which might be traced in pencil on the nasal bones, such that it would indicate their "ridge," or the line marking the join of the right and left sides of the surface formed by the two nasal bones together. On a symmetrical cranium this would coincide with the inter-nasal suture, but the direction of the suture was not considered, presumably, in determining the "ridge" if the nasal bones were asymmetrical. The subtense would then be the *minimum* from this line to the simotic chord, and having fixed its position it would also be the maximum subtense of the section of the nasal bones defined by the three points of contact of the instrument. This method of measuring the simotic subtense appears to have been that followed by Ryley, Bell and Pearson, and by later workers in the Biometric Laboratory who used the simometer. Dr Woo used co-ordinate calipers in place of that instrument, but followed the same convention in making the point of subtense of the subtense arm one on the "median" ridge of the nasal bridge. When the compass arms of the simometer are fixed, as to the simotic chord of a particular cranium, the pivot of the instrument, which is equidistant from their tips, is also a fixed point. The middle, or subtense, arm is then free to rotate round that pivot in such a way that the tips of the three arms and the pivot are always co-planar. It is adjusted so that its tip touches the point on the accepted transverse section of the nasal bones which is nearest to the pivot. The simometer is then removed from the cranium and, while the direction of the

* *Loc. cit.* p. 298.

subtense arm relative to the compass arms is maintained, its tip is moved until it is in line with the tips of the compass arms. The subtense measured is the displacement of the tip of the subtense arm*. If the section in which the simometer measurements are taken is asymmetrical, the subtense may not be perpendicular to the chord and the point of maximum subtense may not coincide with that found by using the co-ordinate calipers. It is clear that the simometer subtense will always be equal to or greater than the subtense found with co-ordinate calipers, and we must ask whether the readings found with the two different instruments are sufficiently close to be considered comparable for practical purposes. To decide the point Dr Woo found the nasal bridge measurements with co-ordinate calipers in the case of a series—the Kerma Egyptian measured by Miss Collett—for which the measurements had previously been recorded by using a simometer. Comparisons are made in the following table, the constants relating to differences having been found from the distributions of differences:

Character	Sex	Measured by	Mean	Mean Difference (C. - W.)	S.D. of Differences
SS	♂	Collett with simometer Woo with co-ordinate calipers	4.006 (94) 3.843 (94)	+ .162 ± .032	.464 ± .023
	♀	Collett with simometer Woo with co-ordinate calipers	3.125 (68) 3.147 (68)	- .022 ± .029	.353 ± .020
SC	♂	Collett Woo	10.81 (94) 10.36 (94)	+ .450 ± .034	.489 ± .024
	♀	Collett Woo	10.22 (68) 9.99 (68)	+ .229 ± .040	.488 ± .028
100 SS/SC	♂	Collett (simometer subtense) Woo (co-ordinate calipers subtense)	37.24 (94) 36.57 (94)	+ .681 ± .276	3.962 ± .195
	♀	Collett (simometer subtense) Woo (co-ordinate calipers subtense)	30.58 (68) 31.76 (68)	- 1.184 ± .232	2.834 ± .164
GB	♂	Collett Woo	95.27 (97) 95.10 (97)	+ .167 ± .038	.549 ± .027
	♀	Collett Woo	90.83 (82) 90.71 (82)	+ .120 ± .044	.587 ± .031

* The simometer is rather a complicated instrument and the reader who is not acquainted with it may find it difficult to follow our remarks without referring to de Mérejkowsky's description and figure in the paper cited.

The mean difference between the male simotic subtenses (*SS*) is seen to be small but quite significant, the mean found by using the simometer being greater than that given by the co-ordinate calipers. The difference between the female means, however, is of the opposite sign though quite insignificant. But some significant differences are also found for the simotic chords (*SC*) and facial breadths (*GB*), though in these cases the two observers followed precisely the same definitions.

For the male simotic index the difference is actually insignificant, while for the female it is quite significant but of the opposite sign to that which would have been expected. We may conclude that the differences between the two methods of determining the simotic subtense only introduce errors which are of the same order as those due to the personal equations of two different workers following the same definition*, and accordingly we have supposed that the simotic measurements found with the co-ordinate calipers may be compared without correction with those previously found with the simometer. For all the series dealt with (except the Kerma Egyptian) which had any of the measurements *IOW*, *SS*, *SC*, *100 SS/SC* and *GB* provided in previously published papers, those values were accepted to give the constants provided in the present paper. It may be observed that none of the differences shown in the table on p. 201 is large enough to vitiate racial comparisons, except possibly the differences for the simotic chord, since they are very small compared with interracial differences. For the simotic index, for example, a difference of about one unit is shewn between the means found by the different observers for the same series, but the range of the mean male simotic indices for different series is from 20·7 to 53·1.

In the next place, we may consider the few measurements collected for the purpose of examining asymmetry. For two male series the direct distances of each median point from the extremities of the corresponding chord were recorded and a comparison of these bilateral measurements is made in the table on p. 203.

Only one of the differences of means can be considered significant and if there be any differentiation of the bilateral measurements considered it is evidently of a small order which would only be revealed by examining larger series.

(3) *Description of the Material Measured.* The object of the present paper is to test the racial significance of the new measurements of facial flatness defined in the previous section. In order to serve this purpose it was necessary to measure as large numbers as possible of crania representing as many races as possible from all parts of the world. The observer (T.L.W.) was obliged to limit his material to that contained in the principal English museums and one Dutch museum. He is greatly indebted to the curators of these institutions for affording him every facility for

* In the paper by Ryley, Bell and Pearson (*loc. cit.* pp. 396—398) comparisons are made between various mean simotic measurements found by the same observer who repeated them on the same series of crania at different times. The largest differences shown are 0·10 for *SS*, 0·36 for *SC* and 1·0 for *100SS/SC*. These are of the same order as, but rather less than, the largest differences shown in the table above for the corresponding measurements.

pursuing this study and particularly to Miss M. L. Tildesley (Royal College of Surgeons), Dr L. H. Dudley Buxton (Oxford), Dr W. L. H. Duckworth (Cambridge), Mr W. P. Pycraft (British Museum), Professor J. A. J. Barge (Leiden) and, finally, to Professor Karl Pearson (Biometric Laboratory). The number of adult crania actually measured was 4266 male, 1630 female, the sexes accepted being in general those provided in previously published papers or museum catalogues, though in the case of a few series the writers had to assign sexes to the specimens themselves. No immature specimens were dealt with. Whenever possible long series representing single racial types were chosen and a number of earlier studies—nearly all of which have appeared in *Biometrika*—dealing with their characters were consulted as a guide to the groups which could be considered racially homogeneous. In the case of a number

Constants	Series	Frontal	Simotic	Rhinal	Premaxillary
Means	Punjabi	<i>R</i> 52.5 ± .18 (81)	6.47 ± .08 (81)	36.5 ± .41 (33)	57.9 ± .25 (67)
		<i>L</i> 52.2 ± .19 (81)	6.44 ± .09 (81)	35.8 ± .44 (33)	58.2 ± .27 (67)
	Hindu (Madras Presidency)	<i>R</i> 51.4 ± .25 (49)	6.05 ± .10 (46)	35.2 ± .74 (10)	57.6 ± .36 (48)
		<i>L</i> 51.3 ± .23 (49)	5.95 ± .11 (46)	34.6 ± .80 (10)	58.0 ± .34 (48)
Standard Deviations	Punjabi	<i>R</i> 2.44 ± .13	1.12 ± .06	3.45 ± .29	3.08 ± .18
		<i>L</i> 2.47 ± .13	1.16 ± .06	3.77 ± .31	3.24 ± .19
	Hindu (Madras Presidency)	<i>R</i> 2.61 ± .18	1.10 ± .07	—	3.69 ± .25
		<i>L</i> 2.44 ± .17	1.13 ± .08	—	3.44 ± .24
Correlations (r_{RL})	Punjabi	.828 ± .024	.826 ± .024	.949 ± .012	.917 ± .013
	Hindu	.953 ± .009	.942 ± .011	—	.863 ± .025
Differences of Means ($\Delta_{MR} - M_L$)	Punjabi	+ .3 ± .108*	+ .03 ± .051	+ .7 ± .140	- .3 ± .107
	Hindu	+ .1 ± .076	+ .10 ± .038	+ .6 ± .255†	- .4 ± .183

of races, however, it was only possible to arrive at means based on sufficiently large numbers by pooling the measurements of short series in different museums. Or means of larger series might be pooled if there was sufficient reason to believe that they represented the same racial type. The test used in such cases was based only on the four indices of facial flatness. If two or more series were thought, from the description of their origins, to represent the same racial type, then their mean indices were calculated and if no significant differences were found between them the series were pooled to give the constants finally used. This practice is only of provisional value, of course. Two samples of skulls may show one character, or group of

* The probable error of the difference of means is given by the formula

$$.67449/\sqrt{n} \times \sqrt{\sigma_R^2 + \sigma_L^2 - 2r_{RL}\sigma_R\sigma_L}.$$

† This approximate probable error was found by using the σ 's and r given for the Punjabi series.

characters, identically the same while differing most significantly in other ways. Several of the series finally used are probably less homogeneous racially than any which could legitimately be used for purposes of rigid racial classification, but they may still be of value in a preliminary study such as the present. All the continental areas are represented, but the races used do not form a random sample from all races of the world. There are several British series, but few long series from other parts of Europe; the races of Africa and Oceania are fairly well represented; the American and Western Asiatic series are particularly meagre, while there are none from Central Asia and Siberia, but India and the Orient are better represented than any other regions. The means are given of a few series which are so short that no use can be made of them. It may be possible to pool these with measurements collected later from other sources.

The museums in which the cranial specimens were measured are: The Museum of the Royal College of Surgeons, London (R.C.S.); the British Museum, Natural History (B.M.); the Biometric Laboratory, University College, London (Biom. Lab.); the University Anatomical Museum at Cambridge (Cambridge); the Oxford University Museum (Oxford); and the Museum of the "Anatomisches Laboratorium," Leiden, Holland (Leiden). The abbreviations given in the brackets are used to denote them in this paper. *Bm.* denotes this *Journal* and *J.A.I.* the *Journal of the Anthropological Institute*.

A. EUROPEAN SERIES.

I. *English: Long Barrow.* Cambridge 18 ♂. The skulls in Thurnam's collection came chiefly from different localities in Wiltshire. They are of the Neolithic period.

II. *Romano-British.* Cambridge 11 ♂, 3 ♀; Oxford 81 ♂, 43 ♀. The specimens at Cambridge came from a single cemetery at White Horse Hill, Berkshire, and those at Oxford came from a number of counties, mainly from Oxfordshire, Berkshire and Wiltshire.

III. *Anglo-Saxon.* (a) Cambridge 13 ♂, 8 ♀; Oxford 41 ♂, 19 ♀. These skulls came from a considerable number of cemeteries in different parts of England. (b) Cambridge 42 ♂, 20 ♀, only the complete skulls being measured. This series was recently excavated by Mr Lethbridge at Burwell, Cambs., from a cemetery of seventh century date. In order to justify the pooling of the measurements of the above two series the male means and standard deviations for the four indices dealt with were calculated separately so that the significance of the differences of the means for these characters between the two series could be satisfactorily tested. The values are given in the Table at the top of p. 205.

It is evident that all the differences between the means are quite insignificant. This justifies the provisional pooling of the two series and the combined means, which have been used for comparative purposes in this paper, are given in Table I.

IV. *English: Mediaeval.* Oxford 43 ♂, 11 ♀. These crania came chiefly from Abingdon, Berkshire, and Sittingbourne, Kent. A few unspecified specimens are also included.

Indices	Frontal		Simotic	
	Mean	σ	Mean	σ
Series (a)	$18.6 \pm .19$ (52)	$2.07 \pm .14$	52.3 ± 1.06 (44)	$10.41 \pm .75$
Series (b)	$18.5 \pm .22$ (40)	$2.06 \pm .16$	$52.4 \pm .87$ (33)	$7.54 \pm .63$
(a) - (b)	$+0.1 \pm .29$	—	-0.1 ± 1.38	—
	Rhinal		Premaxillary	
	Mean	σ	Mean	σ
Series (a)	$45.4 \pm .85$ (16)	$5.04 \pm .60$	$35.1 \pm .36$ (42)	$3.50 \pm .26$
Series (b)	45.2 ± 1.39 (6)*	—	$35.1 \pm .47$ (34)	$4.10 \pm .34$
(a) - (b)	$+0.2 \pm 1.63$	—	$0.0 \pm .60$	—

V. *English: Farringdon Street*. Biom. Lab. 81 ♂, 76 ♀. This series came from a London churchyard used in the seventeenth century. See *Bm.* Vol. xviii. pp. 1—55, 1926.

VI. *English: Spitalfields*. Biom. Lab. 248 ♂, 94 ♀. This series came from a burial-ground at Spitalfields Market, London, and it is of mediaeval or Roman date. See *Bm.* Vol. xxiii. pp. 191—248, 1931.

VII. *Modern Irish*. R.C.S. 19, Cambridge 8, Oxford 38 (soldiers): all ♂. Of these only 15 are from known localities of the Irish Free State. The majority of the crania, including 38 soldiers and 5 seamen, are from unknown localities in Ireland.

VIII. *Modern Scottish*. R.C.S. 19, Cambridge 3, Oxford 13 (soldiers): all ♂. Twenty skulls came from known localities of Scotland and others are unspecified.

IX. *British (Soldiers)*. (a) English soldiers: Oxford 31. Their native places are unknown. (b) British soldiers: Oxford 82. No detailed information relating to the origin of these specimens is available.

Indices	Frontal		Simotic	
	Mean	σ	Mean	σ
Series (a)	$19.1 \pm .29$ (31)	$2.35 \pm .20$	51.5 ± 1.65 (30)	13.36 ± 1.16
Series (b)	$18.9 \pm .18$ (82)	$2.46 \pm .13$	$53.1 \pm .97$ (81)	$12.84 \pm .68$
(a) - (b)	$+ .2 \pm .34$	—	-1.6 ± 1.91	—
	Rhinal		Premaxillary	
	Mean	σ	Mean	σ
Series (a)	$43.2 \pm .90$ (19)	$5.82 \pm .64$	$35.6 \pm .32$ (27)	$2.45 \pm .23$
Series (b)	$44.7 \pm .49$ (65)	$5.89 \pm .35$	$36.3 \pm .24$ (74)	$3.03 \pm .17$
(a) - (b)	-1.5 ± 1.03	—	$-.7 \pm .36$	—

* The probable error of the mean of the rhinal index for the (b) series is based on the standard deviation of the same index for the (a) series.

The mean measurements of the above two series have been pooled to give those in Table I (p. 222). It is evident from the figures shown above that the mean differences for the four indices between the two series are too small to be significant. It is, therefore, legitimate to pool them.

The last three series VII—IX above are probably less homogeneous racially than any other British series used. The means derived from them can only be used provisionally.

X. *North German*. R.C.S. 28, Cambridge 4: all ♂. The specimens came principally from different provinces of Prussia—Brandenburg, Holstein, Westphalia, Hessen and Saxony being represented.

XI. *French*. R.C.S. 46 ♂, 19 ♀. These skulls came mainly from the catacombs of Paris, though a few are from unknown localities. They belong to the post-Merovingian period.

XII. *Italian*. R.C.S. 102 ♂, 33 ♀; Cambridge 1 ♀. These specimens, which previously belonged to G. Nicolucci, came from 12 divisions in the northern and central parts of Italy. A few from the islands of Sicily and Sardinia were excluded. They are all of modern date.

XIII. *Norwegian*. R.C.S. 19 ♂. These specimens came from a Seamen's Hospital at Greenwich, so they are presumably the skulls of seamen. One individual is stated to have come from Bergen, but the home districts of the others are not known.

XIV. *Swedish*. R.C.S. 30 ♂. Of these, 9 crania came from the neighbourhood of Stockholm, and the others are only marked as seamen.

XV. *Finnish*. R.C.S. 22 ♂, Cambridge 1 ♂. Ten of these crania came from various localities of Finland, and 13 from a Seamen's Hospital collection are recorded as Russian Finns. As the series is a short one, no useful comparisons could be made between the two groups.

XVI. *Dutch*. R.C.S. 29 ♂, 10 ♀; Leiden 22 ♂. Two-thirds of these specimens came from different known provinces of Holland, viz. Guelderland, Utrecht, Friesland, Groningen, Zeeland and North Holland: the others are unspecified.

XVII. *Ancient Greek*. R.C.S. 3 ♂, Oxford 9 ♂. These skulls came from Athens, Attica and the Ionian Islands, and they are dated approximately between 500 and 200 B.C.

XVIII. *Modern Greek*. R.C.S. 18 ♂, Oxford 15 ♂. The localities from which these specimens came are: 8 from the Greek mainland, 12 from the Ionian Islands, 4 from the Aegean Islands, and 9 from unknown districts of Greece.

XIX. *Czech*. R.C.S. 23 ♂, 10 ♀. These specimens came from Bohemia, Slovakia and Moravia. The central part of Moravia is better represented than the other regions, so the series may be loosely called a Czech one.

XX. *Russian*. R.C.S. 20 ♂, Leiden 16 ♂. Of these specimens, 22 came from various scattered localities in Russia, and others are marked as "unknown" in locality.

European races, apart from British, are much less satisfactorily represented than are those of the other continental areas except America. This is chiefly due to the fact that there are few European skulls other than British in English museums. The pooling of several short series from various localities of the same country had to be effected in several cases in order to obtain sufficiently large samples.

B. AFRICAN SERIES.

I. *Guanche*. R.C.S. 16 ♂, 19 ♀; Cambridge 5 ♂, 4 ♀; Leiden 4 ♂, 5 ♀. Nearly all these specimens came from Teneriffe.

II. *Nubian*. Biom. Lab. 69 ♂, 61 ♀. This series was excavated by the Archaeological Survey of Nubia in 1907—1908 from sites immediately south of Assuan. It consists only of New Empire (17th—20th Dynasty) crania.

III. *Egyptian: Badari*. Biom. Lab. 36 ♂, 22 ♀. See *Bm.* Vol. XIX. pp. 110—150, 1927. This series is believed to be the earliest of predynastic date that has yet been discovered in Egypt. It has been shown that the earlier Egyptian series bear a closer resemblance to negro types than do the later series from the same country.

IV. *Egyptian: Sedment*. Biom. Lab. 40 ♂, 30 ♀. See *Bm.* Vol. XXII. pp. 65—92, 1930. This series came from a cemetery of 9th Dynasty date at Gebel Sedment, which is about 70 miles South of Cairo.

V. *Egyptian: Kerma*. Biom. Lab. 117 ♂, 97 ♀. These skulls represent the Egyptian population of Kerma (Nubia) in the 12th and 13th Dynasties. See *Bm.* Vol. XXV. pp. 254—284, 1933.

VI. *Negro: Nigeria*. (a) Crania from North Nigeria. R.C.S. 6 ♂, 4 ♀; Cambridge 13 ♂, 3 ♀. These skulls came from various parts of North Nigeria. (b) Crania from South Nigeria. R.C.S. 34 ♂, 12 ♀. These skulls came from several provinces of the country, and the majority of them are recorded as representing the Ibibio and Ekoi tribes of the Calabar region. The male means of these two series for the four indices were calculated separately and are shown below. The probable errors of the North Nigeria means were found by using the South Nigeria standard deviations.

Indices	Frontal	Simotic	Rhinal	Premaxillary
(a) N. Nigeria (Mean)	17.7 ± .29 (18)	30.7 ± 1.68 (17)	30.2 ± 1.16 (11)	36.2 ± .56 (17)
(b) S. Nigeria (Mean)	17.6 ± .21 (34)	28.9 ± 1.22 (32)	29.5 ± 1.11 (12)	36.5 ± .43 (29)
S. Nigeria (σ)	1.82 ± .15	10.24 ± .86	5.70 ± .79	34.2 ± .30
(a) - (b)	+ .1 ± .36	+ 1.8 ± 2.08	+ .7 ± 1.61	- .3 ± .71

It is obvious that none of the differences between the corresponding pairs of indices should be regarded as significant. The pooled means in Table I (p. 222) were used for comparative purposes.

VII. *Gaboon Negro*. (a) Gaboon, 1864. B.M. 50 ♂, 44 ♀. See *Bm.* Vol. VIII. pp. 292—339, 1912. This series was collected by Du Chaillu in Fernand Vaz. (b) Gaboon, 1880. B.M. 18 ♂, 19 ♀. See *ibid.* This series came from the same locality, and it was collected by the same person. It has been shown that series (a) and (b) bear a very close resemblance to one another when all the usual cranial characters are compared. (c) Gaboon, collected at various times. R.C.S. 12 ♂, 6 ♀. These specimens came from the neighbourhood of Fernand Vaz and the Gaboon river. The means of the above three male series for the four indices are shown in the following table.

Indices	Frontal	Simotic	Rhinal	Premaxillary
(a) Gaboon, 1864 (Mean)	17.5 ± .21 (46)	30.9 ± .91 (45)	28.9 ± .41 (32)	38.5 ± .37 (40)
(b) Gaboon, 1880 (Mean)	17.5 ± .34* (18)	29.0 ± 1.52 (16)	29.2 ± .64 (13)	38.5 ± .64 (13)
(c) Gaboon, R.C.S. (Mean)	18.3 ± .42* (12)	28.3 ± 1.92 (10)	28.7 ± 1.16 (4)	39.3 ± .73 (10)
Gaboon, 1864 (σ)	2.13 ± .15	9.01 ± .64	3.43 ± .29	3.42 ± .26
(a) - (b)	0 ± .40	+1.9 ± 1.77	-.3 ± .76	0 ± .74
(a) - (c)	-.8 ± .47	+2.6 ± 2.12	+.2 ± 1.23	-.8 ± .82
(b) - (c)	-.8 ± .54	+.7 ± 2.45	+.5 ± 1.32	-.8 ± .97

It is evident that the differences between all possible pairs of means are quite insignificant, and the combined means given in Table I may therefore justifiably be employed for comparison in this paper.

VIII. *Angoni*. B.M. 25 ♂. See *J.A.I.* Vol. XXVIII. pp. 55—94, 1898. The specimens were obtained from M'ponda's Town at the south end of Lake Nyassa. The donor, Sir H. H. Johnston, describes them as being the crania "of a slightly mixed negro race mainly belonging to the Anyanja stock with a slight Zulu intermixture. It is possible, however, that one or more of them may be Yaos mixed with Arab blood."

IX. *Ashanti*. R.C.S. 12 ♂; Cambridge 2 ♂; Oxford 20 ♂, 31 ♀. Of these, 4 male specimens came from Kumasi, and 3 males were of Ashanti warriors killed in war in 1873—74. Others are merely marked as having been obtained from districts on the West coast of Africa, and no further information is available.

X. *Teita Negro*. Biom. Lab. 57 ♂, 65 ♀. See *Bm.* Vol. XXIII. pp. 271—314, 1931. The Teita, or Wa-Teita, are a small tribe of Bantu-speaking negroes living in the South of Kenya Colony.

XI. *Zulu*. R.C.S. 17 ♂. Most of these skulls are of Zulu warriors. They came from two battle-fields on the Zulu border, Isandhlwana and Ulundi, where fighting took place in 1856 and 1879 respectively.

* The probable errors of the four indices for series (b) and (c) were obtained by using the corresponding standard deviations of series (a).

XII. *Congo Negro*. R.C.S. 58 ♂, 30 ♀; Leiden 2 ♂, 1 ♀. See *Bm.* Vol. VIII. pp. 292—339, 1912. The majority of these specimens represent the Batetela tribe who live at about 24° 20' E., 4° 51' S. near the Lubefu River. A few came from the neighbourhood of the Congo basin and the Upper Congo regions.

XIII. *Bushman*. R.C.S. 9 ♂, 14 ♀; Cambridge 2 ♂; B.M. 1 ♂; Oxford 11 ♂, 3 ♀. Of these, there are 4 males and 1 female from the Cape of Good Hope, 1 male from the Transvaal, 1 male from Namaqualand, and 1 female from the Orange Free State, while the localities from which the others came are unknown.

XIV. *Hottentot*. R.C.S. 2 ♂, 6 ♀; Oxford 7 ♂, 2 ♀. Of these, 3 males and 1 female are known to have come from Namaqualand, Middleburg, the Cape of Good Hope and the Cape of St Francis respectively, while the others are from unknown localities.

XV. *Kaffir*. R.C.S. 22 ♂, 7 ♀; Cambridge 5 ♂, 2 ♀; Oxford 25 ♂, 1 ♀; B.M. 3 ♂. Various tribes of Kaffirs are represented in this series, but most of the specimens represent Bantu negroes inhabiting the South and South-East of Africa. Six male specimens are stated to have been obtained from native burial-grounds during the Kaffir War in 1847.

C. AMERICAN SERIES.

I. *Eskimo*. R.C.S. 33 ♂, 17 ♀; Cambridge 4 ♂, 1 ♀; Oxford 11 ♂, 11 ♀; B.M. 4 ♂, 1 ♀. Of these, 31 males and 18 females are marked as having come from various parts of Greenland and the neighbouring islands; 12 males and 8 females are from several islands situated to the North of Baffin Land and from the North-East coast of Labrador. A few came from unknown localities. All the specimens have been pooled together to make up a sufficient number for the purpose of statistical enquiry.

II. *Peruvian*. This is one of longest series which was measured for the present purpose. In order to examine local differences in the characters dealt with, the material may be first divided into two groups: (a) Specimens from Pacasmayo. R.C.S. 20 ♂, 13 ♀; Cambridge 47 ♂, 22 ♀; Oxford 4 ♂, 4 ♀. These mostly belong to Consul Hutchinson's Collection. (b) Specimens from other localities. R.C.S. 37 ♂, 28 ♀; Cambridge 10 ♂, 5 ♀; Oxford 55 ♂, 42 ♀. These were mostly purchased from Consul Hutchinson. The localities from which the majority of these skulls came are: Cerro del Oro (in the Canete Valley, 100 miles South-East of Callao), Ancon, Huacho (near Callao) and Santo, all situated in the Western part of Peru. The first two named are better represented than the others. The means of the four indices for the two male series are given in the table at the top of p. 210.

From the figures shown it is apparent that the mean differences are all too small to be considered significant except in the case of the simotic index for which the difference is 2.6 times its probable error. It was considered that for the present purpose all these Peruvians might be pooled together although a significant difference occurs in the second index. It should be noted here that among the Peruvian skulls there are many which show a fronto-occipital deformation. It may

Indices	Frontal		Simotic	
Constants	Mean	σ	Mean	σ
(a) Peruvians from Pacasmayo ...	17.2 \pm .13 (69)	1.61 \pm .09	40.3 \pm .77 (67)	9.30 \pm .54
(b) Peruvians from other localities	17.2 \pm .13 (102)	1.98 \pm .09	42.9 \pm .64 (99)	9.49 \pm .45
(a) - (b)	0 \pm .19		- 2.6 \pm 1.00	
	Rhinal		Premaxillary	
(a) Peruvians from Pacasmayo ...	40.0 \pm .84 (16)	4.99 \pm .59	34.6 \pm .26 (61)	2.96 \pm .18
(b) Peruvians from other localities	39.1 \pm .61 (27)	4.72 \pm .43	34.2 \pm .22 (82)	2.98 \pm .16
(a) - (b)	+ .9 \pm 1.04		+ .4 \pm .34	

be asked: Does such a deformation affect their facial measurements? This question can be easily answered by making a comparison of the means for each character from two sets of crania, i.e. crania recorded as having a slight or marked artificial deformation in the frontal region and crania preserved in a natural state. Let us take the male Peruvian specimens at Oxford for illustration as the state of deformation is individually recorded in the catalogue of the Museum there. The indices for the two groups just mentioned were calculated separately and the standard deviations based on the total male Peruvian crania were employed to obtain the probable errors. The constants are:

Indices	Frontal	Simotic	Rhinal	Premaxillary
(a) Normal crania	16.7 \pm .19 (43)	43.9 \pm 1.00 (41)	38.3 \pm .84 (15)	33.0 \pm .36 (32)
(b) Deformed crania	17.0 \pm .31 (16)	41.2 \pm 1.66 (15)	40.0 \pm 1.33 (6)	34.1 \pm .56 (13)
(a) - (b)	- .3 \pm .36	+ 2.7 \pm 1.94	- 1.7 \pm 1.58	- 1.1 \pm .66

It will be seen that the ratio of the difference to its probable error is in no case greater than 1.7. This indicates that our facial measurements are not affected to a significant degree by the artificial deformation practised by the Peruvians as far as can be ascertained from the short series available.

III. *North American Indian*. R.C.S. 38 ♂, 26 ♀; Cambridge 8 ♂. This series represents a miscellaneous collection of North American Indians derived chiefly from Vancouver Island and the West. The different tribes are represented by such small numbers that there is no alternative to pooling all the available material; it gives means which are of small permanent value. Some specimens show a certain degree of frontal deformation, but their facial skeletons were presumably not distorted to any appreciable extent.

IV. *Patagonian*. R.C.S. 21 ♂. These specimens came from various districts in the neighbourhood of the South-East coast of Patagonia, North of Tierra del Fuego. They represent several tribes of native Indians.

V. *Fuegian*. R.C.S. 12 ♂. These came from the South-West coast of Tierra del Fuego.

D. OCEANIC SERIES.

I. *Maori: New Zealand*. R.C.S. 44 ♂, 13 ♀; Cambridge 10 ♂, 5 ♀; Oxford 29 ♂, 6 ♀; B.M. 6 ♂, 1 ♀. Of these, 30 males and 9 females came from the North Island, principally from Auckland, and a few came from the South Island, but there are 60 specimens, male and female, the original localities of which are not specified.

II. *Mori*. R.C.S. 33 ♂, 21 ♀; Cambridge 3 ♂, 1 ♀; Oxford 6 ♂, 4 ♀; B.M. 1 ♂, 1 ♀. These specimens came from various parts of the Chatham Islands and it may be assumed that they form a racially homogeneous sample.

III. *New British*. Cambridge 39 ♂, 24 ♀. The specimens were collected by Mr A. Willey in the Island of New Britain.

IV. *Islanders of the Marquesas*. R.C.S. 22 ♂, 16 ♀. The series came from five main Islands, viz. Nukahiva, Fatuhiva, Hivaoa, Uahuka and Hiau.

V. *Easter Islanders*. R.C.S. 35 ♂, 16 ♀; Leiden 11 ♂, 5 ♀. A detailed description of this material is given in Dr von Bonin's paper: *Bm.* Vol. XXIII. pp. 249—270, 1931.

VI. *Kanaka: Sandwich Islands*. R.C.S. 52 ♂, 54 ♀; Cambridge 15 ♂, 6 ♀. "Kanakas" is the name applied by some anthropologists to the natives of the Polynesian Islands generally. In the native language of the Sandwich Islands "Kanaka" means "man." Here it is restricted to the inhabitants of the Sandwich Islands alone. More than 85 per cent. of the crania of this series came from the Island of Oahu (Woahoo) and the remainder from the Island of Hawaii (Owyhee).

VII. *New Caledonian*. R.C.S. 18 ♂. Of these, 10 specimens came from the mainland of New Caledonia, 4 from the Island of Pines, a small island South-East of New Caledonia, and 4 from the Island of Lifu, the largest of the Loyalty Islands, near New Caledonia.

VIII. *Solomon Islanders*. R.C.S. 21 ♂, Cambridge 6 ♂. Among these 3 specimens are known to have come from the Island of San Cristoval, 1 from Guadalcanal and 1 from Isabel. The others are unspecified and only stated as being natives of the Solomon Islands.

IX. *Islanders of New Hebrides*. R.C.S. 36 ♂. Of these, 24 crania came from various Islands of the New Hebrides ranging from the Cherry (Anuda) Island in the North to the island of Tanna in the South. These include Eromanga, Ambrym, Faté, Api and the Banks Islands. The remaining 12 crania came from the Island of Mallicollo, and these are all remarkable for the depression of the frontal region. It has not yet been ascertained whether this is a natural conformation, or whether

it is due to artificial deformation in infancy. The evidence derived from the Peruvian series leads us to hope that the facial measurements of any deformed crania in the present series would not have been affected appreciably.

X. *Fijian*. R.C.S. 30 ♂, 13 ♀. Out of these, 18 males and 9 females came from various burial-grounds on the Island of Viti Levu, the largest island of the Fiji group. They are mostly of the coastal tribes. Six males and 4 females came from the Island of Ovalau and 6 males from the Island of Vanua-Valavo.

XI. *Papuan*. Cambridge 79 ♂, 49 ♀. The specimens were collected by the Daniels' Expedition, and they came from more than 10 different districts in the territory of British New Guinea.

XII. *Islanders of the Cook Group*. R.C.S. 19 ♂. Of these, 9 came from the Island of Atiu (20° 0' S., 158° 5' W.), 4 from the Island of Mangaia and the remainder from neighbouring small islands.

XIII. *Tasmanian*. R.C.S. 21 ♂, 13 ♀. These specimens were collected by various persons in Tasmania.

XIV—XVI. *Australian*. It was found convenient to group the native Australian skulls measured in the following way:

(a) Australians from the Northern Territory. R.C.S. 10 ♂, Oxford 7 ♂, B.M. 1 ♂. These skulls came chiefly from Port Essington, Port Darwin, the region of Van Diemen Gulf and the banks of the Victoria River, so all are from the North-West of the Northern Territory.

(b) Australians from Queensland. R.C.S. 25 ♂, 10 ♀; Cambridge 2 ♂, 1 ♀; Oxford 9 ♂, 1 ♀. More than half these specimens came from North Queensland and the remainder from South, East and Central Queensland.

(c) Western Australians. R.C.S. 8 ♂, 3 ♀; Cambridge 6 ♂, 6 ♀; B.M. 3 ♂, 2 ♀. These skulls came chiefly from the North-Western district of Western Australia.

(d) Australians from New South Wales. R.C.S. 12 ♂, 8 ♀; Cambridge 2 ♂, 2 ♀; Oxford 17 ♂, 8 ♀. These specimens came mainly from Sydney, Port Stephens, Bathurst, and the neighbourhood of the Hunter, Macleay and Clarence Rivers, i.e. from regions close to the Eastern coast of New South Wales.

(e) South Australians. R.C.S. 35 ♂, 26 ♀; Cambridge 11 ♂, 5 ♀; Oxford 5 ♂, 4 ♀; B.M. 1 ♂, 2 ♀. Most of these came from Adelaide and the banks of the Murray River, near the South coast of South Australia.

(f) Australians from Victoria. R.C.S. 18 ♂, 13 ♀; Cambridge 2 ♂; Oxford 4 ♂, 1 ♀. The majority of these crania came from Melbourne, Geelong, Port Phillip and Port Fairy, on the South coast of Victoria. A few came from Ballarat and the vicinity of the Murray River (Victoria).

We require to know whether there are any significant differences between the characters of the six Australian series. In order to examine this point the means of the male series for the four indices were calculated separately and they are given

below. The differences between all possible pairs of means together with their probable errors are presented in the table below. The standard deviations of the South Australian series were used in computing the probable errors of the means for the Northern Territory, Western Australian, New South Wales and Victoria groups. The differences which exceed 2.5 times their probable errors are in italics. Considering the differences between the means in terms of their corresponding probable errors, which are shown in the second part of the table below, it will at once be noticed that the mean indices of the last four Australian series show no significant differences from one another, but the first two series, i.e. Australians from the Northern Territory and from Queensland, differ quite significantly in one or two characters from one another and also from the last four series mentioned, the single exception being that the Northern Territory series shows no significant differences from the South Australian group. Accordingly, the whole of the Australian specimens measured can be best regrouped in three series, viz. Australians from the Northern Territory (XIV), from Queensland (XV), and from all other parts of the continent (XVI). It may be noticed that the frontal indices show no significant differences. The general characters of Australian crania—according to Morant's recent study

Constants	Series	Frontal	Simotic	Rhinal	Premaxillary
Means	(a) Australians from Northern Territory	18.8 ± .39 (18)	41.9 ± 1.50 (16)	37.2 ± .93 (12)	41.5 ± .60 (17)
	(b) Australians from Queensland	18.5 ± .28 (36)	37.5 ± 1.01 (36)	30.1 ± .67 (18)	39.5 ± .39 (35)
	(c) Western Australians	18.9 ± .41 (16)	43.8 ± 1.45 (17)	33.4 ± 1.08 (9)	38.9 ± .75 (11)
	(d) Australians from New South Wales	18.5 ± .30 (30)	43.2 ± 1.09 (30)	33.7 ± .81 (16)	39.2 ± .45 (30)
	(e) South Australians	18.2 ± .23 (52)	45.9 ± .90 (44)	35.0 ± .78 (17)	40.6 ± .39 (41)
	(f) Australians from Victoria	18.4 ± .34 (23)	43.3 ± 1.25 (23)	33.3 ± 1.14 (8)	39.8 ± .54 (21)
Standard Deviations	(b) Australians from Queensland	2.53 ± .21	8.98 ± .71	4.18 ± .47	3.41 ± .28
	(e) South Australians	2.45 ± .16	8.87 ± .62	4.78 ± .55	3.68 ± .27
Differences of Means	(a) - (b)	+ .3 ± .48	+ 4.4 ± 1.80	+ 7.1 ± 1.19	+ 2.0 ± .72
	(a) - (c)	- .1 ± .57	- 1.9 ± 2.08	+ 3.8 ± 1.42	+ 2.6 ± .96
	(a) - (d)	+ .3 ± .49	- 1.3 ± 1.85	+ 3.5 ± 1.23	+ 2.3 ± .76
	(a) - (e)	+ .6 ± .45	- 4.0 ± 1.75	+ 2.2 ± 1.22	+ .9 ± .72
	(a) - (f)	+ .4 ± .52	- 1.4 ± 1.95	+ 3.9 ± 1.47	+ 1.7 ± .81
	(b) - (c)	- .4 ± .50	- 6.3 ± 1.77	- 3.3 ± 1.26	+ .6 ± .85
	(b) - (d)	0 ± .41	- 5.7 ± 1.49	- 3.6 ± 1.04	+ .3 ± .60
	(b) - (e)	+ .3 ± .38	- 8.4 ± 1.35	- 4.9 ± 1.03	- 1.1 ± .55
	(b) - (f)	+ .1 ± .45	- 5.8 ± 1.61	- 3.2 ± 1.32	- .3 ± .67
	(c) - (d)	+ .4 ± .51	+ .6 ± 1.82	- .3 ± 1.35	- .3 ± .87
	(c) - (e)	+ .7 ± .47	- 2.1 ± 1.71	- 1.6 ± 1.33	- 1.7 ± .84
	(c) - (f)	+ .5 ± .54	+ .5 ± 1.91	+ .1 ± 1.57	- .9 ± .92
	(d) - (e)	+ .3 ± .38	- 2.7 ± 1.41	- 1.3 ± 1.12	- 1.4 ± .60
	(d) - (f)	+ .1 ± .46	- .1 ± 1.66	+ .4 ± 1.40	- .6 ± .71
	(e) - (f)	- .2 ± .41	+ 2.6 ± 1.54	+ 1.7 ± 1.39	+ .8 ± .67

(*Bm.* Vol. XIX. pp. 417—440, 1927)—suggest a division into two groups, viz. Northern Australians, on the one hand, and all the remainder, on the other. This accords with the present grouping based on measurements of facial flattening only, except that we have also to divide off a Queensland group. The pooled means of each character for our three Australian series are given in Table I (p. 222).

E. ASIATIC SERIES.

I. *Chinese.* Two series of Chinese crania were measured in the museums of Holland and England:

(a) *Chinese (Leiden).* Leiden 48 ♂, 4 ♀. Of these, 15 males are marked as having come from the provinces of Kwantung and Fukien, South China, 14 males and 2 females from Hong Kong and Batavia (Java), and the remaining crania are unspecified. There is some reason to believe that these unspecified specimens most probably belong to the Southern Chinese and that no Northern elements are involved.

(b) *Chinese (English museums).* R.C.S. 67 ♂, 12 ♀; Cambridge 6 ♂; Oxford 13 ♂, 2 ♀; B.M. 3 ♂. Nearly half of the specimens measured are marked as having come from Canton, Macao, Amoy, Hong Kong and Batavia (Java), and these almost certainly represent the Southern Chinese. Others are marked as criminals, seamen, or merely inscribed as "Chinese," and no further information is available. These unspecified crania, we believe, may be the heads of Southern Chinese as the spellings of their names given on the skulls are most likely Cantonese or Fukienese. The means and variations of the four indices for both male Chinese series are given in the following table.

Indices	Frontal		Simotic	
	Mean	σ	Mean	σ
(a) Chinese (Leiden)	15.9 ± .22 (48)	2.29 ± .16	32.8 ± .77 (48)	7.92 ± .55
(b) Chinese (England)	15.8 ± .16 (88)	2.15 ± .11	31.1 ± .86 (89)	11.97 ± .61
(a) - (b)	+ .1 ± .28	—	+ 1.7 ± 1.15	—
	Rhinal		Premaxillary	
(a) Chinese (Leiden)	31.9 ± .52 (43)	5.08 ± .37	34.8 ± .32 (44)	3.14 ± .23
(b) Chinese (England)	31.2 ± .43 (66)	5.19 ± .31	33.9 ± .28 (77)	3.57 ± .19
(a) - (b)	+ .7 ± .68	—	+ .9 ± .42	—

Again we find that for all the characters with which we are dealing the two series differ insignificantly from one another, although these samples were collected in various parts of South China and its neighbouring islands. This accords with the belief that the Chinese race is homogeneous in its physical type.

II. *Burmese*. Two series of Burmese skulls were measured.

(a) *Burmese A* Type. Biom. Lab. 44 ♂, 39 ♀. A southern series from the neighbourhood of Moulmein was divided into three groups of which the *Burmese A*, according to Miss Tildesley's study (*Bm.* Vol. XIII. pp. 176—262, 1921), is supposed to be true Burman.

(b) *Burmese* in general. R.C.S. 32 ♂; Oxford 10 ♂. The specimens in these museums are only marked "from Burma." Their localities are mostly unspecified with the exception of 7 cases marked "from Rangoon or Pegu." All skulls known to represent the primitive native races of Burma were omitted.

The differences of the mean indices between the two male series are tested in the usual way.

Indices	Frontal		Simotic	
	Mean	σ	Mean	σ
(a) <i>Burmese A</i>	15.8 ± .28 (42)	2.72 ± .20	32.9 ± .97 (41)	9.24 ± .69
(b) <i>Burmese</i> in general	16.2 ± .16 (42)	1.57 ± .12	33.3 ± .97 (41)	9.25 ± .69
(a) - (b)	-.4 ± .33	—	-.4 ± 1.38	—
	Rhinal		Premaxillary	
(a) <i>Burmese A</i> ...	28.7 ± .52 (21)	3.53 ± .37	32.7 ± .35 (34)	3.00 ± .25
(b) <i>Burmese</i> in general	29.8 ± .43 (36)	3.80 ± .30	33.6 ± .32 (36)	2.85 ± .23
(a) - (b)	-1.1 ± .67	—	-.9 ± .47	—

It is seen that so far as the indices of facial flattening are concerned the two series do not differ significantly from one another. It is, therefore, justifiable to pool them for the present comparative purposes.

III. *Japanese*. R.C.S. 9 ♂, Cambridge 1 ♂, Oxford 4 ♂, B.M. 3 ♂. The series is a short one as there are only a few Japanese specimens in English museums. Only three crania are marked as having come from Tokio, Yokohama and Kyoto respectively, the others being from unknown parts of Japan. They are all of modern date.

IV. *Aino*. R.C.S. 7 ♂, B.M. 1 ♂. Of these, 5 came from the island of Yezo (Hokkaido), the homeland of the Aino, 2 came from the Kurile Islands and 1 from an unknown locality.

V. *Tibetan A*. R.C.S. 17 ♂, B.M. 20 ♂. These specimens came from the South-West of the country and are classified by Morant as of the *Tibetan A* Type. See *Bm.* Vol. XIV. pp. 193—260, 1923, and Vol. XVI. pp. 1—105, 1924.

VI. *Tibetan B*. R.C.S. 15 ♂. This series represents soldiers from the Eastern Province of Kham who are said to belong to the *Tibetan B* Type. See Morant, *loc. cit.* 1923.

VII. *Nepalese*. B.M. 48 ♂. These crania came from different parts of Nepal. See *Bm.* Vol. xvi. pp. 1—105, 1924.

VIII. *Malay*. R.C.S. 10 ♂. These specimens were collected in the Malay States.

IX. *Andamanese*. R.C.S. 17 ♂, 16 ♀; Cambridge 14 ♂, 9 ♀. These specimens came mainly from the Great Andaman Islands although several crania are unspecified. South Andaman is better represented than the other islands.

X. *Nicobarese*. Cambridge 14 ♂. These specimens were collected by J. M. Wright in 1923—24.

XI. *Naga*. Biom. Lab, 23 ♂, 15 ♀. These specimens were on loan to the Biometric Laboratory at the time when they were measured (see *Bm.* Vol. xxv. pp. 1—20, 1933). They represent natives of the Naga Hills in Eastern Assam.

XII. *Javanese*. The Javanese skulls measured may be divided into the four groups below.

(a) Javanese from Bantam and Batavia. Leiden 55 ♂, 12 ♀. This series came from the provinces of Bantam and Batavia in the Western part of the island. See *Bm.* Vol. xxiii. pp. 52—113, 1931.

(b) Javanese from Middle and East Java. Leiden 69 ♂, 15 ♀. These skulls came principally from the Middle and East of the island although a few unspecified skulls are also included. See *ibid.*

(c) Javanese from Madura. R.C.S. 10 ♂, Leiden 13 ♂.

(d) Javanese in general. R.C.S. 30 ♂, 7 ♀. These skulls came from various parts of Java.

Constants	Series	Frontal	Simotic	Rhinal	Premaxillary
Means	(a) Javanese (B.B.)	15.5 ± .20 (54)	33.4 ± .92 (53)	30.4 ± .55 (46)	35.3 ± .38 (51)
	(b) " (M.E.)	15.9 ± .17 (69)	32.7 ± .82 (67)	30.4 ± .37 (61)	34.1 ± .32 (67)
	(c) " (Madura)	16.0 ± .31* (23)	31.6 ± 1.44 (22)	30.7 ± .71 (21)	35.0 ± .57 (22)
	(d) " (General)	16.1 ± .26* (30)	33.4 ± 1.24 (30)	30.5 ± .73 (20)	33.9 ± .52 (26)
Standard Deviations	(a) Javanese (B.B.)	2.18 ± .14	9.88 ± .65	5.54 ± .39	4.01 ± .27
	(b) " (M.E.)	2.08 ± .12	10.13 ± .58	4.25 ± .26	3.87 ± .23
Differences of Means	(a) - (b)	-.4 ± .26	+ .7 ± 1.23	0 ± .66	+ 1.2 ± .50
	(a) - (c)	-.5 ± .37	+ 1.8 ± 1.71	-.3 ± .90	+ .3 ± .68
	(a) - (d)	-.6 ± .33	0 ± 1.54	-.1 ± .91	+ 1.4 ± .64
	(b) - (c)	-.1 ± .35	+ 1.1 ± 1.66	-.3 ± .80	-.9 ± .65
	(b) - (d)	-.2 ± .31	-.7 ± 1.49	-.1 ± .81	+ .2 ± .61
	(c) - (d)	-.1 ± .40	- 1.8 ± 1.90	+ .2 ± 1.01	+ 1.1 ± .77

* The mean weighted value of the standard deviations of the first two series ($\bar{\sigma} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2}}$) for each index was used in calculating the probable errors of the last two series.

As the mean differences of the indices between all possible pairs of these series, which are shown in the table on p. 216, are too small to be considered significant, the four Javanese series have justifiably been pooled for the present purpose and the combined means are given in Table I. It has been shown by von Bonin that cranial series from different parts of Java are very similar to one another in type.

XIII. *Sumatran*. R.C.S. 8♂, Leiden 48♂. These crania came from various parts of the country. A large number of them are said to come from the residencies of Palembang and Lampong and from the West coast of Sumatra. Several from Nias and other neighbouring islands have been included.

XIV. *Natives of Celebes*. R.C.S. 9♂, Leiden 43♂. This series is chiefly made up of the skulls of Macassars and Bugis.

XV. *Natives of Sarawak*. R.C.S. 9♂, Cambridge 46♂, 24♀. The specimens preserved in the two museums were all collected by Sir Charles Hose in Borneo in 1899. They represent various tribes belonging to different districts of Sarawak, chiefly those of the Murut group. They were mostly killed by Long Pata, Long Kiputs or other tribes in the Baram district, Sarawak.

XVI. *Dayak*. R.C.S. 13♂, 8♀, Leiden 42♂, 10♀. See *Bm.* Vol. xxiii. pp. 52—113, 1931, for an account of the Leiden series. As all these skulls came from Borneo and are marked as Dayaks, the two series may justifiably be pooled.

XVII. *Tagal*. Leiden 31♂, 19♀. See *ibid.* The bulk of these crania came from Manila, Mindanao and Central Luzon, Philippine Islands. The Tagals are usually classified as one of the brown, non-negrito tribes of the islands.

XVIII. *Aëta*. Leiden 33♂, 14♀. See *ibid.* The Aëtas (Philippine Islands) are found in the interior of Luzon Island and in the islands of Mindoro, Panay and Negros, and also in the North-Eastern part of Mindanao. The people are very short in stature and are usually classed as Negritoës.

XIX. *Singalese*. R.C.S. 24♂, 10♀; Cambridge 1♂, 2♀; Oxford 9♂, 1♀. The majority of these specimens came from Colombo, Ceylon, and a few from different parts of the Western and Central Provinces of Ceylon. The crania of peoples other than Singalese living in the island were excluded.

XX. *Veddah*. R.C.S. 16♂, 10♀; Cambridge 2♂, 1♀; Oxford 3♂, 3♀. These skulls came mainly from the South-Western part of the Eastern Province of Ceylon.

XXI. *Punjabi*. The Indians living in the Province of Punjab are, according to their religions, chiefly Mohammedan and Hindus. It is supposed that the people of the two groups do not inter-marry. The specimens measured represent: (a) Mohammedans. R.C.S. 39♂, Cambridge 6♂. This series comprises all crania inscribed as Mohammedan or Moslem from the Punjab. (b) Hindus. R.C.S. 42♂, Cambridge 6♂. This series consists of various castes of Hindus among which the low castes are better represented than the others. From the figures shown below it will be seen

that no significant difference is found between any corresponding pair of indices for these two series. They have therefore been pooled.

Series	Frontal		Simotic	
	Mean	σ	Mean	σ
(a) Mohammedans	21.2 \pm .28 (45)	2.79 \pm .20	49.6 \pm 1.28 (45)	12.72 \pm .90
(b) Hindus	20.9 \pm .25 (48)	2.55 \pm .18	49.2 \pm 1.26 (47)	12.79 \pm .89
(a) - (b)	+ .3 \pm .37	—	+ .4 \pm 1.79	—
	Rhinal		Premaxillary	
	Mean	σ	Mean	σ
(a) Mohammedans	43.5 \pm .77 (23)	5.46 \pm .54	36.6 \pm .37 (39)	3.41 \pm .26
(b) Hindus	43.6 \pm 1.44 (20)	9.55 \pm 1.02	35.7 \pm .39 (40)	3.64 \pm .28
(a) - (b)	- .1 \pm 1.63	—	+ .9 \pm .54	—

XXII. *Hindu: Madras Presidency.* R.C.S. 49 ♂. These skulls came from the following localities: (a) 30 crania from Berhampur in the North-East of the Presidency; (b) 6 from the Bellary district, in the Western part of the Presidency; (c) 6 from a burial-ground near Madras, and (d) 7 from the Salem district in the South of the Presidency. All these specimens have been pooled for the purpose of the present paper.

XXIII. *Hindu: North-East India.* There are three more Hindu series which were measured in different museums. These came from the Provinces of Bengal, Bihar and Orissa, and the United Provinces, all in the North-East of India, so they may be conveniently dealt with together. The origin of these series is described as follows:

(a) Bengalese. R.C.S. 88 ♂, 38 ♀; Cambridge 8 ♂, 9 ♀; Oxford 19 ♂, 5 ♀. The majority of these specimens represent various castes of Hindus living in different districts of Bengal. Several crania of Mohammedans which came from the same province are also included.

(b) Hindus from Bihar and Orissa. R.C.S. 37 ♂. These skulls represent various castes of Hindus living in the two provinces mentioned. The specimens came largely from the Patna district in the North-West of Bihar.

(c) Hindus from the United Provinces. R.C.S. 26 ♂; Cambridge 1 ♂. These specimens represent several castes of Hindus who inhabited the United Provinces. The low castes of Hindus are better represented than the high.

The significance of the differences between the mean indices for these three series and those from the Punjab and Madras (Nos. XXI and XXII above) is tested as usual and the constants are given in the Table on p. 220. The differences which exceed 2.5 times their probable errors are in italics.

From the figures shown in the lower part of the table (p. 220) it may be observed that the Punjabi (XXI) and the Southern Hindus (XXII) differ significantly from one another in a slight degree in the case of the first two indices; also, both these series show slightly significant differences, in one or two characters, from all the three series from North-East India, with the single exception that no significant difference is found between the series from Bihar and Orissa and that from Madras Presidency (XXII and XXIII *b*). The last two represent adjoining regions. The largest difference, which is 4.3 times its probable error, is found for the rhinal index between the Punjabi (XXI) and the Bengalese (XXIII *a*), which represent regions as far apart as any pair. In the case of the last three Hindu series (XXIII *a-c*) the differences of the means for all possible pairs of indices are quite insignificant. This shows that the degree of "facial flatness" of these samples from an enormous Indian population is a constant feature as far as can be told from the slender evidence available, and such a result agrees with that of a previous study which was based on the usual cranial measurements of several short Hindu series: see *Bm.* Vol. xxⁿ. pp. 294—300, 1928. In view of the comparisons made above, the five East Indian series measured may be justifiably reduced to three groups, viz. the skulls from the Punjab, those from North-East India (XXIII *a-c*) and those from Madras Presidency. The grouping seems to be a reasonable one which accords with the geographical positions of the peoples compared.

XXIV. *Dravidian*. R.C.S. 36 ♂. These specimens belong to two short series: (a) 18 crania which came chiefly from the Madura district (Madras Presidency), and belong to the Maravar tribe, which is Dravidian; (b) 18 crania from the Tanjore district (Madras Presidency), and no detailed information relating to these is available, but the locality suggests that they are most probably of Dravidian origin.

XXV. *Afghan*. R.C.S. 18 ♂. These crania came from various parts of Afghanistan. Six of them are recorded as belonging to the Yusafzai tribes in the Peshawar Valley of this country.

XXVI. *Arab*. R.C.S. 10 ♂. This collection came from scattered parts of the country including Western (the coast of Midian), Southern (Hadramaut) and Eastern (Oman) Arabia.

XXVII. *Natives of Syria and Palestine*. R.C.S. 10 ♂ from Syria, 11 ♂ from Palestine; Cambridge 4 ♂ from Syria. This series came from several cemeteries in various districts of the two countries. Owing to the small number of specimens measured the pooling of these crania from the two neighbouring countries has been considered desirable for the purpose of the present investigation only.

"Flatness" of the Facial Skeleton in Man

	Indices	Frontal		Simotic		Rhinal		Premaxillary	
	Series	Mean	σ	Mean	σ	Mean	σ	Mean	σ
Means and σ 's	XXI. Punjabi	21.0 ± .20 (98)	2.67 ± .14	49.4 ± .91 (92)	12.94 ± .64	43.5 ± .79 (43)	7.64 ± .56	36.2 ± .28 (79)	3.67 ± .20
	XXII. Hindus from Madras Presidency	20.1 ± .24 (49)	2.06 ± .14	45.7 ± 1.04 (46)	10.44 ± .73	39.6 ± 1.47* (10)	—	36.8 ± .34 (48)	3.44 ± .24
	XXIII (a). Bengalese	20.5 ± .15 (115)	2.44 ± .11	46.3 ± .82 (114)	13.03 ± .58	39.3 ± .59 (52)	6.25 ± .41	35.4 ± .24 (107)	3.74 ± .17
	XXIII (b). Hindus: Bihar and Orissa	20.2 ± .20 (37)	1.78 ± .14	47.9 ± 1.14 (32)	9.55 ± .81	41.5 ± 1.10* (18)	—	35.9 ± .41 (32)	3.43 ± .29
	XXIII (c). Hindus: United Provinces	19.8 ± .32* (27)	—	47.1 ± 1.59* (27)	—	41.9 ± 1.65* (8)	—	35.0 ± .50* (24)	—
Differences of Means	XXI—XXII	+ .9 ± .31		+ 3.7 ± 1.98		+ 3.9 ± 1.67		— .6 ± .44	
	XXI—XXIII (a)	+ .5 ± .25		+ 3.1 ± 1.28		+ 4.2 ± .98		+ .8 ± .37	
	XXI—XXIII (b)	+ .8 ± .28		+ 1.5 ± 1.46		+ 2.0 ± 1.35		+ .3 ± .50	
	XXI—XXIII (c)	+ 1.2 ± .38		+ 2.3 ± 1.83		+ 1.6 ± 1.82		+ 1.2 ± .57	
	XXII—XXIII (a)	— .4 ± .29		— .6 ± 1.32		+ .3 ± 1.58		+ 1.4 ± .41	
	XXII—XXIII (b)	— 1 ± .31		— 2.2 ± 1.54		— 1.9 ± 1.84		+ .9 ± .53	
	XXII—XXIII (c)	+ .3 ± .40		— 1.4 ± 1.90		— 2.3 ± 2.21		+ 1.8 ± .60	
	XXIII (a)—XXIII (b)	+ .3 ± .25		— 1.6 ± 1.40		— 2.2 ± 1.24		— .5 ± .48	
	XXIII (a)—XXIII (c)	+ .7 ± .35		— .8 ± 1.79		— 2.6 ± 1.75		+ .4 ± .56	
	XXIII (b)—XXIII (c)	+ .4 ± .37		+ .8 ± 1.96		— .4 ± 1.98		+ .9 ± .65	

* The probable error of a mean index based on less than 30 skulls was found by using an average σ found from the weighted values of the squared standard deviations $(\bar{\sigma} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + \dots}{n_1 + n_2 + \dots}})$ for all the series in the table having the same index based on more than 30 skulls.

(4) *Sexual Differences of Mean Values.* The material dealt with consists of 83 male and 48 female series, and in general the female series are considerably shorter than the corresponding male series. Table I gives their means for the eight absolute measurements and four indices together with the probable errors. The sexual differences of these constants may be considered first. The sex ratios (i.e. the male means divided by the female means) are given in Table II for the eight absolute measurements in all cases where the means for both sexes are based on 20 or more skulls. Values have been given previously for a number of the larger calvarial arcs and chords in the case of seven European and African series*. These show that the ratio may be peculiar to the particular character considered, but the range found for all of them only extended from 1.024 to 1.066. It can be seen from Table II that all the sex ratios for the internal biorbital breadth (*IOW*)—a calvarial chord—fall within the same range. Little significance can be attached to such a range, of course, since the extreme values may be affected appreciably by errors of random sampling, or the inclusion of any series for which the male and female means do not represent precisely the same racial type is likely to extend the extremes considerably. While showing a few ratios less than 1.024 or greater than 1.066, most of these for *MOW*, *GB*, Sub. *IOW* and Sub. *GB* lie between those values. As far as can be seen the sex ratios for these characters are not peculiar, and the subtenses to *IOW* and *GB* have approximately the same values as the chords. But for the remaining measurements the position is different. For the simotic chord (*SC*) there are seven ratios less than unity, the female mean being greater than the male, and 14 of the 27 ratios are less than 1.024. The simotic subtense (*SS*), however, has only one value less than 1.024, but 24 greater than 1.066. For every series the ratio for the subtense is greater than that for the chord. The last character, the rhinal subtense (Sub. *MOW*), also has high sex ratios, the lowest of the 11 that can be given being 1.078, while again the value for the subtense is invariably greater than the value for the chord (*MOW*) in the case of the same series. We see then that the simotic chord shows peculiarly small sex ratios while the simotic and rhinal subtenses are also distinguished, but on account of their peculiarly large ratios. The interracial averages shown in Table II bear out these conclusions. The breadth of the nasal bones shows relatively small differences between man and woman, but there are relatively large sexual differences between measurements of the projection of the nasal bones, both at the bridge of the nose and at their lower extremity. It may be anticipated that large sexual differences will be found between mean simotic and rhinal indices for the same series, and possibly no markedly significant sexual differences between the mean frontal and premaxillary indices. The fact that these relations do hold can be appreciated most readily from Figs. 1 and 2. The first figure shows the interracial correlation between the simotic and rhinal indices, all available means based on 15 or more skulls being shown, and it can be seen at a glance that in the case of both these indices every male mean is greater than every corresponding female value. This is a particularly striking relation since several of the series are

* See *Biometrika*, Vol. xxiii. pp. 214 and 275, 1931.

TABLE I. Mean Measurements of Facial Flatness for different Racial Series*.

Continent	Serial Number	Series	Sex	Absolute Measurements										Indices			
				IOW	Sub. IOW	SC	SS	MOH	Sub. MOH	GB	Sub. GB	Frontal	Simotic	Rhinal	Pre-maxillary		
A. Europe	I	English: Long Barrow	♂	98.5 (18)	18.4	10.1 (13)	5.3	—	—	94.5 (9)	33.6	18.9 ± 3.4	52.8	—	35.6		
	II	Romano-British	"	98.5 (86)	18.3	9.3 (82)	4.6	53.6 (30)	23.3	94.8 (64)	32.8	18.5 ± 1.6	50.5 ± .93	43.8 ± .69	35.3 ± .25		
	III	Anglo-Saxon	"	98.1 (92)	18.2	9.1 (77)	4.8	52.0 (22)	23.4	95.2 (76)	33.4	18.5 ± 1.5	52.4 ± .71	45.3 ± .83	35.1 ± .28		
	IV	English: Mediaeval	"	98.2 (43)	17.7	9.4 (36)	4.5	54.8 (19)	21.7	95.2 (35)	32.6	17.9 ± 1.9	47.2 ± 1.44	39.9 ± .89	34.3 ± .34		
	V	English: Farringdon St.	"	98.2 (75)	18.6	9.2 (81)	4.6	54.1 (37)	22.9	92.4 (53)	32.7	19.0 ± 1.7	50.7 ± .96	42.7 ± .68	35.6 ± .32		
	VI	English: Spitalfields	"	98.1 (248)	18.0	9.7 (126)	4.5	54.2 (50)	22.0	93.4 (73)	32.2	18.4 ± .07	47.4 ± .71	40.7 ± .49	34.6 ± .24		
	VII	Irish	"	99.0 (65)	19.2	8.6 (65)	4.6	54.2 (44)	24.0	93.8 (57)	34.2	19.3 ± 2.0	53.1 ± .98	44.6 ± .54	36.4 ± .24		
	VIII	Scottish	"	99.1 (35)	18.7	9.1 (32)	4.8	54.4 (16)	24.3	93.2 (31)	33.9	18.8 ± 2.6	52.8 ± 1.12	44.9 ± .97	36.4 ± .38		
	IX	British (Soldiers)	"	97.8 (113)	18.5	8.9 (111)	4.7	54.3 (84)	24.1	93.1 (101)	33.6	19.0 ± 1.6	52.7 ± .84	44.4 ± .45	36.1 ± .21		
	X	North Germans	"	99.5 (32)	18.7	9.6 (30)	4.9	52.9 (13)	25.6	94.7 (31)	33.4	18.8 ± 2.8	51.5 ± 1.47	48.6	35.4 ± .44		
	XI	French	"	98.6 (46)	18.7	9.5 (43)	4.6	53.8 (11)	24.8	91.7 (39)	32.4	19.3 ± 2.1	49.2 ± .92	46.4	35.4 ± .44		
	XII	Italian	"	98.1 (100)	19.3	9.7 (100)	4.5	53.6 (52)	24.2	94.4 (94)	32.9	19.7 ± 1.6	47.2 ± .77	45.4 ± .54	34.8 ± .24		
	XIII	Norwegian	"	97.5 (19)	18.9	9.2 (18)	4.9	53.9 (2)	24.6	92.2 (19)	33.3	19.4 ± 3.3	53.0 ± 1.85	45.7	36.3 ± .51		
	XIV	Swede	"	98.7 (30)	19.3	8.8 (29)	4.5	55.7 (7)	25.4	93.3 (29)	33.0	19.6 ± 2.9	52.2 ± 1.46	45.6	35.7 ± .41		
	XV	Finn	"	100.0 (23)	19.0	10.3 (22)	4.8	52.8 (6)	20.9	97.2 (23)	33.3	19.0 ± 3.0	47.2 ± 1.67	39.7	34.3 ± .47		
	XVI	Dutch	"	99.4 (51)	19.1	9.0 (51)	4.7	53.6 (22)	25.7	92.8 (45)	33.2	19.2 ± 2.4	53.0 ± 1.05	48.3 ± .83	35.7 ± .33		
	XVII	Ancient Greek	"	98.5 (12)	19.4	10.5 (11)	5.3	—	—	97.6 (9)	33.4	19.7	51.4	—	34.4		
	XVIII	Modern Greek	"	97.3 (33)	19.0	9.4 (32)	4.5	56.2 (11)	24.6	95.0 (25)	33.1	19.5 ± 2.7	47.8 ± 1.20	44.3	34.9 ± .45		
	XIX	Czech	"	97.7 (23)	17.9	10.0 (23)	4.6	54.3 (11)	23.0	93.0 (19)	33.6	18.3 ± 3.0	46.0 ± 1.64	42.5	36.4 ± .51		
	XX	Russian	"	98.4 (35)	18.4	9.1 (36)	4.5	53.2 (16)	23.8	95.5 (33)	34.3	18.7 ± 2.3	50.0 ± 1.32	45.2 ± .97	35.9 ± .31		
	II	Romano-British	♀	95.1 (45)	17.7	9.0 (42)	4.2	51.7 (16)	20.6	89.5 (39)	31.4	18.6 ± 2.2	45.1 ± 1.12	40.2 ± .89	35.0 ± .38		
	III	Anglo-Saxon	"	94.0 (45)	17.2	8.7 (41)	4.0	51.9 (11)	21.3	89.9 (40)	32.0	18.2 ± 1.6	45.9 ± .98	41.3	35.6 ± .30		
	IV	English: Mediaeval	"	94.2 (11)	16.4	8.3 (10)	3.5	—	—	90.6 (9)	30.1	17.4	40.0	—	33.4		
	V	English: Farringdon St.	"	94.2 (75)	17.6	9.1 (76)	4.0	52.1 (47)	20.5	86.7 (51)	30.7	18.7 ± 1.7	44.6 ± .79	39.4 ± .52	35.6 ± .38		
	VI	English: Spitalfields	"	93.5 (94)	17.2	9.6 (31)	4.3	53.7 (14)	20.1	87.9 (19)	30.8	18.2 ± 1.2	45.3 ± 1.41	37.8	34.8 ± .55		
	XI	French	"	94.3 (19)	18.8	10.6 (18)	4.6	51.6 (4)	21.1	87.7 (15)	31.5	19.9 ± 3.1	43.3 ± 1.66	40.7	35.9 ± .62		
	XII	Italian	"	92.2 (32)	18.3	9.4 (31)	3.9	50.7 (20)	20.9	87.8 (24)	31.1	19.8 ± 3.2	41.7 ± 1.29	41.2 ± .80	35.4 ± .48		
	XVI	Dutch	"	94.4 (10)	18.3	8.5 (10)	4.2	48.2 (4)	22.8	88.2 (8)	32.2	19.3	50.2	47.4	36.5		
	XIX	Czech	"	92.4 (10)	17.5	8.9 (8)	4.0	54.6 (3)	21.1	86.1 (9)	30.7	18.9	44.7	38.7	35.7		

B. Africa	I	Guanche	♂	99-1 (25)	18-8	10-2 (24)	4-8	59-6 (5)	24-9	98-4 (23)	32-1	19-0 ± 30	47-4 ± 1-28	41-8	32-6 ± 50
	II	Nubian	"	94-4 (69)	17-5	11-7 (55)	4-2	54-1 (6)	18-6	95-6 (40)	33-1	18-3 ± 20	36-3 ± 74	34-6	34-5 ± 39
	III	Egyptian: Badari	"	94-1 (32)	16-6	11-1 (34)	4-2	54-4 (20)	18-8	94-6 (34)	35-1	17-6 ± 25	38-7 ± 82	34-8 ± 63	37-2 ± 39
	IV	Egyptian: Sedment	"	95-5 (40)	17-5	9-5 (37)	3-9	54-1 (21)	21-8	98-9 (37)	36-2	18-3 ± 25	41-1 ± 1-11	40-3 ± 62	38-5 ± 38
	V	Egyptian: Kerma	"	96-2 (117)	17-5	10-4 (99)	3-8	58-8 (22)	19-3	94-9 (103)	33-9	18-2 ± 15	36-5 ± 62	32-9 ± 60	35-8 ± 24
	VI	Negro: Nigeria	"	99-3 (52)	17-5	9-4	2-7	59-0 (23)	17-5	95-0 (46)	34-6	17-6 ± 17	29-5 ± 93	29-8 ± 59	36-4 ± 36
	VII	Gaboon Negro	"	99-3 (76)	17-4	9-3	2-8	62-1 (49)	17-9	95-7 (63)	36-9	17-6 ± 17	29-9 ± 70	28-9 ± 40	38-6 ± 28
	VIII	Angoni	"	99-6 (17)	17-1	8-7 (22)	2-5	63-1 (12)	17-8	95-2 (14)	37-1	17-1 ± 37	30-1 ± 1-34	28-4	39-1
	IX	Ashanti	"	99-4 (33)	17-3	8-8 (34)	2-9	61-1 (18)	18-1	95-3 (30)	36-5	17-4 ± 21	34-0 ± 1-23	30-3 ± 67	38-4 ± 39
	X	Tseta Negro	"	99-1 (55)	17-2	9-4 (39)	2-2	63-5 (27)	17-1	98-5 (31)	33-1	17-3 ± 20	22-4 ± 83	26-9 ± 54	33-7 ± 33
	XI	Zulu	"	102-4 (17)	18-6	9-4 (17)	2-6	60-1 (6)	18-4	97-6 (16)	37-3	18-1 ± 37	28-3 ± 1-52	30-8	38-1 ± 59
	XII	Congo Negro	"	98-6 (60)	17-1	9-5 (60)	2-5	59-8 (17)	17-8	93-9 (42)	33-3	17-3 ± 19	26-5 ± 72	29-9 ± 68	35-5 ± 44
	XIII	Bushman	"	95-6 (21)	14-8	6-8	1-4	58-4 (10)	11-5	92-7 (16)	31-5	15-5 ± 33	20-7 ± 1-37	19-7	34-1 ± 59
	XIV	Hottentot	"	97-3 (9)	16-7	7-9 (9)	2-3	60-1 (4)	13-5	93-7 (8)	32-9	17-0	29-3	22-7	35-1
	XV	Kafir	"	103-0 (54)	17-8	8-5 (54)	2-8	60-9 (25)	18-3	96-5 (51)	35-4	17-3 ± 22	33-6 ± 1-10	30-4 ± 56	36-8 ± 34
C. America	I	Guanche	♀	93-5 (26)	17-5	10-3 (27)	4-0	53-5 (6)	21-1	93-0 (22)	30-0	18-7 ± 27	38-7 ± 1-11	40-0	32-2 ± 45
	II	Nubian	"	92-0 (61)	17-0	10-5 (38)	3-2	58-7 (4)	19-7	91-9 (35)	32-5	18-4 ± 16	30-2 ± 67	33-6	35-3 ± 29
	III	Egyptian: Badari	"	90-3 (19)	16-0	10-6 (18)	4-0	53-6 (9)	18-0	90-9 (15)	33-4	17-7 ± 32	37-5 ± 1-36	33-8	36-7 ± 54
	IV	Egyptian: Sedment	"	90-0 (30)	15-7	10-0 (29)	3-8	51-5 (21)	19-4	89-4 (28)	34-0	17-4 ± 25	37-9 ± 1-08	37-7 ± 60	37-9 ± 40
	V	Egyptian: Kerma	"	92-3 (94)	16-3	9-9 (74)	3-2	54-1 (11)	16-3	90-7 (83)	33-1	17-7 ± 15	32-1 ± 74	30-3	36-5 ± 22
	VI	Negress: Nigeria	"	95-9 (19)	16-9	9-7 (17)	2-1	61-3 (6)	17-8	91-2 (15)	33-5	17-6 ± 32	22-4 ± 1-40	29-6	36-8 ± 54
	VII	Gaboon Negress	"	94-8 (60)	16-7	9-7 (55)	2-5	59-0 (44)	16-6	90-6 (51)	35-2	17-6 ± 18	26-1 ± 77	28-2 ± 40	38-9 ± 31
	IX	Ashanti	"	94-0 (31)	16-9	8-0 (30)	2-1	58-3 (14)	16-1	88-5 (23)	35-1	18-0 ± 25	26-4 ± 1-35	27-6	39-7 ± 44
	X	Tseta Negress	"	94-4 (62)	16-6	10-2 (49)	1-9	61-7 (31)	14-9	94-3 (37)	32-1	17-6 ± 18	18-6 ± 69	24-2 ± 52	34-1 ± 41
	XII	Congo Negress	"	95-2 (30)	16-2	9-5 (25)	2-3	59-3 (9)	16-2	92-9 (23)	32-9	17-0 ± 23	25-7 ± 1-16	27-5	35-4 ± 44
	XIII	Bushman	"	92-3 (17)	14-5	7-9 (16)	1-4	56-7 (9)	10-9	86-6 (15)	29-7	15-7 ± 34	16-9 ± 1-45	19-2	34-6 ± 54
	XIV	Hottentot	"	94-5 (8)	15-7	7-6 (8)	2-0	53-8 (2)	15-0	88-8 (8)	31-7	16-6	27-5	27-9	35-8
	XV	Kafir	"	99-1 (10)	17-1	7-7 (9)	1-8	56-1 (5)	15-8	91-6 (10)	33-2	17-3	24-3	28-3	36-4
	I	Eskimo	♂	99-1 (51)	14-0	5-7 (50)	2-4	51-9 (20)	17-1	103-3 (45)	32-1	14-1 ± 19	42-5 ± 85	33-3 ± 73	31-4 ± 33
		II	Peruvian	"	97-3 (171)	16-7	9-2 (166)	3-8	57-8 (43)	22-7	99-8 (143)	34-2	17-2 ± 09	41-9 ± 50	39-4 ± 50
III		North American Indian	"	100-2 (45)	16-8	8-5 (46)	2-7	55-7 (20)	20-3	101-0 (41)	33-4	16-7 ± 18	31-4 ± 1-13	36-5 ± 73	33-5 ± 31
IV		Patagonian	"	102-0 (20)	18-0	8-4 (20)	4-2	65-6 (2)	23-6	102-8 (19)	37-0	17-6 ± 28	49-0 ± 1-47	36-2	36-0 ± 47
V		Fuegian	"	99-0 (12)	16-5	7-9 (12)	3-2	58-0 (5)	22-9	99-3 (12)	35-4	16-7	42-8	39-4	35-6
I	Eskimo	♀	94-3 (30)	13-6	5-4 (29)	1-9	48-3 (8)	15-1	95-8 (27)	31-6	14-4 ± 21	35-7 ± 92	31-1	33-0 ± 38	
	II	Peruvian	"	93-4 (114)	15-5	9-0 (107)	3-2	53-6 (20)	20-3	94-4 (97)	33-3	16-7 ± 13	36-2 ± 48	38-1 ± 79	35-3 ± 20
	III	North American Indian	"	95-8 (26)	16-5	8-3 (26)	2-4	54-8 (13)	17-7	95-3 (24)	32-6	17-2 ± 27	29-5 ± 97	32-4	34-3 ± 41

* The serial numbers in the second column of this table are those used in the description of the material in section (3) of the text. The numbers in round brackets are the numbers of skulls on which the means are based, and for every series the chord and the corresponding subense and index are for the same skulls, so the number is only given in the case of the chord. The probable errors of mean indices based on 30 or more skulls were found by using the standard deviations for the same series; for those based on 15-29 skulls the standard deviations used are the weighted average values for all series belonging to the same continental area, and where the means are based on fewer than 15 skulls the probable errors are not given. All the standard deviations used are given in Table V.

TABLE I—(continued).

Serial Number	Series	Sex	Absolute Measurements							Indices				
			TOH	Sub. IOW	SC	SS	MOW	Sub. MOW	GB	Sub. GB	Frontal	Simotic	Rhinal	Pre-maxillary
I	Maori	♂	99.1 (89)	17.0	7.2 (87)	3.1 (87)	53.8 (38)	21.1	98.8 (76)	35.4	17.3 ± .14	43.2 ± .88	39.4 ± .56	36.3 ± .27
II	Mori	"	98.9 (43)	17.3	6.5 (42)	2.8 (42)	57.8 (16)	22.1	102.6 (39)	37.0	17.4 ± .19	43.4 ± 1.24	38.6 ± .81	36.2 ± .31
III	New British	"	99.6 (39)	16.1	8.0 (39)	2.7 (39)	59.4 (33)	16.5	96.3 (35)	37.3	16.1 ± .20	33.4 ± 1.14	27.9 ± .41	38.9 ± .27
IV	Marquesas Islands	"	98.5 (22)	17.6	7.4 (22)	2.9 (22)	57.1 (5)	18.7	99.4 (22)	34.5	17.9 ± .33	40.6 ± 1.61	33.6	34.7 ± .48
V	Easter Island	"	98.1 (41)	17.7	7.8 (44)	3.1 (44)	59.5 (21)	21.2	98.0 (36)	35.7	18.0 ± .20	40.0 ± 1.25	36.0 ± .71	36.2 ± .38
VI	Kanaka: Sandwich Is.	"	97.7 (67)	16.0	7.4 (63)	2.7 (63)	58.0 (47)	19.1	98.2 (54)	33.5	16.4 ± .17	36.7 ± 1.00	33.8 ± .53	34.3 ± .26
VII	New Caledonian	"	98.1 (15)	16.9	8.5 (15)	3.0 (15)	60.0 (4)	21.0	97.1 (14)	34.6	17.2 ± .39	35.4 ± 1.95	34.9	35.7
VIII	Solomon Islands	"	98.8 (27)	17.4	8.0 (26)	3.0 (26)	59.0 (10)	18.5	97.0 (23)	34.9	17.6 ± .29	37.6 ± 1.48	31.3	36.0 ± .47
IX	New Hebrides Islands	"	99.2 (36)	17.7	8.6 (34)	3.1 (34)	57.9 (15)	20.0	96.6 (33)	35.7	17.8 ± .25	36.1 ± 1.17	34.9 ± .84	37.0 ± .44
X	Fijian	"	99.8 (30)	18.4	8.6 (30)	3.4 (30)	57.6 (14)	21.1	97.3 (26)	36.3	18.4 ± .24	39.8 ± 1.34	36.8	37.3 ± .44
XI	Papuan	"	97.3 (76)	16.7	8.1 (78)	3.2 (78)	57.8 (47)	18.7	96.7 (65)	34.6	17.2 ± .14	39.6 ± .77	32.6 ± .50	35.9 ± .30
XII	Cook Group Islands	"	97.8 (17)	17.3	8.2 (15)	3.3 (15)	59.1 (4)	20.5	95.9 (19)	34.6	17.6 ± .37	40.1 ± 1.95	34.6	36.2 ± .52
XIII	Tasmanian	"	99.5 (21)	18.6	8.4 (20)	3.4	—	—	91.4 (17)	38.7	18.7 ± .33	41.7 ± 1.69	—	42.3 ± .55
XIV	Australian: Northern Territory	"	98.5 (18)	18.5	8.6 (16)	3.5 (16)	54.2 (12)	20.1	89.8 (17)	37.2	18.8 ± .36	41.9 ± 1.89	37.2	41.5 ± .55
XV	Australian: Queensland	"	101.8 (36)	18.8	9.1 (36)	3.4 (36)	60.0 (18)	18.1	94.3 (35)	37.2	18.5 ± .28	37.5 ± 1.01	30.1 ± .67	39.5 ± .38
XVI	Australian: "Other" regions	"	101.5 (121)	18.7	9.3 (114)	4.1 (114)	57.7 (50)	19.6	92.9 (103)	37.0	18.4 ± .18	44.4 ± .63	34.0 ± .44	39.8 ± .24
I	Maori	♀	93.8 (23)	15.4	7.0 (25)	2.5 (25)	54.2 (11)	17.9	93.5 (20)	34.5	16.4 ± .28	36.8 ± 1.26	33.2	37.0 ± .52
II	Mori	"	94.3 (26)	17.0	7.6 (25)	2.6 (25)	56.3 (7)	20.5	95.9 (23)	34.9	18.0 ± .26	34.6 ± 1.26	36.9	36.4 ± .48
III	New British	"	94.0 (24)	15.9	7.9 (24)	2.1 (24)	55.4 (18)	14.4	91.1 (22)	35.4	16.9 ± .27	26.1 ± 1.28	26.1 ± .63	39.0 ± .49
IV	Marquesas Islands	"	89.9 (16)	15.5	7.2 (16)	2.2 (16)	55.1 (10)	17.3	92.3 (14)	33.7	17.1 ± .33	29.9 ± 1.57	31.5	36.5
V	Easter Island	"	93.3 (18)	16.2	8.1 (21)	2.5 (21)	56.4 (10)	18.6	91.5 (8)	32.6	17.3 ± .31	30.6 ± 1.37	33.0	35.7
VI	Kanaka: Sandwich Is.	"	92.8 (60)	15.0	7.2 (60)	2.1 (60)	54.4 (37)	16.7	91.8 (57)	32.7	16.0 ± .18	28.6 ± .96	30.7 ± .42	35.7 ± .27
X	Fijian	"	93.3 (13)	16.5	7.4 (11)	2.8 (11)	53.0 (4)	19.9	91.0 (13)	34.5	17.6	37.3	37.4	37.9
XI	Papuan	"	93.3 (49)	16.5	8.2 (49)	2.8 (49)	54.9 (34)	16.9	91.6 (40)	33.7	17.6 ± .18	34.0 ± .61	31.1 ± .46	36.9 ± .43
XIII	Tasmanian	"	95.6 (12)	16.9	8.0 (11)	2.1	—	—	89.4 (11)	34.9	17.6	25.1	—	38.7
XV	Australian: Queensland	"	96.1 (12)	17.5	8.5 (12)	3.1 (12)	57.5 (5)	17.2	88.8 (11)	35.3	18.2	35.5	29.9	38.9
XVI	Australian: "Other" regions	"	95.9 (80)	17.3	8.8 (74)	3.0 (74)	55.7 (27)	16.8	88.7 (71)	35.5	18.0 ± .15	34.5 ± .65	30.5 ± .52	40.0 ± .27

I	Chinese	96.1 (136)	15.1	7-6 (137)	2.4	54.6 (109)	17.0	98.3 (121)	34.0	15.8 ± 13	31.7 ± .62	31.5 ± .34	34.2 ± .22
II	Burmese	98.4 (84)	15.7	8-9 (82)	3.0	57.6 (57)	16.9	101.5 (70)	33.7	16.0 ± 17	33.1 ± .67	33.2 ± .24	33.2 ± .24
III	Japanese	98.4 (17)	15.9	7-9 (17)	2.4	55.1 (12)	17.4	99.7 (16)	35.5	16.2 ± 36	31.7 ± 1.73	31.9	35.6 ± .60
IV	Aino	98.5 (8)	17.0	8-6 (8)	3.2	54.1 (4)	17.4	100.9 (7)	34.3	17.2	36.6	32.2	34.0
V	Tibetan A	96.5 (37)	15.5	8.0 (35)	2.5	56.0 (34)	17.3	98.6 (36)	31.5	16.1 ± 23	31.8 ± .95	31.0 ± .66	32.0 ± .36
VI	Tibetan B	99.5 (15)	16.2	7.7 (15)	2.6	56.4 (6)	17.7	100.4 (14)	35.4	16.2 ± 38	34.6 ± 1.85	32.0	35.4
VII	Nepalese	94.3 (46)	15.9	8.1 (47)	3.0	56.3 (39)	18.0	97.4 (44)	32.2	16.9 ± 26	37.7 ± 1.43	31.5 ± .63	33.1 ± .30
VIII	Malay	96.0 (10)	15.0	7.8 (10)	2.7	53.9 (7)	17.0	96.6 (10)	33.7	15.6	33.9	31.5	35.1
IX	Andamanese	91.6 (31)	15.8	9.4 (25)	2.4	52.6 (8)	18.0	94.8 (27)	32.8	17.2 ± 22	25.3 ± 1.43	34.2	34.6 ± .46
X	Nicarabrese	94.7 (14)	15.3	9.2 (14)	2.6	53.5 (11)	14.8	95.7 (10)	32.3	16.2	27.0	26.6	33.8
XI	Naga	96.9 (20)	15.8	8.5 (16)	2.9	—	—	102.2 (10)	34.4	16.2 ± 31	34.9 ± 1.79	—	33.8
XII	Javanese	97.0 (176)	15.4	8.7 (172)	2.8	56.7 (148)	17.3	100.5 (166)	34.6	15.8 ± 11	32.9 ± .50	30.5 ± .27	34.5 ± .20
XIII	Sumatran	96.4 (56)	15.9	8.4 (56)	2.7	55.6 (52)	18.0	99.5 (50)	35.5	16.5 ± 14	32.6 ± .91	32.8 ± .55	35.8 ± .34
XIV	Celebes (Natives)	99.0 (52)	16.3	8.7 (52)	2.7	55.6 (45)	17.6	100.6 (49)	34.3	16.4 ± 21	31.7 ± .70	31.7 ± .43	34.2 ± .33
XV	Sarawak (Natives)	96.4 (55)	16.2	8.0 (55)	2.6	55.7 (25)	17.3	98.7 (43)	32.5	16.8 ± 19	33.7 ± .91	31.3 ± .70	32.9 ± .47
XVI	Dayak	96.5 (55)	15.9	8.5 (50)	2.7	56.6 (44)	17.6	101.2 (46)	34.2	16.5 ± 18	31.8 ± .94	31.3 ± .45	33.8 ± .33
XVII	Tagal	97.6 (31)	16.5	9.6 (27)	3.4	59.2 (11)	17.0	100.1 (23)	34.1	16.9 ± 20	36.0 ± 1.38	28.9	34.1 ± .50
XVIII	Aëta	96.4 (33)	15.9	8.7 (26)	3.1	58.3 (19)	16.4	98.8 (27)	34.2	16.5 ± 19	35.3 ± 1.40	28.1 ± .81	34.7 ± .46
XIX	Singalese	96.1 (34)	20.1	8.8 (34)	4.0	56.8 (28)	22.8	94.0 (26)	36.0	21.0 ± 26	45.3 ± 1.20	40.3 ± .66	38.4 ± .47
XX	Veddah	93.7 (21)	19.3	8.5 (20)	3.8	57.0 (12)	20.8	91.7 (13)	33.8	20.6 ± 32	44.8 ± 1.60	36.6	37.6
XXI	Punjabi	95.8 (93)	20.2	9.1 (92)	4.5	55.6 (43)	24.1	93.8 (79)	33.7	21.0 ± 20	49.4 ± .91	43.5 ± .79	36.2 ± .28
XXII	Hindu: Madras	94.7 (49)	19.0	8.7 (46)	3.9	55.8 (10)	22.0	93.0 (48)	34.2	20.1 ± 24	45.7 ± 1.04	39.6	36.8 ± .34
XXIII	Hindu: N.E. India	95.4 (179)	19.4	8.7 (173)	4.1	57.1 (78)	22.7	94.5 (163)	33.4	20.3 ± 12	46.7 ± .57	39.9 ± .38	35.4 ± .18
XXIV	Dravidian	93.2 (36)	18.9	8.7 (35)	3.7	54.8 (2)	21.6	93.0 (31)	34.1	20.3 ± 24	43.1 ± 1.19	39.5	36.6 ± .40
XXV	Afghan	97.6 (18)	20.0	9.8 (17)	5.0	55.3 (5)	24.0	96.9 (16)	35.5	20.5 ± 35	52.0 ± 1.73	42.5	36.7 ± .60
XXVI	Arab	97.5 (9)	18.9	9.4 (9)	4.2	54.1 (5)	22.8	92.3 (9)	34.3	19.4	45.5	43.5	37.2
XXVII	Syrian and Palestinian (Natives)	95.5 (25)	18.7	10.0 (20)	5.2	56.7 (4)	27.3	92.4 (16)	33.3	19.5 ± 30	52.4 ± 1.60	48.4	36.1 ± .60

I	Chinese	91.3 (17)	14.6	7-0 (16)	1.8	52.1 (12)	15.8	93.5 (16)	33.2	16.0 ± 32	25.8 ± 1.54	30.4	35.5 ± .53
II	Burmese	93.3 (38)	14.3	8.5 (37)	2.5	55.2 (15)	14.0	97.1 (30)	31.3	15.3 ± 22	29.8 ± .94	25.4 ± 1.00	32.5 ± .34
IX	Andamanese	89.1 (25)	15.3	9-2 (24)	2.2	51.8 (12)	16.8	89.7 (25)	31.8	17.1 ± 26	23.9 ± 1.26	32.4	35.4 ± .42
XI	Naga	92.2 (15)	14.7	8-2 (14)	2.2	55.8 (3)	14.0	94.6 (10)	32.0	15.9 ± 34	25.5	25.6	33.9
XII	Javanese	93.3 (34)	14.3	8.7 (34)	2.4	55.6 (30)	15.7	95.5 (30)	33.6	15.4 ± 20	27.7 ± .89	28.5 ± .58	33.2 ± .45
XV	Sarawak (Natives)	91.9 (24)	15.7	8-6 (23)	2.6	51.8 (9)	13.8	92.5 (16)	31.4	16.9 ± 27	31.7 ± 1.28	26.7	33.9 ± .53
XVI	Dayak	93.0 (18)	15.8	8-0 (17)	2.3	53.9 (9)	16.1	95.4 (12)	33.1	17.0 ± 31	28.4 ± 1.49	29.8	34.6
XVII	Tagal	91.2 (19)	14.9	8-4 (17)	2.4	50.5 (6)	14.8	93.8 (17)	32.2	16.3 ± 30	28.1 ± 1.49	31.6	34.3 ± .51
XVIII	Aëta	92.8 (14)	14.7	8-0 (13)	2.3	56.3 (8)	14.8	93.4 (11)	31.7	15.8	29.7	26.9	34.1
XIX	Singalese	92.8 (13)	19.0	8-6 (13)	3.3	54.9 (11)	19.4	92.2 (11)	34.7	20.6	37.5	35.4	37.6
XX	Veddah	88.7 (14)	17.5	8-3 (12)	3.1	54.2 (6)	20.8	86.5 (10)	32.1	19.6	38.2	38.5	37.0
XXIII	Hindu: N.E. India	91.8 (50)	18.1	8-5 (48)	3.5	53.3 (25)	18.8	92.2 (46)	31.8	19.7 ± 20	42.5 ± 1.02	35.7 ± .78	35.3 ± .29

TABLE II.

Sex Ratios of the Absolute Measurements (all means used based on Twenty or more Skulls).

Continental Area	Serial No.	Series *	Subtenses				Chords			
			Sub. IOW	SS	Sub. MOW	Sub. GB	IOW	SC	MOW	GB
A. European Series	III	Anglo-Saxon ...	1·058	1·200	—	1·044	1·044	1·046	—	1·059
	V	English: Farringdon St. ...	1·057	1·150	1·117	1·065	1·042	1·011	1·039	1·065
	XII	Italian ...	1·055	1·154	1·158	1·058	1·064	1·032	1·057	1·075
	VI	English: Spitalfields ...	1·047	1·047	—	—	1·049	1·010	—	—
	II	Romano-British ...	1·034	1·095	—	1·045	1·036	1·033	—	1·059
		Average ratios ...	1·050	1·129	1·137	1·053	1·047	1·026	1·048	1·065
B. African Series	IV	Egyptian: Sediment ...	1·115	1·026	1·124	1·065	1·061	·950	1·050	1·050
	I	Guanche ...	1·074	1·190	—	1·070	1·060	·990	—	1·058
	V	Egyptian: Kerma ...	1·074	1·188	—	1·024	1·042	1·051	—	1·046
	XII	Congo Negro ...	1·056	1·087	—	1·012	1·036	1·000	—	1·011
	VII	Gaboon Negro ...	1·042	1·104	1·078	1·048	1·047	·959	1·053	1·057
	X	Negro: Teita Hills ...	1·036	1·158	1·148	1·031	1·050	·922	1·029	1·045
	II	Nubian ...	1·029	1·313	—	1·018	1·026	1·114	—	1·040
	IX	Ashanti ...	1·024	1·381	—	1·040	1·057	1·100	—	1·077
		Average ratios ...	1·056	1·181	1·117	1·039	1·047	1·011	1·044	1·048
C. American Series	II	Peruvian ...	1·077	1·188	1·118	1·027	1·042	1·022	1·078	1·057
	I	Eskimo ...	1·029	1·263	—	1·016	1·051	1·056	—	1·078
	III	North American Indian ...	1·018	1·125	—	1·025	1·046	1·024	—	1·060
		Average ratios ...	1·041	1·192	1·118	1·023	1·046	1·034	1·078	1·065
D. Oceanic Series	I	Maori ...	1·104	1·240	—	1·026	1·056	1·029	—	1·057
	XI	Papuan ...	1·112	1·143	1·107	1·027	1·044	·988	1·053	1·056
	XVI	Australian: "Other regions" ...	1·081	1·367	1·167	1·042	1·058	1·057	1·036	1·047
	VI	Kanaka ...	1·067	1·286	1·144	1·024	1·053	1·028	1·066	1·070
	II	Mori ...	1·018	1·077	—	1·060	1·049	·855	—	1·070
	III	New British ...	1·013	1·286	—	1·054	1·060	1·012	—	1·057
		Average ratios ...	1·049	1·233	1·139	1·039	1·053	·955	1·052	1·059
E. Asiatic Series	II	Burmese ...	1·098	1·200	—	1·077	1·055	1·047	—	1·045
	XII	Javanese ...	1·077	1·167	1·102	1·030	1·040	1·000	1·020	1·052
	XXIII	Hindu: N.E. India ...	1·072	1·171	1·207	1·050	1·039	1·024	1·071	1·025
	IX	Andamanese ...	1·033	1·091	—	1·031	1·028	1·022	—	1·057
	XV	Sarawak (Natives) ...	1·032	·963	—	—	1·049	·930	—	—
		Average ratios ...	1·062	1·126	1·155	1·047	1·042	1·005	1·045	1·045
		Average of all Series ...	1·053 (27)	1·173 (27)	1·134 (11)	1·050 (25)	1·048 (27)	1·012 (27)	1·040 (11)	1·055 (25)

* The series for each continental area are arranged in order according to the size of the sex ratios for the first character.

small, and it may be confidently asserted that for the same series no one of the usual cranial indices for which sexual differences are found would have shown a like uniformity. The fact that marked sexual differences are found for the simotic index was first noted by de Mérejkowsky, and it was confirmed by Ryley, Bell and Pearson. Table III gives the absolute differences of the mean indices and these quantities expressed in terms of their probable errors. It will be seen that most of the sexual differences of the simotic and rhinal indices are significant, and many are markedly significant. The degree of significance tends to be greater for the simotic than for the rhinal index, but it must be remembered that the means for the former are based on considerably larger numbers than the means for the latter in the case of most of the series. For these two characters there are evidently marked sexual differences for all modern races. Fig. 2 and Table III enable us to examine the frontal and premaxillary indices in the same way. For the former there is really no suggestion of sexual differentiation. For 15 series the male index is the greater, for 11 the position is reversed and there is absolute equality in one case. The most significant sexual difference is only 3.09 times its probable error. The premaxillary index is a measurement of shape, which indicates less similarity between the sexes. There are six series having the male mean greater, 18 having the female mean greater, and equality is shown for one series. No differences are markedly significant, but the three which exceed three times their probable errors all have the female mean greater. It is quite likely that the female index would be found invariably greater than the male in the case of large samples, but sexual differentiation here is obviously much less than that observed between the shapes of the nasal bones.

We may conclude that the male nasal bones are, on the average, decidedly more protruding than the female, while the premaxillary region tends to be less protruding in the male. The last relation will clearly not help in aiding the sexing of individual crania, and the rhinal index is of little value for the same purpose since the tips of the nasal bones are so often found to be defective. It may be doubted, too, whether the simotic index is of much value in aiding sexual discrimination owing to the fact that the intraracial variability of the character is very large. The difference between the means for the two sexes (Table III) is in all cases but one (Australians: "Other" regions) less than both male and female standard deviations for the same series (Table V), and in most cases very decidedly less. In these circumstances the simotic index can be of little value by itself in aiding "mathematical sexing," though the form of the nasal bridge is a character which may profitably be considered in conjunction with many others in anatomical sexing.

These data may be considered from another point of view. Do they suggest that sexual differences are greater for "advanced" or for "primitive" races? More and larger samples would obviously be needed to give any decisive answer to this question. The sex ratios for the simotic subtense and the sexual differences for the simotic index need only be considered in this connection, and it may be seen from

TABLE III.
Sexual Differences of the Mean Indices of Facial Flattening (all means used based on Twenty or more Skulls).

Continental Area	Serial No.	Series*	Frontal		Simotic		Rhinal		Premaxillary	
			$\Delta\delta - \varphi \pm p.e.\Delta$	$\frac{\Delta\delta - \varphi}{p.e.\Delta}$	$\Delta\delta - \varphi \pm p.e.\Delta$	$\frac{\Delta\delta - \varphi}{p.e.\Delta}$	$\Delta\delta - \varphi \pm p.e.\Delta$	$\frac{\Delta\delta - \varphi}{p.e.\Delta}$	$\Delta\delta - \varphi \pm p.e.\Delta$	$\frac{\Delta\delta - \varphi}{p.e.\Delta}$
A. European Series	III	Anglo-Saxon	+3 ± .22	+1.36	+6.5 ± 1.21	+ 5.37	—	—	—	—
	V	English: Farringdon St.	+3 ± .24	+1.25	+6.1 ± 1.24	+ 4.92	+3 ± .84	—	—	—
	II	Romano-British	—1 ± .27	— .37	+5.4 ± 1.46	+ 3.70	+3 ± .86	+3 ± .46	+ 0 ± .51	0
	XII	Italian	—1 ± .35	— .29	+5.5 ± 1.50	+ 3.67	+4 ± 2 ± .96	+3 ± .46	+ 3 ± .46	+ .65
	VI	English: Spitalfields	+2 ± .14	+1.43	+2.1 ± 1.58	+ 1.33	—	+4 ± .54	+ 6 ± .54	+ 1.11
B. African Series	II	Nubian	—1 ± .26	— .38	+6.1 ± 1.00	+ 6.10	—	—	—	—
	I	Guanche	+3 ± .41	+ .73	+8.7 ± 1.69	+ 5.15	—	—	—	—
	V	Egyptian: Kerna	+5 ± .21	+2.38	+4.4 ± .97	+ 4.54	—	—	—	—
	IX	Ashanti	—6 ± .33	—1.82	+7.6 ± 1.83	+ 4.15	—	—	—	—
	VII	Gaboon Negro	0 ± .24	0	+3.8 ± 1.03	+ 3.69	+1 ± .56	+3 ± .41	+3 ± .41	+ .73
	X	Negro: Teika Hills	—3 ± .27	—1.11	+3.8 ± 1.08	+ 3.52	+7 ± .45	+4 ± .56	+4 ± .56	+ .73
	IV	Egyptian: Sedment	+9 ± .35	+2.57	+3.2 ± 1.55	+ 2.06	+2.7 ± .75	+6 ± .55	+6 ± .55	+1.09
	XII	Congo Negro	+3 ± .30	+1.00	+8 ± 1.36	+ .59	+2.6 ± .86	+1 ± .62	+1 ± .62	+ .16
	II	Peruvian	+5 ± .16	+3.09	+5.7 ± .69	+ 8.26	+1.3 ± .94	+9 ± .26	+9 ± .26	+3.46
	I	Eskimo	—3 ± .28	—1.07	+6.8 ± 1.25	+ 5.44	—	—	—	—
	III	North American Indian	—5 ± .32	—1.56	+1.9 ± 1.49	+ 1.28	—	—	—	—
C. American Series										
	XVI	Australian: "Other" regions	+4 ± .23	+1.74	+9.9 ± .91	+10.88	+3.5 ± .68	+5 ± .15	+2 ± .36	+ .56
	VI	Kanaka	+4 ± .25	+1.60	+8.1 ± 1.39	+ 5.83	+3.1 ± .68	+4 ± .56	+1.4 ± .37	+ .379
	II	Mori	—6 ± .32	—1.88	+8.8 ± 1.76	+ 5.00	—	—	—	—
	XI	Papuan	—4 ± .23	—1.74	+5.6 ± 1.12	+ 5.00	+1.5 ± .68	+2 ± .21	+1.0 ± .52	+ .35
	I	Maori	+9 ± .31	+2.90	+6.4 ± 1.54	+ 4.16	—	—	—	—
D. Oceanic Series	III	New British	—8 ± .34	—2.35	+7.3 ± 1.71	+ 4.23	—	—	—	—
	XII	Javanese	+4 ± .23	+1.74	+5.2 ± 1.02	+ 5.10	+2.0 ± .64	+3 ± .13	+7 ± .48	+1.46
	XXIII	Hindu: N.E. India	+6 ± .23	+2.61	+4.2 ± 1.17	+ 3.60	+4.2 ± .87	+4 ± .83	+1 ± .35	+ .29
	II	Burmese	+7 ± .27	+2.59	+3.3 ± 1.15	+ 2.87	—	—	+7 ± .52	+1.35
	XV	Sarawak (Natives)	—1 ± .33	— .30	+2.0 ± 1.57	+ 1.27	—	—	—	—
E. Asiatic Series	IX	Andamanese	+1 ± .34	+ .29	+1.4 ± 1.90	+ .74	—	—	—	—
		Average difference for all Series	+0.11		+5.21		+2.65		—	—
									—	—

* The series for each continental area are arranged in order according to the size of the sex ratio for the simotic index.

the constants in Tables II (p. 226) and III (p. 229) that the races of Asia and Europe show a slight tendency to be less differentiated sexually than those of the other regions, while the Oceanic races tend to show the greatest sexual differences.

(5) *Sexual and Racial Differences in Variability.* Table IV gives the standard deviations and coefficients of variation for all the racial distributions of absolute measurements made up by 50 or more individuals: Table V gives the standard deviations of the indices for all the distributions made up by 30 or more individuals. The sexual differences in variability may be considered first. Few comparisons can be made in the case of the absolute measurements, but the data suggest that female variation for the characters considered shows a slight tendency to be greater than male variation. Combining all the measurements, there are 18 cases in which the male coefficient of variation exceeds the female and 26 in which the position is reversed. The most significant difference is only 3·4 times its probable error, and among the four cases showing this ratio greater than 3·0 the female index is the greater in three. It is curious that the position appears to be different when the indices are considered. In 44 comparisons the male standard deviation is the greater and for the remaining 14 the female constant is greater than the male. In six cases the sexual difference in variability exceeds 3·0 times its probable error, the greatest ratio being 6·4, and for all these the male constant is the greater. The dominance of male variation for the indices may also be appreciated by comparing the weighted mean standard deviations given in Table V. Out of all possible sexual comparisons between these which can be made for the four indices and five continental areas, there are 16 cases in which the male constant is in excess of the female and only four for which the position is reversed. We must conclude that for these measurements of shape male variation shows a distinct tendency to be greater than female, which for the measurements of size, judging by coefficients of variation, there is either sexual equality in variability, or females show a slight tendency to be more variable than males of the same race. The material is obviously too meagre to investigate whether there are any differences in the preponderance of variability of one sex over the other between different continental areas. Racial comparisons of the constants of variability may be considered next and it will be sufficient if we confine our attention to the male indices in this connection. The series in Table V are arranged in order of their standard deviations for the frontal index in the case of each continental area. The range of values for each area is quite wide and there is no difficulty in selecting pairs belonging to the same group which differ quite significantly. This may not indicate in every case that the racial populations from which the samples were drawn really differ in variability for this particular character, as several of the samples may not have been randomly chosen owing to reasons of which we are ignorant. If the distributions of the standard deviations for the five continental areas are compared it will be seen that they are all overlapping and the inter-group means—excluding the American which can only be based on three series—are almost identical. The same is found in the case of the premaxillary index, and so few series are available for the rhinal index that

TABLE IV.
Variabilities of Absolute Measurements for Distributions of Fifty or more Skulls.

Continental Areas	Serial No.	Race	Sex	Standard Deviations							
				Frontal		Simotic		Rhinal		Premaxillary	
				Chord	Subtense	Chord	Subtense	Chord	Subtense	Chord	Subtense
A. Europe	II	Romano-British ...	♂	4.04 ± .21 (86)	2.33 ± .12	1.79 ± .09 (82)	1.27 ± .07	—	—	4.63 ± .28 (64)	2.92 ± .17
	III	Anglo-Saxon ...	"	3.78 ± .19 (92)	2.23 ± .11	1.94 ± .11 (77)	1.17 ± .06	—	—	5.22 ± .29 (76)	3.42 ± .19
	V	English: Farringdon St. ...	"	3.88 ± .21 (75)	1.94 ± .11	1.92 ± .10 (81)	1.10 ± .06	—	—	6.12 ± .40 (83)	3.27 ± .27
	VI	English: Spitalfields ...	"	3.55 ± .11 (248)	1.83 ± .06	1.90 ± .08 (126)	1.21 ± .05	4.54 ± .30 (50)	2.37 ± .16	3.93 ± .22 (73)	2.53 ± .14
	IX	British (Soldiers) ...	"	4.81 ± .22 (113)	2.41 ± .11	1.59 ± .07 (111)	1.15 ± .05	4.74 ± .25 (84)	3.18 ± .17	5.03 ± .24 (101)	2.58 ± .12
	VII	Modern Irish ...	"	3.28 ± .19 (65)	2.57 ± .15	1.68 ± .10 (65)	1.07 ± .06	—	—	3.64 ± .23 (57)	2.63 ± .17
	XII	Italian ...	"	4.25 ± .20 (100)	2.38 ± .11	2.12 ± .10 (100)	1.15 ± .05	4.60 ± .30 (52)	2.64 ± .17	4.80 ± .24 (94)	3.23 ± .16
	XVI	Dutch ...	"	4.28 ± .29 (51)	2.79 ± .19	1.99 ± .13 (51)	1.19 ± .08	—	—	—	—
	V	English: Farringdon St. ...	♀	3.87 ± .21 (75)	2.23 ± .12	1.74 ± .09 (76)	.99 ± .05	—	—	4.33 ± .29 (51)	2.38 ± .16
	VI	English: Spitalfields ...	"	3.31 ± .16 (94)	1.68 ± .08	—	—	—	—	—	—
A. Europe	Serial No.	Race	Sex	Coefficients of Variation							
				Chord	Subtense	Chord	Subtense	Chord	Subtense	Chord	Subtense
				4.10 ± .21	12.73 ± .67	19.26 ± 1.05	27.50 ± 1.55	—	—	4.88 ± .29	8.91 ± .54
				3.86 ± .19	12.36 ± .62	21.33 ± 1.21	24.33 ± 1.40	—	—	3.49 ± .30	10.24 ± .57
				3.95 ± .22	10.41 ± .58	20.87 ± 1.15	23.91 ± 1.33	—	—	6.62 ± .44	8.79 ± .58
				3.62 ± .11	10.19 ± .31	19.59 ± .86	26.89 ± 1.21	8.37 ± .57	10.77 ± .73	4.21 ± .24	7.86 ± .44
				4.92 ± .22	13.01 ± .60	17.85 ± .83	24.37 ± 1.17	8.72 ± .46	13.20 ± .70	5.40 ± .26	7.67 ± .37
				3.31 ± .20	13.37 ± .81	19.58 ± 1.20	23.20 ± 1.44	—	—	3.89 ± .25	7.69 ± .49
				4.33 ± .21	12.84 ± .60	21.81 ± 1.09	25.48 ± 1.29	8.57 ± .57	10.91 ± .73	5.08 ± .25	9.89 ± .49
				4.31 ± .29	14.63 ± 1.00	22.15 ± 1.55	25.34 ± 1.80	—	—	4.99 ± .33	7.74 ± .52
				4.11 ± .23	12.68 ± .71	19.12 ± 1.07	24.75 ± 1.43	—	—	—	—
				3.54 ± .17	9.78 ± .49	—	—	—	—	—	—

				Standard Deviations						
B. Africa	II	Nubian	♂	3.25 ± 19 (69)	2.43 ± .14	2.18 ± .14 (55)	1.05 ± .07	—	—	—
	V	Egyptian: Kerma	"	3.70 ± 16 (117)	2.08 ± .09	1.92 ± .09 (99)	1.19 ± .06	—	4.53 ± .21 (103)	3.31 ± .16
	VI	Nigerian (Natives)	"	4.04 ± 27 (52)	1.83 ± .12	—	—	—	—	—
	VII	Gaboon Negro	"	3.61 ± 20 (76)	2.55 ± .14	2.63 ± .15 (71)	.98 ± .06	—	4.76 ± .29 (63)	2.86 ± .17
	X	Negro: Teita Hills	"	3.78 ± 24 (55)	2.38 ± .13	—	—	—	—	—
	XII	Congo Negro	"	3.52 ± 22 (60)	2.28 ± .14	2.08 ± .13 (60)	.86 ± .05	—	—	—
	XV	Kafir	"	3.94 ± 26 (54)	2.66 ± .17	2.43 ± .16 (54)	1.11 ± .07	—	4.87 ± .33 (51)	3.35 ± .22
	II	Nubian	♀	3.84 ± 23 (61)	2.12 ± .13	—	—	—	—	—
	V	Egyptian: Kerma	"	3.18 ± 16 (84)	2.16 ± .11	2.04 ± .11 (74)	1.07 ± .06	—	4.18 ± .22 (83)	3.15 ± .16
	VII	Gaboon Negro	"	4.58 ± 28 (60)	1.99 ± .12	2.48 ± .16 (55)	.83 ± .05	—	5.77 ± .38 (51)	3.75 ± .25
X	Negro: Teita Hills	"	3.12 ± 19 (62)	1.99 ± .12	—	—	—	—	—	—
				Coefficients of Variation						
B. Africa	II	Nubian	♂	3.45 ± 20	13.89 ± .81	18.60 ± 1.24	25.10 ± 1.71	—	—	—
	V	Egyptian: Kerma	"	3.83 ± 17	11.88 ± .53	18.48 ± .92	31.24 ± 1.64	—	4.78 ± .23	9.76 ± .46
	VI	Nigerian (Natives)	"	4.06 ± 27	10.44 ± .70	—	—	—	—	—
	VII	Gaboon Negro	"	3.64 ± 20	14.66 ± .82	28.27 ± 1.72	35.59 ± 2.26	—	4.97 ± .30	7.75 ± .47
	X	Negro: Teita Hills	"	3.82 ± 25	13.84 ± .91	—	—	—	—	—
	XII	Congo Negro	"	3.57 ± 22	13.32 ± .83	21.88 ± 1.41	34.40 ± 2.36	—	—	—
	XV	Kafir	"	3.82 ± 25	14.93 ± .99	28.54 ± 2.00	29.53 ± 2.94	—	5.08 ± .34	9.47 ± .64
	II	Nubian	♀	4.12 ± 25	12.47 ± .77	—	—	—	—	—
	V	Egyptian: Kerma	"	3.45 ± 17	13.27 ± .66	20.64 ± 1.19	33.50 ± 2.06	—	4.61 ± .24	9.53 ± .50
	VII	Gaboon Negro	"	4.83 ± 30	11.89 ± .74	25.57 ± 1.75	33.30 ± 2.37	—	6.36 ± .43	10.67 ± .72
X	Negro: Teita Hills	"	3.30 ± 20	11.97 ± .74	—	—	—	—	—	—

Continental Areas	Serial No.	Race	Sex	Standard Deviations							
				Frontal		Simotic		Rhinal		Premaxillary	
				Chord	Subtense	Chord	Subtense	Chord	Subtense	Chord	Subtense
C. America	I	Eskimo	♂	3.45 ± .23 (51)	2.06 ± .14	1.84 ± .12 (50)	.95 ± .06	—	—	—	—
	II	Peruvian	"	3.31 ± .12 (171)	1.98 ± .07	1.78 ± .07 (166)	.86 ± .03	—	—	4.33 ± .17 (143)	3.05 ± .12
	II	Peruvian	♀	3.44 ± .15 (114)	1.97 ± .09	1.44 ± .07 (107)	.79 ± .04	—	—	4.32 ± .21 (97)	2.99 ± .14
				Coefficients of Variation							
C. America	I	Eskimo	♂	3.48 ± .23 (51)	14.72 ± 1.00	32.30 ± 2.40	39.61 ± 3.06	—	—	—	—
	II	Peruvian	"	3.41 ± .12	11.87 ± .44	19.37 ± .73	22.65 ± .88	—	—	4.35 ± .17	8.92 ± .37
	II	Peruvian	♀	3.69 ± .17	12.72 ± .58	15.94 ± .75	24.56 ± 1.20	—	—	4.58 ± .22	8.98 ± .44
				Standard Deviations							
D. Oceania	I	Maori: New Zealand	♂	4.66 ± .24 (89)	2.09 ± .11	1.51 ± .08 (87)	1.04 ± .05	—	—	5.46 ± .30 (76)	3.43 ± .19
	VI	Kanaka	"	3.66 ± .21 (67)	2.33 ± .14	1.55 ± .09	.91 ± .05	—	—	5.04 ± .33 (54)	3.00 ± .19
	XI	Papuan	"	4.02 ± .22 (76)	1.93 ± .11	2.05 ± .11	.86 ± .05	—	—	5.77 ± .34 (65)	2.82 ± .17
	XVI	Australian: "Other" regions	"	3.95 ± .17 (121)	2.45 ± .11	1.78 ± .08 (114)	.95 ± .04	4.31 ± .41 (50)	2.51 ± .17	5.08 ± .24 (103)	2.98 ± .14
	VI	Kanaka	♀	4.04 ± .25 (60)	2.02 ± .12	1.61 ± .10	.90 ± .06	—	—	5.14 ± .32 (57)	3.04 ± .19
	XVI	Australian: "Other" regions	"	3.47 ± .19 (80)	1.96 ± .10	1.93 ± .11 (74)	.91 ± .05	—	—	5.58 ± .32 (71)	3.22 ± .18
				Coefficients of Variation							
D. Oceania	I	Maori: New Zealand	♂	4.70 ± .24	12.28 ± .63	20.93 ± 1.12	33.46 ± 1.89	—	—	5.52 ± .30	9.68 ± .53
	VI	Kanaka	"	3.74 ± .22	14.54 ± .87	20.96 ± 1.31	33.57 ± 2.23	—	—	5.13 ± .33	8.94 ± .58
	XI	Papuan	"	4.13 ± .23	11.54 ± .64	22.30 ± 1.45	26.80 ± 1.55	—	—	5.96 ± .35	8.16 ± .49
	XVI	Australian: "Other" regions	"	3.39 ± .15	13.08 ± .58	19.16 ± .89	23.06 ± 1.08	7.48 ± .51	13.15 ± .90	5.47 ± .26	8.06 ± .38
	VI	Kanaka	♀	4.36 ± .27	13.46 ± .84	22.39 ± 1.45	42.81 ± 3.08	—	—	5.59 ± .35	9.31 ± .59
	XVI	Australian: "Other" regions	"	3.62 ± .19	11.33 ± .61	21.89 ± 1.27	30.23 ± 1.83	—	—	6.29 ± .36	9.08 ± .52

				Standard Deviations							Coefficients of Variation						
E. Asia	I	Chinese ...	♂	4.07 ± .17 (136)	2.33 ± .10	1.94 ± .08 (137)	.94 ± .04	4.99 ± .23 (109)	2.71 ± .12	4.51 ± .20 (121)	3.27 ± .14						
	II	Burmese ...	"	4.79 ± .25 (84)	2.51 ± .13	2.32 ± .12 (82)	1.01 ± .05	5.06 ± .32 (57)	2.57 ± .16	4.97 ± .28 (70)	2.98 ± .17						
	XII	Javanese ...	"	4.52 ± .16 (176)	2.20 ± .08	1.98 ± .07 (172)	1.00 ± .04	4.60 ± .18 (148)	2.57 ± .10	4.70 ± .17 (166)	3.53 ± .13						
	XIII	Sumatran ...	"	3.43 ± .22 (56)	1.56 ± .10	2.27 ± .14 (56)	1.01 ± .06	5.30 ± .35 (52)	2.87 ± .19	4.71 ± .32 (50)	3.45 ± .23						
	XIV	Celebes (Natives) ...	"	4.11 ± .27 (52)	2.36 ± .16	2.32 ± .15 (52)	.85 ± .06	—	—	—	—						
	XV	Sarawak (Natives) ...	"	4.01 ± .26 (55)	2.16 ± .14	1.86 ± .12 (55)	.87 ± .06	—	—	—	—						
	XVI	Dayak ...	"	4.21 ± .27 (55)	2.12 ± .14	2.19 ± .15 (50)	.88 ± .06	—	—	—	—						
	XXI	Punjabi ...	"	3.70 ± .18 (93)	2.65 ± .13	1.85 ± .09 (92)	1.27 ± .06	—	—	4.73 ± .25 (79)	3.27 ± .18						
	XXIII	Hindu: N.E. India ...	"	3.92 ± .14 (179)	2.38 ± .08	1.68 ± .06 (173)	1.10 ± .04	4.26 ± .23 (78)	2.42 ± .13	5.01 ± .19 (163)	3.52 ± .13						
	XXIII	Hindu: N.E. India ...	♀	3.67 ± .25 (50)	2.28 ± .15	—	—	—	—	—	—						
				Coefficients of Variation													
E. Asia	I	Chinese ...	♂	4.23 ± .17	15.43 ± .65	25.57 ± 1.11	39.08 ± 1.82	9.15 ± .42	15.92 ± .75	4.54 ± .20	9.63 ± .42						
	II	Burmese ...	"	4.87 ± .25	15.96 ± .85	26.04 ± 1.46	33.73 ± 1.97	8.79 ± .56	15.22 ± .98	4.89 ± .28	8.85 ± .51						
	XII	Javanese ...	"	4.66 ± .17	14.31 ± .52	22.75 ± .87	35.65 ± 1.45	8.11 ± .32	14.83 ± .59	4.67 ± .17	10.21 ± .38						
	XIII	Sumatran ...	"	3.55 ± .23	9.82 ± .63	27.07 ± 1.85	37.25 ± 2.68	9.54 ± .64	15.97 ± 1.08	4.73 ± .32	9.71 ± .66						
	XIV	Celebes (Natives) ...	"	4.15 ± .27	14.46 ± .98	26.70 ± 1.89	31.46 ± 2.28	—	—	—	—						
	XV	Sarawak (Natives) ...	"	4.16 ± .27	13.32 ± .87	23.29 ± 1.58	33.29 ± 2.37	—	—	—	—						
	XVI	Dayak ...	"	4.36 ± .28	13.34 ± .87	25.79 ± 1.85	32.56 ± 2.42	—	—	—	—						
	XXI	Punjabi ...	"	3.86 ± .19	13.14 ± .66	20.36 ± 1.05	28.13 ± 1.51	—	—	5.04 ± .27	9.70 ± .53						
	XXIII	Hindu: N.E. India ...	"	4.11 ± .15	12.26 ± .44	19.32 ± .73	26.90 ± 1.04	7.47 ± .41	10.65 ± .58	5.30 ± .20	10.54 ± .40						
	XXIII	Hindu: N.E. India ...	♀	4.00 ± .27	12.61 ± .86	—	—	—	—	—	—						

TABLE V. *Standard Deviations of the Indices of Facial Flattening**.

Continental Area	Serial No.	Race	Sex	Frontal Index	Simotic Index	Rhinal Index	Premaxillary Index
A. Europe	VI	English: Spitalfields ...	♂	1.70 ± .05 (248)	11.79 ± .50 (126)	5.15 ± .35 (50)	3.08 ± .17 (73)
	IV	English: Mediaeval ...	"	1.87 ± .14 (43)	12.79 ± 1.02 (36)	—	3.67 ± .30 (36)
	XX	Russian ...	"	2.03 ± .16 (35)	11.72 ± .93 (36)	—	2.66 ± .22 (33)
	III	Anglo-Saxon ...	"	2.08 ± .10 (92)	9.29 ± .51 (77)	—	3.65 ± .20 (76)
	XI	French ...	"	2.11 ± .15 (46)	8.98 ± .65 (43)	—	4.11 ± .31 (39)
	II	Romano-British ...	"	2.18 ± .11 (86)	12.51 ± .66 (82)	5.56 ± .48 (30)	2.96 ± .18 (64)
	V	English: Farringdon St. ...	"	2.19 ± .12 (75)	12.75 ± .68 (81)	6.16 ± .48 (37)	3.67 ± .24 (53)
	XVIII	Modern Greek ...	"	2.26 ± .19 (33)	10.02 ± .85 (32)	—	—
	VIII	Modern Scottish ...	"	2.30 ± .19 (35)	9.40 ± .79 (32)	—	3.14 ± .27 (31)
	X	North German ...	"	2.31 ± .20 (32)	11.90 ± 1.04 (30)	—	3.65 ± .31 (31)
	XII	Italian ...	"	2.33 ± .11 (100)	11.40 ± .54 (100)	5.78 ± .38 (52)	3.44 ± .17 (94)
	XIV	Swedish ...	"	2.36 ± .21 (30)	—	—	—
	VII	Modern Irish ...	"	2.37 ± .14 (65)	11.70 ± .69 (65)	5.35 ± .39 (44)	2.65 ± .17 (57)
	IX	British (Soldiers) ...	"	2.45 ± .11 (113)	13.13 ± .59 (111)	6.05 ± .31 (84)	3.12 ± .15 (101)
	XVI	Dutch ...	"	2.49 ± .17 (51)	11.15 ± .74 (51)	—	3.32 ± .24 (45)
		Weighted Mean ($\bar{\sigma}$) ...	"	2.14 [15]	11.64 [14]	5.77 [6]	3.32 [13]
	III	Anglo-Saxon ...	♀	1.59 ± .11 (45)	9.32 ± .69 (41)	—	2.82 ± .21 (40)
	VI	English: Spitalfields ...	"	1.76 ± .09 (94)	11.63 ± .97 (31)	—	—
	II	Romano-British ...	"	2.14 ± .15 (45)	10.73 ± .79 (42)	—	3.56 ± .29 (39)
	V	English: Farringdon St. ...	"	2.15 ± .12 (75)	10.19 ± .56 (76)	5.27 ± .37 (47)	3.99 ± .27 (51)
	XII	Italian ...	"	2.64 ± .22 (32)	10.67 ± .91 (31)	—	—
		Weighted Mean ($\bar{\sigma}$) ...	"	2.00 [5]	10.42 [5]	5.27 [1]	3.53 [3]
B. Africa	IX	Ashanti ...	♂	1.79 ± .15 (33)	10.66 ± .87 (34)	—	3.15 ± .27 (30)
	VI	Negro: Nigeria ...	"	1.82 ± .12 (52)	9.66 ± .66 (49)	—	3.60 ± .25 (46)
	III	Egyptian: Badari ...	"	2.09 ± .18 (32)	7.07 ± .58 (34)	—	3.37 ± .28 (34)
	VII	Gaboon Negro ...	"	2.18 ± .12 (76)	8.70 ± .49 (71)	4.18 ± .28 (49)	3.27 ± .20 (63)
	XII	Congo Negro ...	"	2.21 ± .14 (60)	8.23 ± .51 (60)	—	4.26 ± .31 (42)
	X	Teita Negro ...	"	2.24 ± .14 (55)	7.69 ± .59 (39)	—	2.71 ± .23 (31)
	IV	Egyptian: Sedment ...	"	2.33 ± .18 (40)	10.01 ± .79 (37)	—	3.38 ± .27 (37)
	V	Egyptian: Kerma ...	"	2.41 ± .11 (117)	9.19 ± .44 (99)	—	3.62 ± .17 (103)
	XV	Kaffir ...	"	2.43 ± .16 (54)	11.94 ± .78 (54)	—	3.63 ± .24 (51)
	II	Nubian ...	"	2.49 ± .14 (69)	8.09 ± .52 (55)	—	3.62 ± .27 (40)
		Weighted Mean ($\bar{\sigma}$) ...	"	2.25 [10]	9.31 [10]	4.18 [1]	3.52 [10]
	XII	Congo Negro ...	♀	1.87 ± .16 (30)	—	—	—
	II	Nubian ...	"	1.90 ± .12 (61)	6.15 ± .48 (38)	—	2.56 ± .21 (35)
	IV	Egyptian: Sedment ...	"	2.00 ± .17 (30)	—	—	—
	VII	Gaboon Negro ...	"	2.01 ± .12 (60)	8.42 ± .54 (55)	3.96 ± .29 (44)	3.24 ± .22 (51)
	IX	Ashanti ...	"	2.08 ± .18 (31)	10.96 ± .95 (30)	—	—
	X	Teita Negro ...	"	2.09 ± .13 (62)	7.16 ± .49 (49)	4.26 ± .37 (31)	3.69 ± .29 (37)
	V	Egyptian: Kerma ...	"	2.19 ± .11 (94)	9.49 ± .53 (74)	—	3.00 ± .16 (83)
		Weighted Mean ($\bar{\sigma}$) ...	"	2.05 [7]	8.58 [5]	4.09 [2]	3.12 [4]

* The numbers in round brackets denote the numbers of skulls on which the standard deviations are based and none was calculated for series made up by fewer than 30 specimens. The weighted mean $\bar{\sigma}$'s for the different continental areas were found from the formula $\bar{\sigma} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + \dots}{n_1 + n_2 + \dots}}$, and the numbers in square brackets denote the number of series on which they are based. The series for a particular continental area are arranged in order of the standard deviations of the frontal indices.

TABLE V—(continued).

Continental Area	Serial No.	Race	Sex	Frontal Index	Simotic Index	Rhinal Index	Premaxillary Index
C. America	III	North American Indian ...	♂	1.74 ± .12 (45)	11.31 ± .80 (46)	—	2.93 ± .22 (41)
	II	Peruvian	"	1.81 ± .07 (171)	9.51 ± .35 (166)	4.84 ± .35 (43)	3.00 ± .12 (143)
	I	Eskimo	"	1.96 ± .13 (51)	8.86 ± .60 (50)	—	3.23 ± .23 (45)
		Weighted Mean ($\bar{\sigma}$) ...	"	1.83 [3]	9.73 [3]	4.84 [1]	3.03 [3]
	II	Peruvian	♀	2.09 ± .09 (114)	7.36 ± .34 (107)	5.25 ± .56 (20)	2.96 ± .14 (97)
	I	Eskimo	"	1.72 ± .15 (30)	—	—	—
		Weighted Mean ($\bar{\sigma}$) ...	"	2.02 [2]	7.36 [1]	5.25 [1]	2.96 [1]
D. Oceania	II	Mori	♂	1.84 ± .13 (43)	11.89 ± .88 (42)	—	2.88 ± .22 (39)
	XI	Papuan	"	1.84 ± .10 (76)	10.04 ± .54 (78)	5.06 ± .35 (47)	3.57 ± .21 (65)
	III	New British	"	1.85 ± .14 (39)	10.54 ± .81 (39)	3.46 ± .38 (33)	2.40 ± .19 (35)
	V	Easter Island	"	1.93 ± .10 (41)	12.24 ± .88 (44)	—	3.37 ± .27 (36)
	X	Fijian	"	1.98 ± .17 (30)	10.85 ± .95 (30)	—	—
	I	Maori	"	2.01 ± .12 (89)	12.20 ± .62 (87)	5.11 ± .40 (38)	3.46 ± .19 (76)
	VI	Kanaka: Sandwich Islands	"	2.09 ± .12 (67)	11.81 ± .71 (63)	5.39 ± .38 (47)	2.80 ± .18 (54)
	IX	New Hebridean	"	2.23 ± .18 (36)	10.13 ± .83 (34)	—	3.74 ± .31 (33)
	XV	Australian: Queensland ...	"	2.53 ± .20 (36)	8.98 ± .71 (36)	—	3.41 ± .28 (35)
	XVI	Australian: "Other" regions	"	2.09 ± .13 (121)	10.04 ± .45 (114)	4.62 ± .31 (50)	3.60 ± .17 (103)
		Weighted Mean ($\bar{\sigma}$) ...	"	2.26 [10]	11.22 [10]	4.83 [5]	3.34 [9]
	XI	Papuan	♀	1.86 ± .13 (49)	8.45 ± .58 (49)	3.95 ± .32 (34)	4.03 ± .30 (40)
	VI	Kanaka: Sandwich Islands	"	2.05 ± .13 (60)	11.05 ± .68 (60)	3.81 ± .30 (37)	2.98 ± .19 (57)
	XVI	Australian: "Other" regions	"	1.94 ± .10 (80)	8.26 ± .46 (74)	4.22 ± .39 (27)	3.41 ± .19 (71)
		Weighted Mean ($\bar{\sigma}$) ...	"	1.96 [3]	9.31 [3]	3.97 [3]	3.43 [3]
E. Asia	XIII	Sumatran	♂	1.58 ± .10 (56)	10.14 ± .65 (56)	5.87 ± .39 (52)	3.61 ± .24 (50)
	XVIII	Aëta	"	1.64 ± .14 (33)	—	—	—
	XVII	Tagal	"	1.65 ± .14 (31)	—	—	—
	IX	Andamanese	"	1.83 ± .16 (31)	—	—	—
	XVI	Dayak	"	2.00 ± .13 (55)	9.85 ± .67 (50)	4.46 ± .32 (44)	3.32 ± .23 (46)
	XV	Sarawak (Natives)	"	2.04 ± .13 (55)	10.05 ± .65 (55)	—	4.56 ± .33 (43)
	V	Tibetan A	"	2.06 ± .16 (37)	8.33 ± .67 (35)	5.71 ± .47 (34)	3.21 ± .26 (36)
	XXII	Hindu: Madras	"	2.06 ± .14 (49)	10.44 ± .73 (46)	—	3.44 ± .24 (48)
	XII	Javanese	"	2.13 ± .08 (176)	9.75 ± .36 (172)	4.82 ± .19 (148)	3.83 ± .14 (166)
	XXIV	Dravidian	"	2.15 ± .17 (36)	10.46 ± .72 (35)	—	3.33 ± .29 (31)
	I	Chinese	"	2.19 ± .09 (136)	10.79 ± .44 (137)	5.18 ± .24 (109)	3.51 ± .15 (121)
	XIV	Celebes (Natives)	"	2.19 ± .15 (52)	7.44 ± .49 (52)	4.29 ± .31 (45)	3.42 ± .23 (49)
	II	Burmese	"	2.24 ± .12 (84)	8.95 ± .47 (82)	3.81 ± .24 (57)	2.99 ± .17 (70)
	XIX	Singalese	"	2.24 ± .18 (34)	10.37 ± .85 (34)	—	—
	XXIII	Hindu: N.E. India	"	2.35 ± .08 (179)	11.17 ± .41 (173)	5.02 ± .27 (78)	3.44 ± .13 (163)
	VII	Nepalese	"	2.56 ± .18 (46)	14.48 ± 1.01 (47)	5.79 ± .44 (39)	2.97 ± .21 (44)
	XXI	Punjabi	"	2.87 ± .14 (93)	12.94 ± .64 (92)	7.64 ± .56 (43)	3.67 ± .20 (78)
		Weighted Mean ($\bar{\sigma}$) ...	"	2.20 [17]	10.60 [14]	5.20 [10]	3.54 [13]
	XII	Javanese	♀	1.75 ± .14 (34)	7.68 ± .64 (34)	4.72 ± .41 (30)	3.68 ± .32 (30)
	II	Burmese	"	1.99 ± .15 (38)	8.49 ± .67 (37)	—	2.75 ± .24 (30)
	XXIII	Hindu: N.E. India	"	2.07 ± .14 (50)	10.46 ± .72 (48)	—	2.96 ± .21 (46)
		Weighted Mean ($\bar{\sigma}$) ...	"	1.96 [3]	9.13 [3]	4.72 [1]	3.13 [3]

no conclusion can be drawn. The position is almost the same for the simotic index except that the group mean for the African races is appreciably lower than the others. It may be seen from Fig. 1 that the lowest simotic indices found are for African races. We can only conclude that the variabilities of the indices of facial flattening for different races are not peculiar to continental areas and that there is no evidence that the variations of advanced and primitive peoples are differentiated in this respect.

(6) *Intra- and Interracial Correlations of the Measurements of Facial Flattening.* All four of the indices considered in the present paper are of the same type, as each is formed by expressing a "median" subtense as a percentage of a corresponding transverse breadth. It might be anticipated that for both intra- and interracial samples they would be quite highly correlated with one another, and this could be supposed an expression of the fact that they might all be presumed to measure the same feature of the cranium (viz. the transverse flattening of the facial skeleton) in rather different ways. For an intraracial sample positive correlations are to be expected between all pairs of the absolute measurements, owing to the effect of a common size factor*, and it would not be surprising to find that some pairs of the breadths and some pairs of the subtenses show high positive correlations. The correlations actually found do not coincide with all our anticipations in the present case. Table VI gives 72 male and female intraracial correlation coefficients for the longest series available. They range from -0.176 to $+0.434$ and there are only 29 values (all positive) which can be considered to differ significantly from zero. The frontal index is seen to be quite uncorrelated with the simotic, no significant correlation being found for any series, while the closest association is that between the simotic and rhinal indices which both measure the prominence of the nasal bones, all the values being significant in this case. Both frontal and premaxillary indices show a more marked, but still low, degree of association with the rhinal than with the simotic index. We must conclude that the last, which gives a measure of the prominence of the root of the nasal bones, is practically uncorrelated in the individual with other measurements of facial flattening which do not involve the nasal bones. The same might be true for an index confined to the lower extremities of these bones, the rhinal not being of that type since the breadth from which it is calculated is between points on the malar and maxillary bones. But the correlations between the frontal and premaxillary indices are also of a low order and the supposition that all our indices were measuring the same feature of the facial skeleton in rather different ways is seen to be entirely belied by the facts. The intraracial correlations of the absolute measurements will clearly be of interest and these are given in Table VII for the male Javanese series—the longest available—for which the indicial correlations in Table VI are quite typical. As might have been anticipated, all the correlations

* See the standard paper on cranial correlations by Professor Karl Pearson and Miss Adelaide G. Davin: "On the Biometric Constants of the Human Skull," *Biometrika*, Vol. xvi. (1924), pp. 328—363.

TABLE VI.

Intraracial Correlations of the Indices of Facial Flattening.*

Race	Sex	$r_{F,S}$ (Frontal and Simotic)	$r_{F,R}$ (Frontal and Rhinal)	$r_{F,P}$ (Frontal and Premaxillary)	$r_{S,R}$ (Simotic and Rhinal)	$r_{S,P}$ (Simotic and Premaxillary)	$r_{R,P}$ (Rhinal and Premaxillary)
British (Soldiers) ...	♂	$+0.83 \pm 0.064$ (111)	$+0.299 \pm 0.067$ (84)	$+0.242 \pm 0.063$ (101)	$+0.248 \pm 0.069$ (84)	$+0.199 \pm 0.065$ (98)	$+0.243 \pm 0.073$ (76)
Italian ...	"	-0.089 ± 0.067 (101)	$+0.279 \pm 0.086$ (52)	$+0.203 \pm 0.066$ (96)	$+0.303 \pm 0.080$ (51)	$+0.237 \pm 0.066$ (95)	$+0.288 \pm 0.087$ (50)
Chinese ...	"	$+0.083 \pm 0.058$ (134)	$+0.301 \pm 0.059$ (108)	$+0.178 \pm 0.059$ (121)	$+0.381 \pm 0.055$ (109)	$+0.242 \pm 0.058$ (121)	$+0.331 \pm 0.060$ (100)
Hindu: N.E. India ...	"	$+0.076 \pm 0.051$ (173)	$+0.434 \pm 0.062$ (78)	$+0.243 \pm 0.050$ (163)	$+0.373 \pm 0.065$ (78)	$+0.169 \pm 0.052$ (158)	$+0.371 \pm 0.068$ (73)
Javanese ...	"	$+0.064 \pm 0.052$ (165)	$+0.153 \pm 0.054$ (149)	$+0.169 \pm 0.051$ (166)	$+0.399 \pm 0.047$ (143)	$+0.119 \pm 0.052$ (163)	$+0.127 \pm 0.056$ (142)
Egyptian: Kerma ...	"	$+0.002 \pm 0.069$ (96)	—	$+0.074 \pm 0.067$ (99)	—	$+0.086 \pm 0.071$ (88)	—
Gaboon Negro ...	"	$+0.115 \pm 0.078$ (72)	-0.069 ± 0.096 (49)	$+0.113 \pm 0.083$ (64)	$+0.272 \pm 0.089$ (49)	$+0.093 \pm 0.086$ (60)	$+0.147 \pm 0.102$ (42)
Peruvian ...	"	$+0.137 \pm 0.051$ (166)	$+0.141 \pm 0.101$ (43)	$+0.253 \pm 0.053$ (143)	$+0.413 \pm 0.085$ (43)	$+0.020 \pm 0.056$ (143)	$+0.189 \pm 0.106$ (38)
Maori ...	"	$+0.173 \pm 0.070$ (88)	$+0.035 \pm 0.111$ (37)	-0.052 ± 0.077 (76)	$+0.351 \pm 0.096$ (38)	$+0.182 \pm 0.075$ (75)	-0.038 ± 0.114 (35)
Australian: "Other" regions	"	-0.038 ± 0.064 (111)	$+0.047 \pm 0.096$ (49)	$+0.082 \pm 0.066$ (102)	$+0.282 \pm 0.088$ (50)	-0.024 ± 0.067 (101)	$+0.400 \pm 0.086$ (43)
Egyptian: Kerma ...	♀	-0.176 ± 0.077 (72)	—	$+0.141 \pm 0.074$ (80)	—	-0.061 ± 0.081 (69)	—
Gaboon Negro ...	"	-0.144 ± 0.091 (53)	$+0.089 \pm 0.101$ (44)	$+0.007 \pm 0.095$ (50)	$+0.426 \pm 0.082$ (44)	$+0.082 \pm 0.099$ (46)	$+0.101 \pm 0.107$ (39)
Peruvian ...	"	$+0.096 \pm 0.065$ (107)	—	$+0.156 \pm 0.063$ (96)	—	-0.109 ± 0.070 (92)	—
Australian: "Other" regions	"	-0.035 ± 0.079 (73)	—	$+0.093 \pm 0.080$ (70)	—	$+0.142 \pm 0.080$ (69)	—

The coefficients in italics are the only ones which differ from zero by more than three times their probable errors.

in Table VII are positive. It has been shown by Pearson and Davin that nearly all the intraracial values found between pairs of absolute measurements are positive owing to the influence of a common size factor. This could be supposed to account for a number of correlations found for the Egyptian series with which they were dealing of the order $+0.1$ to $+0.4$, but when the measurements compared also partly "covered" one another higher values were generally found[†]. Among the 28 coefficients in Table VII there are 10 less than $+0.2$, and most of these do not

[†] E.g. the coefficients found for the male Egyptian *E* series are $+0.3971 \pm 0.0191$ for *L* and *B*, $+0.3844 \pm 0.0208$ for *B* and *H*, $+0.2758 \pm 0.0229$ for *G'H* and *J*, and $+0.2358 \pm 0.0218$ for *NB* and *L*.

[†] E.g. the coefficients found for the male Egyptian *E* series are $+0.341 \pm 0.0187$ for *B* and *U*, $+0.7059 \pm 0.0117$ for *NH* and *G'H*, and $+0.4170 \pm 0.0199$ for *B'* and *J*.

TABLE VII.

Intracracial Correlations of Absolute Measurements for the Male Japanese Series, all values based on the same 137 Skulls.

Characters	Sub. IOW	SS	Sub. MOW	Sub. GB	IOW	SC	MOW	GB
Sub. IOW	—	+ .303 ± .052	+ .396 ± .049	+ .225 ± .055	+ .445 ± .046	+ .437 ± .047	+ .366 ± .050	+ .141 ± .056
SS	+ .303 ± .052	—	+ .367 ± .050	+ .087 ± .057	+ .192 ± .056	+ .539 ± .042	+ .071 ± .057	+ .044 ± .058
Sub. MOW	+ .396 ± .049	+ .367 ± .050	—	+ .259 ± .054	+ .201 ± .055	+ .091 ± .057	+ .237 ± .054	+ .151 ± .056
Sub. GB	+ .225 ± .055	+ .087 ± .057	+ .259 ± .054	—	+ .267 ± .054	+ .047 ± .058	+ .213 ± .055	+ .143 ± .056
IOW	+ .445 ± .046	+ .192 ± .056	+ .201 ± .055	+ .267 ± .054	—	+ .305 ± .052	+ .450 ± .046	+ .611 ± .036
SC	+ .437 ± .047	+ .539 ± .042	+ .091 ± .057	+ .047 ± .058	+ .305 ± .052	—	+ .222 ± .055	+ .025 ± .058
MOW	+ .366 ± .050	+ .071 ± .057	+ .237 ± .054	+ .213 ± .055	+ .450 ± .046	+ .222 ± .055	—	+ .418 ± .048
GB	+ .141 ± .056	+ .044 ± .058	+ .151 ± .056	+ .143 ± .057	+ .611 ± .036	+ .022 ± .058	+ .418 ± .048	—

TABLE VIII.

Interracial Indicical Constants for Series made up by Fifteen or more Crania.*

Indices	Sex	Means	Standard Deviations	Correlations of Indices			
				Frontal	Simotic	Rhinal	Premaxillary
Frontal	♂♂♂	17.95 ± .109 (76)	1.412 ± .077 (76)	—	+ .699 ± .040 (75)	+ .679 ± .054 (45)	+ .320 ± .072 (70)
Simotic	♂♂♂	17.35 ± .132 (37)	1.189 ± .093 (37)	—	+ .574 ± .075 (36)	—	+ .244 ± .110 (33)
	♂♂♂	40.40 ± .649 (75)	8.338 ± .459 (75)	+ .699 ± .040 (75)	—	+ .868 ± .025 (45)	+ .136 ± .079 (70)
	♂♂♂	32.58 ± .832 (36)	7.399 ± .588 (36)	+ .574 ± .075 (36)	—	—	—
Rhinal	♂♂♂	35.97 ± .414 (45)	4.113 ± .292 (45)	+ .679 ± .054 (45)	+ .868 ± .025 (45)	—	—
	♂♂♂	32.64 ± 1.024 (14)	5.679 ± .724 (14)	—	—	—	—
	♂♂♂	35.79 ± .161 (70)	1.994 ± .114 (70)	+ .320 ± .072 (70)	+ .136 ± .079 (70)	+ .070 ± .100 (45)	—
Premaxillary	♂♂♂	35.76 ± .225 (33)	1.917 ± .159 (33)	+ .244 ± .110 (33)	—	—	—

* In calculating the interracial constants in Tables VIII—X no account was taken of the fact that the means are of unequal value since the numbers of individuals on which they are based differ widely. Since no system of weighting the means was employed, the usual formulae were applied to give the probable errors, and these would not be applicable, of course, if weighted means were used to determine the interracial constants. The procedure adopted is probably accurate enough for the purpose in view.

differ significantly from zero, while there are only two greater than .5*. The different subtenses show no high correlations with one another and the simotic chord is lowly correlated with the other facial breadths. We are dealing, in fact, with characters which are but little associated intraracially, and the conception of there being a single factor of facial features which can be measured in the individual in slightly different ways by the different subtenses or indices considered is seen to be of no practical utility.

Since the indices we are considering were devised for the purpose of discriminating racial types, the interracial correlations are of more importance than the intraracial values dealt with above. Unfortunately the material collected is quite insufficient to give reliable estimates of the former. Four of the scatter diagrams given by the mean values are in Figs. 1—4, and the constants are in Tables VIII—X. In calculating the last, all means available based on 15 or more skulls were used in order to obtain a sufficient number of series, and it must be admitted that the sampling errors of the less reliable means are probably large enough to distort the resulting interracial constants. Owing to this fact, the standard deviations in Table VIII are probably larger than they should be. Nevertheless, those for the frontal and premaxillary indices are less than any intraracial standard deviations found (cf. Table V), and for the simotic and rhinal indices the interracial values are less than nearly all the intraracial. For all these indices interracial variability tends to be decidedly less than intraracial variability, and the same is true for the absolute measurements (cf. Tables IV and IX) and for nearly all of the more usual cranial measurements. The interracial correlations in Table VIII are probably somewhat too low, owing to the fact that several unreliable means were included in computing them. They are surprisingly different from the intraracial values in Table VI. All of the former class are decidedly higher than the corresponding coefficients of the latter class, except in the case of the premaxillary with the simotic and rhinal indices respectively for which the interracial and intraracial coefficients are not clearly differentiated. This demonstrates again the impossibility of deriving any knowledge of interracial correlation from a knowledge of intraracial constants only. The coefficients for absolute measurements in Tables VII and X can be used to demonstrate the same point. While all the values in Table VII are positive, there are several quite high negative values in Table X, and it is of interest to note that these are nearly all with the maxillary facial breadth (*GB*). There is a clear indication that *in the type* the frontal and nasal subtenses and the simotic breadth decrease with an increase in *GB*.

* The relative values of the correlation coefficients will depend partly on the racial homogeneity of the series. The Egyptian *E* series is known to be more homogeneous than most available series. Comparisons with the Javanese can only be made in the case of three male coefficients of variation, the values being:

	<i>GB</i>	<i>SS</i>	<i>SC</i>
Egyptian <i>E</i>	$4.90 \pm .08$ (877)	23.10 ± 1.64 (50)	16.84 ± 1.17 (50)
Javanese	$4.87 \pm .17$	35.65 ± 1.45	$22.75 \pm .87$

GB for Egyptian *E* is given by Pearson and Davin in *Biometrika*, Vol. xvi. 1924, pp. 828—863, and *SS* and *SC* by Ryley, Bell and Pearson in the same *Journal* for 1918, Vol. ix. pp. 391—445.

TABLE IX. *Interracial Means and Variabilities of Absolute Measurements for Series made up by Fifteen or more Crania.*

Sex	Male				Female		
	Mean	Standard Deviation	Coefficient of Variation		Mean	Standard Deviation	Coefficient of Variation
Frontal ...	17.58 ± 1.04 (76)	1.338 ± 0.73 (76)	7.61 ± .419 (76)		16.19 ± 1.34 (37)	1.213 ± .095 (37)	7.49 ± .591 (37)
Simotic ...	97.90 ± 1.61 (76)	2.084 ± 1.14 (76)	2.13 ± .117 (76)		93.13 ± 1.86 (37)	1.680 ± .132 (37)	1.80 ± 1.412 (37)
Chord ...	3.55 ± 0.70 (75)	.895 ± 0.49 (75)	25.21 ± 1.474 (75)		2.85 ± .085 (36)	.847 ± .087 (36)	29.72 ± 2.563 (36)
Subtense ...	8.81 ± 0.74 (75)	.949 ± 0.52 (75)	10.77 ± 1.474 (75)		8.73 ± 1.29 (36)	1.147 ± .081 (36)	7.61 ± .608 (36)
Rhinal ...	20.13 ± 2.71 (45)	2.687 ± 1.92 (45)	13.40 ± .970 (45)		17.61 ± 4.38 (14)	2.432 ± .310 (14)	13.81 ± 1.794 (14)
Chord ...	56.57 ± 2.71 (45)	2.691 ± 1.91 (45)	4.76 ± .280 (45)		54.63 ± 5.16 (14)	2.862 ± .365 (14)	5.24 ± .670 (14)
Subtense ...	34.32 ± 1.49 (70)	1.843 ± 1.05 (70)	5.37 ± .307 (70)		32.68 ± 1.80 (33)	1.529 ± 1.27 (33)	4.68 ± .389 (33)
Premaxillary ...	96.90 ± 2.46 (70)	3.048 ± 1.74 (70)	3.17 ± .181 (70)		91.57 ± 3.33 (33)	2.833 ± 2.35 (33)	3.09 ± .257 (33)

TABLE X. *Interracial Correlations of Absolute Measurements for Series made up by Fifteen or more Crania.*

Characters	Sex	Sub. IOW	SS	Sub. MOW	Sub. GB	IOW	SC	MOW	GB
Sub. IOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	+ .744 ± .035 (75)	+ .680 ± .053 (45)	+ .156 ± .079 (70)	+ .124 ± .076 (76)	+ .462 ± .061 (75)	—	—
SS	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .744 ± .035 (75)	—	—	—	+ .313 ± .100 (37)	+ .579 ± .075 (36)	—	—
Sub. MOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .550 ± .078 (36)	+ .862 ± .026 (45)	—	—	—	+ .578 ± .052 (75)	—	—
Sub. GB	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .680 ± .053 (45)	+ .862 ± .026 (45)	—	—	—	+ .602 ± .072 (36)	—	—
IOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .156 ± .079 (70)	—	—	—	+ .124 ± .076 (76)	+ .462 ± .061 (75)	—	—
SC	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .124 ± .076 (76)	—	—	—	+ .313 ± .100 (37)	+ .579 ± .075 (36)	—	—
MOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	—	—	—	—	—	—	—
GB	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	—	—	—	—	—	—	—
Sub. IOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	+ .744 ± .035 (75)	+ .680 ± .053 (45)	+ .156 ± .079 (70)	+ .124 ± .076 (76)	+ .462 ± .061 (75)	—	—
SS	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .744 ± .035 (75)	—	—	—	+ .313 ± .100 (37)	+ .579 ± .075 (36)	—	—
Sub. MOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .550 ± .078 (36)	+ .862 ± .026 (45)	—	—	—	+ .578 ± .052 (75)	—	—
Sub. GB	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .680 ± .053 (45)	+ .862 ± .026 (45)	—	—	—	+ .602 ± .072 (36)	—	—
IOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .156 ± .079 (70)	—	—	—	+ .124 ± .076 (76)	+ .462 ± .061 (75)	—	—
SC	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	+ .124 ± .076 (76)	—	—	—	+ .313 ± .100 (37)	+ .579 ± .075 (36)	—	—
MOW	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	—	—	—	—	—	—	—
GB	♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂ ♂	—	—	—	—	—	—	—	—

(7) *Racial Comparisons of the Measurements of Facial Flattening.* A description of the material measured is given in Section (3) above, and the mean measurements from which racial comparisons can be made are in Table I. Enough series are available, representing all the principal groups of modern races, to give a definite estimate of the value of the new criteria for anthropological purposes. Figs. 1—4 give scatter distributions of the mean indices taken in pairs, and no detailed descriptions of, or comments on, them need be made. But it is necessary to supplement the diagrams by discussing the order of differences which may be considered to indicate statistical differentiation in the case of each index.

Fig. 1 shows the interracial correlation of the rhinal and simotic indices. All series for which both means are based on 15 or more skulls are shown, but the majority of them are made up by considerably more than 15 individuals, the actual numbers being given in Table I. In order to obtain an estimate of differences which are statistically significant, it will be sufficient to consider the 11 male European series which are all in the top right-hand corner of the diagram. Taking the rhinal index first, it is found that out of the 55 possible differences between the means, 31 are less than 2·5 times their probable errors, 9 show values of the ratio of the difference to its probable error between 2·5 and 4·0, and for the remaining 15 the ratios are greater than 4·0. If we suppose that a difference is significant when it exceeds 2·5 times its probable error, then by the rhinal index the Dutch series is differentiated from all the others except the Russian; the Spitalfields and Mediaeval English from all except one another and the Farringdon Street English; while the Italian and Farringdon Street English series are also differentiated. Six of the series—viz. the Russian, Anglo-Saxon, Scottish, Irish, Romano-British and that of British Soldiers—cannot be distinguished from one another by their rhinal indices, and neither the Italian series, on the one hand, nor the Farringdon Street English, on the other, can be distinguished from the same group. Turning to the simotic index, it is found that 36 of the 55 differences are less than 2·5 times their probable errors; 9 show ratios between 2·5 and 4·0, and the remaining 10 show ratios greater than 4·0. Taking the limit again at 2·5, the Italian series is differentiated from all others except the Russian, the Spitalfields and the Mediaeval English; the Spitalfields is differentiated from all others except the Italian, Russian and Mediaeval English; and the Mediaeval English is differentiated from all others except the Italian, Russian, Spitalfields and Farringdon Street English and the Romano-British series. Eight of the 11 series—viz. the Dutch, Russian, Anglo-Saxon, Scottish, Irish, Romano-British, Farringdon Street English and that of the British Soldiers—thus show no significant differences between their mean simotic indices. If larger numbers of individuals were available, it is quite probable, of course, that some of the mean simotic indices for these racial types would be differentiated. For the material available, however, it is quite evident that statistically significant differences are found between points which are separated, in the case of either variate, by distances which are a small fraction of the total range shown by the same variate. It will be quite safe to assume, without further examination, that all the European series differ most

significantly from all the Oriental and African negro series in the case of both rhinal and simotic indices.

The frontal and premaxillary indices, used in Fig. 2, may be treated in the same way, and it will be sufficient to restrict attention to the group of five male Indian series at the top of the diagram. For these, only three of the possible ten comparisons of mean frontal indices show differences in excess of 2.5 times their probable errors, and the greatest ratio found is 3.0. The Singalese series shows a difference from that of Hindus from Madras which may be considered just significant, and the same is true for the Punjabi series from those of Hindus from North-East India and Madras respectively. More significant differences are found between the mean pre-maxillary indices, only three of the ratios being less than 2.5, five being between 2.5 and 4.0, and two being greater than 4.0. Both the means for Singalese and Hindus from North-East India are differentiated from all for the other three series. All the mean frontal indices of the Indian series will clearly be differentiated most markedly from all the means for the Oriental series, and quite markedly from nearly all the European means. The premaxillary index is less able to differentiate these three groups of races, but significant differences for it will be shown between means which are separated by quite a small fraction—less than one-tenth, say—of the range shown by all races in the world.

The fact that the last statement is true for all four indices indicates—since many of the series with which we are dealing are very small—that the indices are well-defined racial characters and their uses for purposes of classification may now be considered. The characters of this kind with which anthropologists are familiar may be divided somewhat arbitrarily into the following classes:

(a) Characters which appear to be of little use for any purposes of racial classification, although they may differ quite markedly from race to race.

(b) Characters which serve to differentiate and provide suggestive arrangements of racial types belonging to any single family of races, but which are practically valueless for the purpose of classifying the different families of races. Such are, notably, the cephalic index and stature, the ranges for these two being almost as great for European races, for example, as for all races in the world.

(c) Characters which are practically constant for races belonging to any single family of races, but which provide suggestive orders when the different families are compared. Such are, notably, skin colour, the nasal index and the nasal angle or other measurements of prognathism. Characters of this kind alone are capable of approximately grading all races in the world in order of their "primitiveness*."

Fig. 1 suggests forcibly that both the rhinal and simotic indices belong to the last of these classes. The interracial correlation between them for male means was

* See comments on this point by G. M. Morant in *Annals of Eugenics*, Vol. II. (1927), pp. 334—336, where it is concluded that the nasal index and nasal angle are the best fitted of the usual cranial characters to serve the purpose mentioned.

found to be +.868, and it is clear that the orders in which the two considered singly will arrange the races are very similar. African negro types have the most depressed nasal bones, then the Oriental races, which are remarkably similar to one another in this respect, follow and lead on to the American, Oceanic and Egyptian groups which are not distinguished from one another; the three Indian series all have more prominent nasal bones, and finally come the European types with the most prominent to be found in the world as far as our evidence can tell. This character of nasal prominence evidently tends to be fairly constant for races belonging to the same family—such as the European, Oriental or African negro groups—while it makes some very clear inter-group distinctions. Anyone unaccustomed to dealing with quantitative anthropological characters in this way may fail to appreciate the significance of the diagram we are considering. The vast majority of the measurements of size or shape usually provided fail entirely to supply any suggestive arrangements, when considered singly or in pairs, in the case of a sample drawn from all parts of the world. The cephalic index or stature, for example, will give an apparently meaningless jumble for such a sample of racial types. But the rhinal and simotic indices appear to be as capable of making inter-group distinctions as any other craniological characters known. The only other ones which can compare with them in this respect are the nasal index and the nasal angle or other measurements of prognathism. These last two are known to be highly correlated interracially with one another*, and it is probable that both are highly correlated with the rhinal and simotic indices. But the measurements of nasal prominence seem to give rather more suggestive orders than either the nasal index or angle†. The rhinal subtense was found to be rather an unsatisfactory measurement, since the tips of the nasal bones are so often broken, and it is sometimes difficult to decide whether they are defective or not. Since the resulting index has now also been shown to be highly correlated with the simotic index we do not recommend its further use.

The order in which the mean frontal indices arrange the racial types can be appreciated from Fig. 2. This is still a suggestive one, though it has evidently not the significance of those given by the rhinal and simotic indices. The Indian, European and Oriental groups are kept quite distinct and the African negroes occupy a restricted range between the last two. It is interesting to note that the Indian series all fall at one extreme of the distribution, while all belonging to the second Asiatic group—the Oriental—are at the other extreme. The primitive Oceanic and African negro series occupy intermediate positions and several of them are not differentiated by this character from European types. It is evident that the frontal index can give no indication whatever of "primitiveness." The position with regard to the premaxillary index is similar to that found in the case of most anthropological characters. It has been shown above that a small fraction of the total range may be taken to indicate a significant difference, so the index is as capable of differentiating races as most measurements. But it fails entirely to

* Morant found a coefficient of +.747 between the nasal index and nasal angle, *loc. cit.* p. 336.

† Cf. our Fig. 1 with the tables of nasal indices and angles given by Morant, *loc. cit.* pp. 335 and 336.

arrange them in an order which can be supposed to aid classification. No two of the principal families of races are distinguished and the range for African negroes is almost as great as for all races in the world. The premaxillary index is seen to be the least valuable of the four we are considering for purposes of inter-group comparisons and, like the others, it does not provide any interesting intra-group arrangements. It thus falls into class (a) defined above. What interest there is in the arrangement shown in Fig. 2 is given to it by the frontal index alone.

Fig. 3. shows the interracial correlation of the two most suggestive indices—the frontal and simotic. The coefficient between them for male types was found to be +.699. Four of the principal families of races are seen to occupy discrete areas with somewhat of a jumble of Oceanic, American and Egyptian types falling between them. It may be doubted whether any other pair of craniological characters that have hitherto been investigated can effect a more interesting arrangement than this. The compactness of the points representing the Oriental series and their wide separation from the Indian group are particularly interesting facts. It has been shown by coefficients of racial likeness that when all the more important of the usual craniometric characters are considered together no sharp line of division can be drawn between these two Asiatic families of races, since the Nepalese type closely resembles that of Hindus of Bengal, on the one hand, and Tibetans of the A type on the other, while the last has close affinities with several Oriental types*. The relationships suggested by Fig. 3 alone are entirely different from these; the Tibetans A are as widely separated from the Indians as any other Oriental series, and, though the Nepalese point does diverge slightly in the direction of the Indians, it might well be considered a true member of the Oriental group. The frontal and simotic indices thus provide a means of distinguishing the two Asiatic families of races in the clearest possible manner, although a considerable number of other measurements fail to make a sharp division between them. This example warns us not to attempt to base any scheme of racial classification on the evidence of the two indices *alone*. They clearly provide valuable indications for the purpose in view—perhaps as valuable as those which can be given by any pair of characters known—but racial relationships can only be estimated by taking into account the evidence provided by a considerable number of characters which may not be highly correlated interracially with one another. It would be a gross mistake to assume that the New British are closely allied to any Oriental people merely because its point falls well inside the restricted area covered by all the Oriental races in Fig. 3, though mistakes of that kind are not uncommonly made.

Fig. 4 shows the interracial correlation of the premaxillary and simotic indices and it is obviously of far less interest than Fig. 3. The fact that the African Negro, Oriental and European groups are separated is seen to be due almost entirely to the separation made between them by the simotic indices. The premaxillary index may be supposed to give a measure of sub-nasal prognathism

* See T. L. Woo and G. M. Morant: "A Preliminary Classification of Asiatic Races Based on Cranial Measurements." *Biometrika*, Vol. xxiv. (1932), pp. 108—134.

Fig. 2). But the frontal breadth (*IOW*) is only able to distinguish the African Negro and Indian groups. The simotic subtense makes absolute distinctions between the African Negro and European, African Negro and Indian, Oriental and European, Oriental and Indian, and Oceanic and European groups, while the mean simotic indices make all these distinctions and also one between the African Negro and Oceanic groups as well (see Fig. 3). But the simotic chord is not able to distinguish any of the groups. These rough comparisons suggest forcibly that the frontal and simotic indices are valuable racial criteria by virtue of differences between the subtenses, but not between the chords involved. The case is different for the premaxillary characters, which were not valuable racial criteria, since the index makes no distinctions between any of the groups (see Fig. 2), but the breadth (*GB*) differentiates the Oriental group from both the European and Indian (see Table XI).

TABLE XI.

Ranges of Male Mean Absolute Measurements (all based on Fifteen or more Crania for different Groups of Races.*

Racial Groups	Sub. <i>IOW</i>	<i>SS</i>	<i>IOW</i>	<i>SC</i>	<i>GB</i>
African Negro†	17.1—18.6 (8)	2.2—2.9 (8)	98.6—103.0 (8)	8.5—9.5 (8)	93.9—98.5 (7)
Oriental‡	15.1—16.5 (14)	2.4—3.4 (14)	94.3—99.5 (14)	7.6—9.6 (14)	97.4—101.5 (12)
Oceanic	16.0—18.8 (16)	2.7—4.1 (16)	97.3—101.8 (16)	6.5—9.3 (16)	89.8—102.6 (15)
European	17.7—19.3 (19)	4.5—4.9 (18)	96.6—100.0 (19)	8.6—10.3 (18)	91.7—97.2 (18)
Indian§	18.9—20.2 (7)	3.7—5.0 (7)	93.2—97.6 (7)	8.5—9.8 (7)	93.0—96.9 (6)

* The American series were omitted since very few are available, as are some others, such as the Egyptian series, which cannot be placed in any of the groups included. The numbers in brackets are the numbers of series from which the extremes were deduced.

† Excluding the Bushman series. ‡ Excluding the Andamanese but including the Celebes series.

§ Including the Afghan series.

8. *Conclusions.* The transverse "flatness" of the facial skeleton has long been recognised to be a valuable criterion for the purpose of aiding the discrimination and classification of the races of modern man. This research was undertaken with the object of examining it by quantitative methods on an adequate number of crania. Four pairs of absolute measurements were taken, whenever possible, on each specimen giving four indices which express a "median" subtense as a percentage of a corresponding transverse breadth. One of these indices—the simotic—had been used previously and the others are termed the frontal, rhinal and premaxillary indices. All the subtenses were found with a pair of co-ordinate calipers made for the purpose. The material measured (preserved in various English and one Dutch museum) consists of 4266 male and 1630 female specimens divided into 83 series which represent all the principal groups of races of modern man. An analysis of the statistical constants leads to the following conclusions:

(i) Five of the eight absolute measurements show sex ratios of the usual order, but for the simotic chord the ratios are small and for the simotic and rhinal sub-

tenses they are particularly large. The simotic and rhinal indices hence show marked sexual differences between means, while the premaxillary index shows slight sexual differentiation and the frontal none at all. The nasal bones are, on the average, more protruding in the male than in the female of the same race, but the premaxillary region tends to be less protruding in the male.

(ii) In the case of the indices male variation shows a distinct tendency to be greater than female, but for absolute measurements—judging by coefficients of variation—female variation is either greater than male, or else there is sexual equality in variability.

(iii) There is no evidence that the variabilities of the characters examined differentiate "primitive" and "advanced" races.

(iv) The intraracial correlations of the absolute measurements and indices are decidedly lower than would have been anticipated.

(v) The interracial variabilities tend to be decidedly smaller than the intraracial variabilities.

(vi) The interracial correlations of the indices are clearly differentiated from, and they tend to be decidedly higher than, the corresponding intraracial correlations. The correlations of the two classes in the case of the absolute measurements show many marked differences.

(vii) All four indices show significant differences between the majority of the means that are available, so they are well-marked racial characters.

(viii) While showing no marked differences between races belonging to the same family, or group, of races, the rhinal and simotic indices—which are highly correlated interracially—appear as capable of making inter-group distinctions as any other craniological characters that have been examined (see Fig. 1).

(ix) The frontal index also shows small intra-group differences, and it makes several clear inter-group distinctions, though these are not so well-defined as those given by the rhinal and simotic indices (see Fig. 2).

(x) The premaxillary index fails to make distinctions between the different groups of races and it does not provide any intra-group arrangements to which significance can be attached (see Fig. 2).

(xi) There would be little advantage in determining the rhinal and premaxillary indices for larger numbers of skulls, but the simotic and frontal indices are valuable characters which should be recorded in the routine descriptions of racial series of crania. The arrangement provided by the last two (see Fig. 3) is as interesting as that given by any other pair of cranial measurements. Some distinctions are made by the simotic and frontal indices which are not effected by any other cranial characters that have been investigated.

(xii) It is shown that the simotic and frontal indices are valuable racial criteria by virtue of racial differences between the subtenses—but not between the chords—from which they are formed.

MISCELLANEA.

(1) On certain Measures of Dependence between Statistical Variables.

By J. F. STEFFENSEN, Copenhagen.

1. In a lecture delivered at "Institut Henri-Poincaré" and published in the *Annales* of that Institute (1933, pp. 319—331) I have suggested a modification of Karl Pearson's Mean Square Contingency. The object of the modification was to present the measure in such a form that it remains applicable also when the number of distinct values which the statistical variables can assume becomes infinite. It has been pointed out to me recently by Mrs Pollaczek that the measure in the form I had proposed does not always, as I had somewhat incautiously asserted, assume the value 1 in the case of complete dependence between the variables. A slight modification suffices, however, to meet this objection, and the measure is thus reduced to its original form which I communicated to Professor Cramér at the beginning of 1930 without publishing it. In the present Note I intend to establish the measure from first principles in its correct form, and thereafter to propose a new measure which presents certain advantages over the other.

2. With a view to comparisons with the lecture, it is preferable to retain the same notation, and as this differs somewhat from notations employed by other authors, I shall begin by explaining the symbols preferred.

Let there be a compound event depending on two statistical variables x and y of which x must assume one of the distinct values

$$x_1, x_2, x_3, \dots,$$

and y one of the distinct values

$$y_1, y_2, y_3, \dots,$$

each with a certain non-vanishing probability, the sum of the probabilities being unity for each sequence of values. The number of values may be finite or not, but it is assumed to exceed one in each sequence, as otherwise x or y would not be a statistical variable.

The probability that x assumes the value x_i and y the value y_j is denoted by p_{ij} . The probability that x assumes the value x_i irrespective of what value y assumes is obtained by summing p_{ij} with respect to all j . A summation of this nature is denoted by replacing the letter with respect to which the summation is performed by an asterisk. We thus have

$$p_{i*} = \sum_j p_{ij} \text{ and similarly } p_{*j} = \sum_i p_{ij}.$$

We also note

$$p_{**} = \sum_{ij} p_{ij} = \sum_i p_{i*} = \sum_j p_{*j} = 1.$$

It was assumed that p_{i*} and p_{*j} cannot vanish; they are also smaller than unity, because there are more than one possible values for each of the variables. We have, therefore,

$$0 < p_{i*} < 1, \quad 0 < p_{*j} < 1 \dots \dots \dots (1),$$

for all values of i and j .

The probability that y assumes the value y_j when it is known that x assumes the value x_i is denoted by $p_{(i)j}$. We thus have

$$p_{ij} = p_{i*} p_{(i)j} \dots \dots \dots (2),$$

and similarly

$$p_{ij} = p_{*j} p_{i(j)} \dots \dots \dots (3).$$

The extreme case where x and y are *completely independent* of each other is characterised by the relation

$$p_{ij} = p_{i*} p_{*j} \dots\dots\dots (4),$$

valid for all i and j .

The other extreme case occurs if one of the statistical variables is a function of the other. In this case, each value of x is always associated with the same value of y , and we may therefore, without loss of generality, associate x_i with y_i . The condition of *complete dependence* may then be written

$$p_{ii} = p_{i*} = p_{*i}; \quad p_{ij} = 0 \text{ for } i \neq j \dots\dots\dots (5).$$

3. After these preliminaries, we may proceed to the measure. It is denoted by ψ and defined by the relation

$$\psi^2 = \sum p_{ij} \phi_{ij}^2 \dots\dots\dots (6),$$

where

$$\phi_{ij}^2 = \frac{(p_{ij} - p_{i*} p_{*j})^2}{p_{i*} (1 - p_{i*}) p_{*j} (1 - p_{*j})} \dots\dots\dots (7).$$

We propose to show that ψ^2 possesses the following properties:

- (I) ψ^2 is always comprised between 0 and 1.
- (II) ψ^2 vanishes if the variables are completely independent and only in that case.
- (III) ψ^2 assumes the value 1 in the case of complete dependence, and only in that case.

As regards the property (I), we observe that ψ^2 is an arithmetical mean, with positive or zero coefficients, of the quantities ϕ_{ij}^2 . It therefore suffices to prove that $\phi_{ij}^2 \leq 1$ for all values of i and j .

Now, in the cases where $p_{ij} \geq p_{i*} p_{*j}$, we write ϕ_{ij}^2 in the form

$$\phi_{ij}^2 = \frac{p_{ij} - p_{i*} p_{*j}}{p_{i*} (1 - p_{*j})} \cdot \frac{p_{ij} - p_{i*} p_{*j}}{p_{*j} (1 - p_{i*})} \dots\dots\dots (8).$$

As $p_{ij} \geq p_{i*} p_{*j}$, we have

$$p_{ij} - p_{i*} p_{*j} \leq p_{i*} (1 - p_{*j}),$$

so that the first factor in (8) cannot exceed unity.

Further, as $p_{ij} \leq p_{*j}$, we have

$$p_{ij} - p_{i*} p_{*j} \leq p_{*j} (1 - p_{i*}),$$

so that the second factor in (8) cannot exceed unity. We therefore have $\phi_{ij}^2 \leq 1$, if $p_{ij} \geq p_{i*} p_{*j}$.

Considering, next, the cases where $p_{ij} \leq p_{i*} p_{*j}$, we write ϕ_{ij}^2 in the form

$$\phi_{ij}^2 = \frac{p_{i*} p_{*j} - p_{ij}}{p_{i*} p_{*j}} \cdot \frac{p_{i*} p_{*j} - p_{ij}}{(1 - p_{i*})(1 - p_{*j})} \dots\dots\dots (9).$$

Here, the first factor obviously cannot exceed unity. As regards the second factor, we begin by observing that

$$p_{i*} - p_{ij} \leq 1 - p_{*j},$$

because the right-hand side is obtained by summation of the left-hand side with respect to i . We have therefore

$$p_{i*} p_{*j} - p_{ij} \leq 1 - p_{i*} - p_{*j} + p_{i*} p_{*j}$$

or

$$p_{i*} p_{*j} - p_{ij} \leq (1 - p_{i*})(1 - p_{*j}),$$

so that the second factor in (9) cannot exceed unity.

We therefore have $\phi_{ij}^2 \leq 1$ also if $p_{ij} \leq p_{i*} p_{*j}$; and hence, in all cases.

Proceeding to the property (II), it is obvious that ψ^2 vanishes if $p_{ij} = p_{i*} p_{*j}$ for all values of i and j , that is, if the variables are completely independent.

In order to prove the converse proposition, we assume that ψ^2 vanishes, and have to show that then $p_{ij} = p_{i*} p_{*j}$ for all values of i and j . We begin by proving that if $\psi^2 = 0$, no p_{ij} can vanish*.

Let us for a moment assume that, for instance, $p_{rs} = 0$; this assumption leads to a contradiction if $\psi^2 = 0$. We have, in fact,

$$\sum_y (p_{iy} - p_{i*} p_{*y}) = 0,$$

and as the term

$$(p_{rs} - p_{r*} p_{*s})$$

is negative, because $p_{rs} = 0$, there must be at least one positive term in the sum. Let, for instance, the term

$$(p_{nm} - p_{n*} p_{*m})$$

be positive. In that case ϕ_{nm}^2 does not vanish, and ψ^2 contains therefore the positive term $p_{nm} \phi_{nm}^2$. Therefore $\psi^2 > 0$; but we had assumed $\psi^2 = 0$, and we have thus arrived at a contradiction.

As no p_{ij} can vanish, if $\psi^2 = 0$, it follows from (6) that all ϕ_{ij}^2 vanish if $\psi^2 = 0$, that is, we have $p_{ij} = p_{i*} p_{*j}$ for all values of i and j .

Turning, finally, our attention to the property (III), we first have to prove that if the dependence is complete, then $\psi^2 = 1$. Now, making use of (5), (6) reduces to

$$\sum_i p_{ii} \frac{(p_{ii} - p_{i*}^2)^2}{p_{ii}(1 - p_{ii}) p_{ii}(1 - p_{ii})} = \sum_i p_{ii} = 1.$$

Conversely, let us assume that $\psi^2 = 1$, that is

$$\sum_{ij} p_{ij} \phi_{ij}^2 = 1.$$

As $\sum_{ij} p_{ij} = 1$, and as $\phi_{ij}^2 \leq 1$, this relation cannot hold, unless $\phi_{ij}^2 = 1$ for all values of i and j for which p_{ij} does not vanish. Now, if $p_{ij} > 0$, we cannot at the same time have $p_{ij} \leq p_{i*} p_{*j}$ and $\phi_{ij}^2 = 1$, as the first factor in (9) is less than unity. We therefore have $p_{ij} > p_{i*} p_{*j}$, and as each of the factors in (8) must equal unity, if $\phi_{ij}^2 = 1$, we have $p_{ij} = p_{i*} p_{*j}$, that is, y_j occurs certainly, if x_i occurs.

These considerations show that if $\psi^2 = 1$, then there is complete dependence between the variables.

4. In order to approximate to ψ by means of a given experience, we replace the probabilities by the corresponding relative frequencies. Let H_{ij} be the absolute frequency of the combination (x_i, y_j) , and let us write

$$H_{i*} = \sum_j H_{ij}, \quad H_{*j} = \sum_i H_{ij}, \quad N = H_{**},$$

so that N is the total number of observations. If Ψ denotes the approximation to ψ obtained in this way, we find by (6) and (7)

$$\Psi^2 = \frac{1}{N} \sum_{ij} \frac{H_{ij} (NH_{ij} - H_{i*} H_{*j})^2}{H_{i*} (N - H_{i*}) H_{*j} (N - H_{*j})} \dots \dots \dots (10).$$

For the reasons explained in the lecture, no attempt has been made to estimate the mean error of this expression.

5. The calculation of Ψ by (10), although not prohibitive, is more laborious than desirable, and I have therefore endeavoured to establish another measure which leads to simpler calculations. The new measure which I shall denote by ω is defined as

$$\omega = \frac{\sum |p_{ij} - p_{i*} p_{*j}|}{\sum (p_{ij} - p_{ij}^2) + \sum p_{i*} p_{*j}} \dots \dots \dots (11),$$

* I owe this observation to Mr N. P. Bertelsen with whom I have discussed various points in this paper.

where Σ denotes summation with respect to all i and j , $\bar{\Sigma}$ summation with respect to all i and j for which $p_{ij} > p_{i*}p_{*j}$, and $\underline{\Sigma}$ summation with respect to the remaining i and j , that is, the values for which $p_{ij} \leq p_{i*}p_{*j}$.

It can be proved that ω possesses the same three fundamental properties (I), (II) and (III) as ψ . Before proceeding to the proof, it is useful to point out that ω may be written in various other forms. For the sake of argumentation it is preferable to write

$$\omega = \frac{\bar{\Sigma}(p_{ij} - p_{i*}p_{*j}) + \underline{\Sigma}(p_{i*}p_{*j} - p_{ij})}{\bar{\Sigma}(p_{ij} - p_{ij}^2) + \underline{\Sigma}p_{i*}p_{*j}} \dots\dots\dots(12).$$

Further, as

$$\Sigma(p_{ij} - p_{i*}p_{*j}) = 0,$$

the aggregate of positive terms in this sum must cancel the aggregate of negative ones; hence (11) may be written

$$\omega = 2 \frac{\bar{\Sigma}(p_{ij} - p_{i*}p_{*j})}{\bar{\Sigma}(p_{ij} - p_{ij}^2) + \underline{\Sigma}p_{i*}p_{*j}} \dots\dots\dots(13).$$

If, in this expression, we eliminate $\underline{\Sigma}$ by the relation $\Sigma = \bar{\Sigma} + \underline{\Sigma}$ and observe that $\Sigma p_{i*}p_{*j} = 1$, we obtain

$$\omega = 2 \frac{\bar{\Sigma}(p_{ij} - p_{i*}p_{*j})}{\bar{\Sigma}(p_{ij} - p_{i*}p_{*j}) + 1 - \bar{\Sigma}p_{ij}^2} \dots\dots\dots(14).$$

The convergence of all the sums is obvious, as

$$\Sigma|p_{ij} - p_{i*}p_{*j}| < \Sigma p_{ij} + \Sigma p_{i*}p_{*j} = 2,$$

and

$$\bar{\Sigma}(p_{ij} - p_{ij}^2) + \underline{\Sigma}p_{i*}p_{*j} < \Sigma p_{ij} + \Sigma p_{i*}p_{*j} = 2,$$

$$\bar{\Sigma}p_{ij}^2 < \Sigma p_{ij} = 1.$$

As regards the proof of the first property, or (I), we observe that $p_{ij} \leq p_{i*}$, $p_{ij} \leq p_{*j}$, so that $p_{ij}^2 \leq p_{i*}p_{*j}$. It follows that each term in the nominator of (12) cannot exceed the corresponding term in the denominator, so that $0 \leq \omega \leq 1$.

The second property (II) is obvious, as (11) shows that ω vanishes if $p_{ij} = p_{i*}p_{*j}$ for all values of i and j , and only then.

In order to prove the third property (III), we must first show that $\omega = 1$ if there is complete dependence between x and y . Now in that case we have, according to (5), $p_{ij} = 0$ if $i \neq j$, and the corresponding terms which belong to $\bar{\Sigma}$ in (12) become equal in nominator and denominator. If, on the other hand, $i = j$, we have $p_{ii} = p_{i*} = p_{*i}$, and the corresponding terms which belong to $\bar{\Sigma}$ in (12) become equal to $(p_{ii} - p_{ii}^2)$ both in nominator and denominator of (12). Hence $\omega = 1$, if the dependence is complete.

We finally have to prove that if $\omega = 1$, then the dependence is complete. But, observing that all the terms in the sums in (12) are positive or zero, and that each term in the nominator cannot exceed the corresponding term in the denominator, it becomes clear that, if $\omega = 1$, then each term in the nominator must be equal to the corresponding term in the denominator. Hence, we have either $p_{ij} = 0$ or else $p_{ij}^2 = p_{i*}p_{*j}$; but on multiplying (2) and (3) with each other it is seen that we can only have $p_{ij}^2 = p_{i*}p_{*j}$ if $p_{i(j)} = 1$, $p_{i(j)} = 1$, so that

$$p_{ij} = p_{i*} = p_{*j}.$$

The dependence is, therefore, complete.

6. In order to approximate to ω by means of a given experience, we replace, as in paragraph 4, the probabilities by the corresponding relative frequencies. The simplest formula to employ for the numerical work is (14). If Ω denotes the approximation to ω obtained in this way, we find

$$\Omega = 2 \frac{\bar{\Sigma}NH_{ij} - \bar{\Sigma}H_{i*}H_{*j}}{\bar{\Sigma}NH_{ij} - \bar{\Sigma}H_{i*}H_{*j} + N^2 - \bar{\Sigma}H_{ij}^2} \dots\dots\dots(15),$$

where $\bar{\Sigma}$ means summation with respect to all values of i and j for which $NH_{ij} > H_{i*}H_{*j}$.

It is seen that the calculation of Ω by (15) is considerably easier than the calculation of Ψ by (10).

7. In order to apply the measures ψ and ω to continuous probabilities, we choose a constant interval h for x , and a corresponding interval k for y , and put

$$p_{ij} = \int_{jk}^{jk+k} \int_{ih}^{ih+h} f(x, y) dx dy,$$

$$p_{i*} = \int_{ih}^{ih+h} f(x, *) dx, \quad p_{*j} = \int_{jk}^{jk+k} f(*, y) dy,$$

where the asterisk means integration over the whole range. Inserting these values in (6) and (11), it is seen that the values of ψ and ω depend on the values chosen for h and k . This dependence is very pronounced in the case of ψ , less so in the case of ω , so that ω is also in this respect preferable to ψ .

An entirely satisfactory measure in the case of continuous probabilities has not yet been found, as far as I am aware. The measures proposed by H. Cramér* and by Mrs Pollaczek† are subject to inconveniences to which these authors themselves draw attention.

(II) Remarks on Professor Steffensen's Measure of Contingency.

EDITORIAL.

Let us consider a bivariate contingency table in which m_{ij} individuals are in the ij internal cell, and for which $m_{i.}$ individuals fall in the i th cell of one marginal total and $m_{.j}$ in the j th cell of the other marginal total.

Let

$$M = S_{ij} (m_{ij}) = S_i (m_{i.}) = S_j (m_{.j}).$$

Consider the expression

$$\psi^2 = S_{ij} \frac{m_{ij} \left(m_{ij} - \frac{m_{i.} m_{.j}}{M} \right)^2}{M m_{i.} m_{.j} \left(1 - \frac{m_{i.}}{M} \right) \left(1 - \frac{m_{.j}}{M} \right)} \dots\dots\dots(i).$$

Then from (i) it is possible to show that, if

(a) $m_{ij} = m_{i.} m_{.j} / M$ for all values of i and j , then the two variates are independent and $\psi^2 = 0$, and conversely if $\psi^2 = 0$, the variates are independent;

(b) $m_{ij} = m_{i.} = m_{.j}$, the two variates are such that if a value of one occurs in the i th marginal cell, the corresponding value of the second will always fall into the j th marginal cell, and the value of ψ^2 will be 1, and conversely if the value of ψ^2 be unity, then the above relation holds;

(c) the value of ψ^2 always lies between 0 and 1.

Now let us write $p_{ij} = m_{ij} / M$ and $p_{i.} = m_{i.} / M$ and $p_{.j} = m_{.j} / M$, then (i) becomes

$$\psi^2 = S_{ij} \frac{p_{ij} (p_{ij} - p_{i.} p_{.j})^2}{p_{i.} p_{.j} (1 - p_{i.}) (1 - p_{.j})} \dots\dots\dots(ii).$$

(ii) is clearly in form identical with Professor Steffensen's ψ^2 value.

We may define p_{ij} as the chance of drawing an individual out of the ij th cell and $p_{i.}$ and $p_{.j}$ as the chances of drawing individuals from i th and j th marginal cells, when an individual must be returned, before a second drawing, to the cell from which it has been extracted.

* *Skandinavisk Aktuarietidskrift*, Vol. VII. (1924), p. 231.

† *Zeitschrift für angewandte Mathematik*, Vol. 13 (1933), p. 122.

Now there seems in this method of approaching the subject no restriction on the number of cells or on the smallness or largeness of the p 's, or it seems allowable to pass to differentials.

Suppose then we have a frequency surface of x, y variates, where

$$z_1 = Mf_1(x, y),$$

and for the distributions of x and y alone, i.e. marginal total curves,

$$z_2 = Mf_2(x), \quad z_3 = Mf_3(y).$$

We can take

$$m_{ij} = Mf_1(x, y) dx dy,$$

$$m_{i.} = Mf_2(x) dx, \quad m_{.j} = Mf_3(y) dy,$$

and accordingly

$$\begin{aligned} \psi^2 &= S_{ij} \frac{dx^2 dy^2 f_1(x, y) (f_1(x, y) - f_2(x) f_3(y))^2}{f_2(x) f_3(y) dx dy (1 - f_2(x) dx) (1 - f_3(y) dy)} \\ &= \iint dx^2 dy^2 \left\{ \frac{f_1(x, y) (f_1(x, y) - f_2(x) f_3(y))^2}{f_2(x) f_3(y)} \right\} \text{ in the limit.} \end{aligned}$$

The expression in curled brackets will be finite and accordingly the integral, owing to the presence of $dx^2 dy^2$ instead of $dx dy$, will vanish, however close or loose the bond between x and y .

The mean square contingency as I define it is given by

$$\begin{aligned} \phi^2 &= S_{ij} \frac{\left(m_{ij} - \frac{m_{i.} m_{.j}}{M}\right)^2}{\frac{M m_{i.} m_{.j}}{M}} \\ &= S_{ij} \frac{(p_{ij} - p_{i.} p_{.j})^2}{p_{i.} p_{.j}} \\ &= \iint dx dy \frac{(f_1(x, y) - f_2(x) f_3(y))^2}{f_2(x) f_3(y)} \text{ in the limit,} \end{aligned}$$

and remains finite when we pass to the limit. My coefficient of mean square contingency, C_2 , is given by

$$C_2 = \sqrt{\frac{\phi^2}{1 + \phi^2}},$$

and this always lies between 0 and $\sqrt{1 - \frac{1}{n}}$, where n is the number of cells occupied if there be only one occupied cell in each row and in each column of the table. Only when n is indefinitely large, does C_2 range from 0 to unity, for example in cases of continuous distribution. So far from considering this a disadvantage of C_2 , I hold it an essential property of a contingency coefficient; for as long as the cells are not indefinitely small, it is possible to arrange the individuals in them so as to increase the contingency. I cannot therefore hold that in choosing ψ^2 in the form (i) instead of C_2 , we should gain any advantage from the fact that its upper limit is unity. I do not think a good coefficient of contingency ought to be unity until we introduce the conception of continuity, namely every y having attached to it a definite x , so that y is a continuous function of x . The fact that Professor Steffensen's ψ^2 is necessarily always zero when we pass to continuity, suggests that it will take a very low value in cases where we should expect a high value of the variate relationship. Indeed, the presence of the factor m_{ij} in the numerator of (i), or of p_{ij} in Professor Steffensen's expression, means there can be no contribution to ψ^2 from the ij th cell when $m_{i.}$, $m_{.j}$, or $p_{i.}$, $p_{.j}$ are finite but p_{ij} is zero. This seems a serious disadvantage in practical working, for if $m_{i.}$ and $m_{.j}$ are considerable, and yet m_{ij} zero, we should expect a considerable addition to the contingency, which is not provided by the Steffensen ψ^2 .

(6) Let us now see what happens when we increase the scatter. We will take four .5's to a row. Table VI suggests a high degree of relationship, if it were based on a continuous variation.

TABLE VI.

[illegible]

Steffensen's ψ^2 sinks to .04770, while my $\phi^2 = 2.25$, giving as Coefficient of Mean Square Contingency $C_2 = .83205$. We shall find that this drop down of the ψ^2 as we get lower and lower probabilities is the influence of the p_{ii} term in the numerator, and that, if we reach continuous variates, the influence of the approach of $\delta x \delta y$ to $dx dy$ and zero would be more and more conspicuous.

(7) Table VII shows a still greater increase in the scatter. We should anticipate lower values of ψ^2 and C_2 , but still there is a very considerable degree of relationship between the two variates. We find $\psi^2 = 0.1916$ and $\phi^2 = 1.5$, giving a value .77460 for C_2 .

TABLE VII.

[illegible]

It is clear that the two methods differ completely in the positions in their respective scales, both running from 0 to 1, which are taken by the same table. It certainly seems as if the distributions dealt with in Tables I to VII ought to show high relationships between the variates, but actually judged by ψ^2 they give in all cases, but Table III, remarkably low values. We

I think, on the contrary, that both Piaggio (p. 96, last line) and I (p. 424, Section 5) have shown explicitly that the indeterminateness of g is of exactly this type. For we have shown that the indeterminate factor (ki in Piaggio's notation, t in mine) cannot be determinate because it need satisfy only $K+1$ equations and it is itself a set of N unknowns, and N is greater than $K+1$. Similarly, Spearman's x , y , z cannot be determinate because they need to satisfy only one equation and they are themselves a set of 3 unknowns, and 3 is greater than 1. It seems to me impossible to recognise a difference in *type* between these two cases. It *might* be asserted that if N , the number of individuals tested, is small, the *range* of indeterminateness is limited by the numerical possibilities, because the standard deviation of ki is fixed, but these limits are exceedingly broad, and they increase indefinitely as N is increased*. Therefore such a claim could have no meaning from the psychological standpoint, for it cannot be true that the question whether individual Smith possesses a certain mental quality depends in some mathematical or occult way on how many Joneses are dragged in to be tested with him.

Secondly, Spearman says that the arbitrary term (ki) in the formula for g represents the "inexactitude" of one's *determination* of g , not a multiple nature in g itself. The error that he appears to me to be making here cannot be described so simply, but, I think, it is very important, especially as it is also implicit in the remarks of several other authors. It has been shown that, for the given set of $K-1$ tests and N individuals, there exists not one set of Ng 's but an arbitrarily large group of such sets. Spearman appears to think that, because the variability of g *within each of these sets* is limited, it follows that the variability of a g from set to set for each fixed individual is also limited. This is not true, except in those extreme cases when $\sigma_{ki}=0$, exactly. Unfortunately Piaggio's language also encourages one to fall into this error, for in discussing his illustrative examples he implies that g can be shown to be practically unique if σ_{ki} for each set is negligibly small.

But it is easy to select a group of sets of ki 's in which the variability within each set is as small as he indicates but for which the variability from set to set for any individual chosen arbitrarily in advance is very, very large. It is clearly, however, this last variability which must be proved to be small if it is to be shown in this manner that each individual has a characteristic g .

The gist of the matter is really contained in the mathematician's notion of "order of choice." If, before looking at Smith's scores on the tests, one may choose a number at random (subject only to the broad limitations mentioned before), and can then demonstrate that this number can be assigned as Smith's g , as well as any other number, and in perfect harmony with all the other hypotheses, then it is meaningless to assert that Smith *has* a g . In the numerical illustration at the close of my *Biometrika* paper, two such Smiths were exhibited (individuals 11 and 12).

It is not, of course, contended that a unique general factor does not, in truth, exist, but that its existence does not follow from Spearman's hypotheses. Moreover, it would seem inherently impossible that it could follow from his hypotheses, for into them he has introduced only group averages, of means, σ 's, and r 's. Out of them, therefore, whatever the mathematics, one could expect to obtain nothing more significant than characteristics of the group as a whole, and indeed this much, to repeat a statement in my other paper, has actually been established in the "almost unique" cases.

* E.g., it is numerically impossible to choose 8 numbers for which the total range of variation is greater than 4σ , or 50 numbers for which it is greater than 10σ .

(iv) Note on the Recurrence Formulae for the Moments of the Point Binomial.

By A. A. KRISHNASWAMI AYYANGAR, Mysore University, Mysore, S. India.

§ 1. It is the object of this note to indicate that the recurrence formulae for the moments of the point binomial $(p+q)^n$ about the mean nq can be most simply obtained by a little ingenious transformation of the term $(t-nq)^s p^{n-t} q^t \binom{n}{t}$ in the series for μ_s , the s th moment. This transformation explains why the recurrence formulae for incomplete moments are as easy to derive as those for complete moments, which are particular cases of the former.

We may write

$$(t-nq)^s p^{n-t} q^t \binom{n}{t} = nq(t-1-\overline{n-1} \cdot q+p)^{s-1} p^{n-t} q^{t-1} \binom{n-1}{t-1} - nq(t-nq)^{s-1} p^{n-t} q^t \binom{n}{t} \quad [\text{since } p+q=1] \dots\dots(1).$$

Let

$$\nu_s^\rho = \sum_{t=\rho}^{t=n} (t-nq)^s p^{n-t} q^t \binom{n}{t}.$$

Then

$$\begin{aligned} \frac{1}{nq} \cdot \nu_s^\rho &= \sum_{t=\rho}^{t=n} (t-1-\overline{n-1} \cdot q+p)^{s-1} p^{n-t} q^{t-1} \binom{n-1}{t-1} - \sum_{t=\rho}^{t=n} (t-nq)^{s-1} p^{n-t} q^t \binom{n}{t} \\ &= \sum_{t'=\rho-1}^{t'=n-1} (t'-\overline{n-1} \cdot q+p)^{s-1} p^{n-t'} q^{t'} \binom{n-1}{t'} - \sum_{t=\rho}^{t=n} (t-nq)^{s-1} p^{n-t} q^t \binom{n}{t}, \end{aligned}$$

i.e. $\nu_s^\rho = \{ (E+p)^{s-1} \nu_{s-1}^{\rho-1} - \nu_{s-1}^\rho \} nq \dots\dots\dots(2),$

where $E(\nu_{s-1}^{\rho-1}) = \nu_{s-1}^{\rho-1}$ and $\nu_{s-1}^{\rho-1}$ is the s th incomplete moment defined for the binomial $(p+q)^{n-1}$. Putting $\rho=0$, we get the recurrence formula for complete moments

$$\mu_s = -nq\mu_{s-1} + nq(E+p)^{s-1}\mu_0' \dots\dots\dots(3).$$

The formulae (2) and (3) are believed to be new.

Application:

For brevity we write ν_s and ν_s'' for ν_s^ρ and $\nu_s^{\rho-1}$. Putting $s=1$ in (2), we have

$$\nu_1 = (\nu_0'' - \nu_0) nq, \text{ i.e. } \nu_0'' = \nu_0 + \frac{\nu_1}{nq} \dots\dots\dots(4).$$

But

$$\begin{aligned} \nu_0'' &= \sum_{t=\rho-1}^{t=n-1} p^{n-1-t} q^t \binom{n-1}{t} \\ &= \sum_{t=\rho-1}^{t=n-2} \left[p^{n-1-t} q^{t+1} \binom{n}{t+1} + p^{n-t} q^t \binom{n-1}{t} - p^{n-t-1} q^{t+1} \binom{n-1}{t+1} \right] + q^n \binom{n}{n} + p q^{n-1} \binom{n-1}{n-1} \\ &\quad \text{since } \binom{n-1}{t} = \binom{n-1}{t+1} p + \binom{n}{t+1} q - \binom{n-1}{t+1} q \\ &= \sum_{t'=\rho}^{t'=n} p^{n-t'} q^{t'} \binom{n}{t'} + \sum_{t=\rho-1}^{t=n-2} \left[p^{n-t} q^t \binom{n-1}{t} - p^{n-t-1} q^{t+1} \binom{n-1}{t+1} \right] + p q^{n-1} \binom{n-1}{n-1}, \end{aligned}$$

i.e. $\nu_0'' = \nu_0 + p^{n-\rho+1} q^{\rho-1} \binom{n-1}{\rho-1} \dots\dots\dots(5).$

Comparing (4) and (5), we get

$$\nu_1 = n p^{n-\rho+1} q^\rho \binom{n-1}{\rho-1} = \rho p^{n-\rho+1} q^\rho \binom{n}{\rho}^* \dots\dots\dots(6).$$

* This result, for which we have given here a simple proof, is due to Ragnar Frisch. See *Scandinavian Actuarietidskrift* (1924), No. 8, p. 161, quoted in *Biometrika*, Vol. xvii. (1925) p. 170.

Hence

$$\nu_1'' = \nu_1(\rho - 1)/nq \dots\dots\dots(7).$$

Similarly by setting $s=2, 3, \dots$ in (2) we calculate in succession

$$\left. \begin{aligned} \nu_2 &= \nu_1(\rho - \overline{n+1} \cdot q) + \nu_0 npq^* \\ \nu_2'' &= \nu_1 \left\{ p - \frac{p}{n} + \frac{(\rho-1)^2}{nq} - \rho + 1 \right\} + \nu_0 pq(n-1) \\ \nu_3 &= \nu_1 \{ (\rho - \overline{n+1} \cdot q)^2 + pq(2n-1) \} + \nu_0 npq(p-q)^* \\ \nu_3'' &= \frac{\nu_1}{nq} \{ (\rho-1) [(\rho-1-nq)^2 + pq \cdot \overline{2n-3}] + pq(p-q)(n-1) \} \\ &\quad + \nu_0(n-1)pq(p-q) \\ \nu_4 &= \nu_1 [(\rho - \overline{n+1} \cdot q)^3 + 3pq(n-1)(\rho - \overline{n+1} \cdot q) + pq(p-q)(3n-1)] \\ &\quad + \nu_0 [npq(1+3pq \cdot \overline{n-2})] \end{aligned} \right\} \dots\dots\dots(8).$$

and so on

In the above expressions, the coefficients of ν_0 are the complete moment coefficients and follow the recurrence-relation (3); and it must be possible to discover a recurrence relation for the coefficients of ν_1 also (see the following section).

§ 2. We will now give a simple proof of Pearson's recurrence formula†, which enables us also to write down immediately the corresponding recurrence formula for incomplete moments given by Ragnar Frisch‡.

We transform $(t-nq)^{s-1} p^{n-t} q^t \binom{n}{t}$ in two different ways:

$$(1) \quad pt(t-nq)^{s-1} p^{n-t} q^t \binom{n}{t} = p(t-nq)^s p^{n-t} q^t \binom{n}{t} + nqp(t-nq)^{s-1} p^{n-t} q^t \binom{n}{t},$$

$$\begin{aligned} (2) \quad pt(t-nq)^{s-1} p^{n-t} q^t \binom{n}{t} &= (t-nq)^{s-1} p^{n-t+1} q^{t-1} \binom{n}{t-1} (n-t+1)q \\ &= (t-nq)^{s-1} p^{n-t+1} q^{t-1} \binom{n}{t-1} (nqp - q \cdot \overline{t-1-nq}) \end{aligned}$$

since $p+q=1$.

$$\therefore p \{ (t-nq)^s + nq(t-nq)^{s-1} \} p^{n-t} q^t \binom{n}{t} = (t-nq)^{s-1} p^{n-t-1} q^{t-1} \binom{n}{t-1} (nqp - q \cdot \overline{t-1-nq}) \dots\dots\dots(9).$$

$$\begin{aligned} \therefore p \sum_{t=\rho}^{t=n} \{ (t-nq)^s + nq(t-nq)^{s-1} \} p^{n-t} q^t \binom{n}{t} \\ &= \sum_{t'=\rho-1}^{t'=n-1} \left\{ (t'-nq+1)^{s-1} p^{n-t'} q^{t'} \binom{n}{t'} \right\} (nqp - q \cdot \overline{t'-nq}) \\ &= \sum_{t'=\rho}^{t'=n} \left\{ (t'-nq+1)^{s-1} p^{n-t'} q^{t'} \binom{n}{t'} \right\} (nqp - q \cdot \overline{t'-nq}) \\ &\quad + (\rho-nq)^{s-1} p^{n-\rho+1} q^{\rho-1} \binom{n}{\rho-1} (nqp - q \cdot \overline{\rho-1-nq}) \\ &= \sum_{t'=\rho}^{t'=n} \left\{ (t'-nq+1)^{s-1} p^{n-t'} q^{t'} \binom{n}{t'} \right\} (nqp - q \cdot \overline{t'-nq}) + (\rho-nq)^{s-1} \rho p^{n-\rho+1} q^{\rho} \binom{n}{\rho}, \end{aligned}$$

$$\text{i.e.} \quad p(E^s + nqE^{s-1})\nu_0 = (E+1)^{s-1}(nqp\nu_0 - q\nu_1) + \nu_1(\rho-nq)^{s-1},$$

$$\text{i.e.} \quad \{ (1+E)^{s-1} - E^{s-1} \} (nqp\nu_0 - q\nu_1) = \nu_s - \nu_1(\rho-nq)^{s-1} \dots\dots\dots(10).$$

If we put here $\rho=0$, then $\nu_1=0$ and we get Pearson's formula.

* These results were given previously by Ragnar Frisch.

† *Biometrika* (1924), Vol. xvi. p. 160.

‡ *Ibid.* (1925), Vol. xvii. p. 171.

From (8) we see that ν_s is of the form $\nu_1 \lambda_s + \nu_0 \mu_s$, where $\lambda_0 = 0$ and $\lambda_1 = 1$. Using this expression in (10) and equating the coefficients of ν_1 , we get the recurrence relation for λ_s , viz.

$$\{(1+E)^{s-1} - E^{s-1}\} (nqp\lambda_0 - q\lambda_1) = \lambda_s - (\rho - nq)^{s-1} \dots\dots\dots(11),$$

which was not obvious from the other formula (2).

(v) A Note on the Incomplete Moments of the Hypergeometrical Series.

By A. A. KRISHNASWAMI AYYANGAR, Maharajah's College, Mysore, South India.

1. The aim of this note is to indicate an extremely simple and elementary method, applicable alike to complete and to incomplete moments, of deriving the recurrence formula* due to Karl Pearson for the moments of the hypergeometrical series defined by

$$\frac{pn(pn-1)\dots(pn-r+1)}{n(n-1)\dots(n-r+1)} \left[1 + \frac{r}{1} \cdot \frac{qn}{pn-r+1} x + \frac{r(r-1)}{1 \cdot 2} \cdot \frac{qn(qn-1)}{(pn-r+1)(pn-r+2)} x^2 + \dots \right].$$

Here the coefficient of x^t , which we may denote by $\phi(t)$ for brevity, is the probability of drawing t individuals without some mark in a sample of r individuals selected at random from a population n containing pn marked and qn unmarked individuals. The mean of the samples is evidently rq .

The incomplete moment of the s th order about the mean is defined by

$$\nu_s = \sum_{t=\rho}^r (t-rq)^s \phi(t).$$

2. The following identities are self-evident:

$$(i) \quad t(pn-r+t) = n(t-rq) + (r-t)(qn-t);$$

$$(ii) \quad (r-t)(qn-t) = (t-rq)^2 - c_1(t-rq) + c_2,$$

$$\text{where} \quad c_1 = nq + r(p-q) \quad \text{and} \quad c_2 = rpg(n-r);$$

$$(iii) \quad t(pn-r+t)\phi(t) = (r-t-1)(qn-t-1)\phi(t-1).$$

Multiplying both sides of (iii) by $(t-rq)^{s-1}$ and summing for values of t from ρ to r , we have using (i) and (ii),

$$\begin{aligned} & \sum_{t=\rho}^r \phi(t) (t-rq)^{s-1} \{n(t-rq) + (t-rq)^2 - c_1(t-rq) + c_2\} \\ &= \sum_{t=\rho}^r \phi(t-1) (t-rq)^{s-1} (r-t-1)(qn-t-1) \\ &= \sum_{t'=\rho-1}^{r-1} \phi(t') (t'-rq+1)^{s-1} (r-t')(qn-t'), \quad \text{where } t'=t-1, \\ &= \sum_{t'=\rho}^r \phi(t') (t'-rq+1)^{s-1} \{(t'-rq)^2 - c_1(t'-rq) + c_2\} \\ & \quad + \phi(\rho-1) \cdot (\rho-rq)^{s-1} (r-\rho+1)(qn-\rho+1). \end{aligned}$$

Introducing the operator E to denote the operation

$$E(\nu_s) = \nu_{s+1},$$

we write the above result in the form

$$n\nu_s + E^{s-1}(\nu_s - c_1\nu_1 + c_2\nu_0) = (1+E)^{s-1}(\nu_s - c_1\nu_1 + c_2\nu_0) + (\rho-rq)^{s-1} \rho(pn-r+\rho)\phi(\rho),$$

$$\text{i.e.} \quad \{(1+E)^{s-1} - E^{s-1}\}(\nu_s - c_1\nu_1 + c_2\nu_0) = n\nu_s - (\rho-rq)^{s-1} \rho(pn-r+\rho)\phi(\rho).$$

Put $s=1$, $n\nu_1 = \rho(pn-r+\rho)\phi(\rho)$,

$$\text{i.e.} \quad \nu_1 = \frac{\rho(pn-r+\rho)}{n} \phi(\rho)^* \dots\dots\dots(1).$$

Hence, we may write the recurrence formula for incomplete moments in the form

$$\{(1+E)^{s-1} - E^{s-1}\}(\nu_s - c_1\nu_1 + c_2\nu_0) = n\nu_s - n(\rho-rq)^{s-1}\nu_1 \dots\dots\dots(2).$$

This shows that we may put

$$\nu_s = \mu_s\nu_0 + \lambda_s\nu_1,$$

where

$$\lambda_0 = \mu_1 = 0, \quad \lambda_1 = \mu_0 = 1^\dagger,$$

in (2) and equate the coefficients of ν_1 and ν_0 and obtain recurrence formulae separately for λ_s and μ_s .

In fact, the recurrence formula thus obtained for μ_s is the same as Pearson's for complete moments, while that for λ_s is

$$\{(1+E)^{s-1} - E^{s-1}\}(\lambda_s - c_1\lambda_1 + c_2\lambda_0) = n\{\lambda_s - (\rho-rq)^{s-1}\} \dots\dots\dots(3).$$

Put

$$s=2, \quad \lambda_2 - c_1 = n\lambda_2 - n(\rho-rq),$$

$$\text{i.e.} \quad \lambda_2 = \frac{n(\rho-rq) - r(\rho-rq)}{n-1} \dots\dots\dots(4).$$

$$\text{Put } s=3, \quad (1+2E)(\lambda_3 - c_1\lambda_1 + c_2\lambda_0) = 2\lambda_3 + (1-2c_1)\lambda_2 + (2c_2 - c_1) = n\lambda_3 - n(\rho-rq)^2.$$

$$\text{Therefore} \quad (n-2)\lambda_3 = n(\rho-rq)^2 + (1-2c_1)\frac{n(\rho-rq) - c_1}{n-1} + 2c_2 - c_1,$$

$$\begin{aligned} \text{i.e.} \quad \lambda_3 = & \frac{1}{(n-1)(n-2)} [n^2\{(\rho-r+1q)^2 + pq(2r-1)\} \\ & - n\{(\rho-r+1:q)^2 + (\rho-r+1.q)(2r-1)(p-q) \\ & + pq(2r^2-2r-1)+r\} + 2r^2(1-3pq)] \dots\dots\dots(5). \end{aligned}$$

When $\rho=0$, it follows from (1) that $\nu_1=0$ and obviously $\nu_0=1$, so that $\nu_s=\mu_s$, which satisfies Pearson's recurrence formula.

(vi) On Simometers and their Handling.

By K. PEARSON.

The original purpose of the simometer was to measure the flatness of the nasal bridge by the ratio of its subtense to the chord. For this purpose we may take either the simotic or the rhinal index, or indeed both. But for the purposes of this note we are not concerned with what terminals may be chosen for our fundamental chord; we are concerned only with what is the definition of our "subtense." Probably to most mathematicians the subtense is the portion of the perpendicular bisector of the chord intercepted between the chord and the arc. It was from this conception of a subtense that Newton proceeded to his definition of curvature. If the arc while still an arc of continuous curvature be not symmetrical about the chordal bisector but skew to it, then the line from the point of chordal bisection to the point of contact of the tangent parallel to the chord has been considered—as in the cases of parabola and ellipse—as the subtense.

Now in the case of the nasal bridge when the two sides of the nose are not mirror images the one of the other, the length measured as "subtense," either by the simometer of Mérejkowsky or by the co-ordinate calipers of Woo (see pp. 197—98 of this issue), is not the mathematical subtense as just defined. What, however, is of more importance, the two instruments do not, or should not, give the same result, and even the same instrument will give different results according

* This is a generalisation of Frisch's result given in *Biometrika*, Vol. xvii. (1925) p. 170 (18).

† These values have to be used only after the operator E has had its effect.

to the manner in which its use is prescribed. Both Mérejkowsky's and Woo's simometers possess two degrees of freedom for the measuring tip of the mid-arm. We will consider their geometrical aspects separately.

Let ab be the chord and $afdb$ the arc of which the subtense is to be measured. Then when the tips of the curved outer arms of Mérejkowsky's simometer are placed on a and b there is a fixed point g of the instrument at equal distances from a and b . Through g a line passes (actually a rod), the end of which, d (the tip of the rod or mid-leg), can be adjusted to any point of the area agb by a turn round g and a forward motion of the tip d produced by a screw reading to $\frac{1}{10}$ mm. If gd be taken to bisect the angle agb , then Mérejkowsky's will enable us to read off the subtense cd , that is the length of the line through the bisection of ab , perpendicular to ab , and reaching to the curve. This requires two readings of the instrument, first to measure gd directly, and second to measure gc ; the latter can only be obtained by retaining the outer legs at the distance ab , removing the simometer from the nasal bridge and, without rotating gd , bringing d down to c so that the tips of the three tips of the legs all touch a plane.

If the tip d be not kept to the perpendicular bisector cg , but directed to some point h on the arc, then Mérejkowsky's simometer measures hi , which has no pretence to be a subtense, and may be greater than the maximum distance fe of the arc from the chord. The true maximum subtense will only be given by this instrument when the nose is symmetrical and care is taken that the mid-leg bisects the angle agb . This simometer provides no means of ascertaining directly the point f of the arc most remote from ab , and if the tip of the mid-leg be brought to f the subtense fg

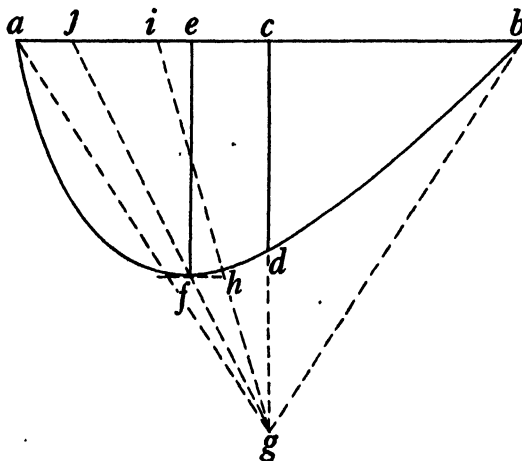


Fig. 1. $ag=gb$, cg is a perpendicular bisector of ab , f is highest point of arc afb , and h is an arbitrary point between f and d .

might in a very skew nasal bridge be such that j did not even fall inside ab ! Further, we have seen that f cannot be ascertained from this instrument. What the simometer can achieve is the ascertainment of the minimum distance from ab of the trace of the nasal bridge on the plane perpendicular to and bisecting the chord ab .

We now turn to Dr Woo's simometer or his co-ordinate calipers*.

It has three legs perpendicular to a bar, of which the second (gf) and third (mb) can be shifted along the bar and the second or mid-leg moved also perpendicular to the bar: all three legs and the bar may be discussed geometrically as coplanar lines. Fixing the external la and mb with their

* This name does not seem very well chosen.

tips at a and b , the mid-leg, gnf , has two degrees of freedom, and there is no difficulty in bringing it into the position f which enables the maximum subtense of the chord ab to be read off. By turning the instrument round ab and reading off a number of maximum subtenses like ef , we can find their minimum, which is probably the quantity of most value to the anthropologist. But unfortunately Dr Woo does not appear to have used his simometer in this manner, any more than the later workers in the Biometric Laboratory have used Mérejkowsky's simometer with the mid-leg bisecting the angle between the chords of the external legs. The procedure in both cases has been apparently to draw on the nasal bridge a line supposed to represent the nasal bridge,

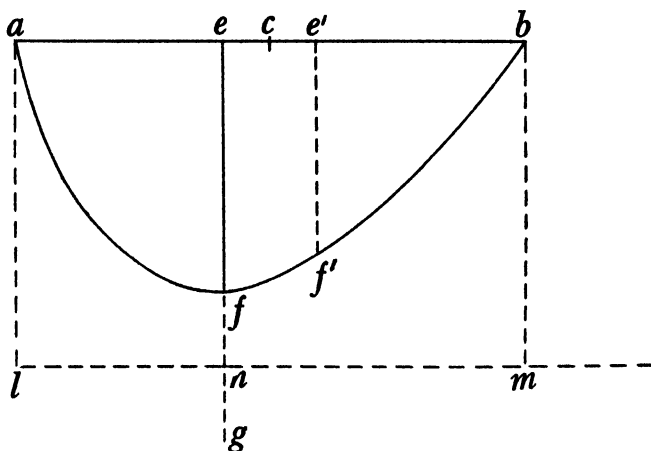


Fig. 2. c is mid-point of ab , $al = bm$, f is highest point of asymmetrical arc afb , $efng$ is perpendicular to ab , f' is an arbitrary point, $f'e'$ is perpendicular to ab , g is a point on efn produced, and lm is produced beyond m to some distance.

and this seems to have been done without any reference to the chord ab . Its obvious purpose is to get rid of one of the two degrees of freedom, which both instruments possess when working in a definite plane. But if we pay no regard to the chord ab , f may well be at f' instead of the point at maximum distance from ab , and accordingly Dr Woo's simometer would measure $f'e'$, not fe . Even with a nasal bridge which without regard to sutures had perfect mirror symmetry, both simometers would fail to give the maximum subtense if a and b were not images one of the other in the mirror plane of that nasal bridge, and we aimed our mid-leg tip at the point where the arc met the mirror plane.

To sum up: Both instruments seem to have been used in a manner which, at any rate *theoretically*, invalidates their purpose, if that purpose be to find the minimum subtense (perpendicular to ab) of all the maximum subtenses of sections of the nasal bridge by planes through ab *.

Woo's instrument could give the required result, but with more labour than is necessary to obtain a doubtful result from an arbitrary median nasal ridge. Mérejkowsky's simometer could never give in asymmetrical nasal bridges (including the asymmetry of a and b as a part of the nasal) the minimum of the maximum subtenses, but if we start by defining the median nasal ridge to be the trace on the nasal bridge of the plane perpendicularly bisecting ab , it will give the minimum subtense in this plane by simply causing the mid-leg to bisect the angle at g . No system which does not define the median nasal ridge with reference to ab can it seems to me be really satisfactory.

* Optically we may suppose the eye placed at a great or infinite distance along the line ab produced. Such an eye would see the nasal bridge as a plane curve (although all the points of the curve might not be coplanar), and the point on this curve nearest to the line ab would give the required subtense.

The optical conception could be carried out excellently with the cranial co-ordinatograph*. Place the skull by aid of it with ab perpendicular to the drawing board, and trace with the knife edge projector the contour of the nasal bridge; the minimum distance from the plan of the line ab to this contour would be the subtense sought. This frees us from any arbitrary choice of a median ridge for the nasal bridge or from the supposition that the maximum subtenses of all the sections through the chord lie in one plane, the trace of which can be determined by personal appreciation.

I am inclined to believe that the simometer of Mérejkowsky, with its choice of the plane of bisection of the chord as plane for measuring the subtense, is at least as accurate as Dr Woo's calipers applied to an arbitrary median ridge which pays no attention to the chord. What the latter method would achieve if used as a three-degree of freedom instrument to determine the minimum of the maximum subtenses it is difficult to say without experimenting. What theoretically seems to me erroneous is the determination of a nasal ridge without direct reference to the chord ab . That the two instruments do not lead to widely, although they do to significantly different results is consoling to the craniologist, if not to the theorist. There are certainly some erratic differences in the tables on p. 200, which require clearing up.

It may not be out of place to recall here a method I introduced in 1911 for determining the flatness of the nasal bridge. A fine celluloid tape or even a thread may be used to measure the shortest arcual or geodesic distance from the terminals of the chord ab across the nasal bridge. A steel tape is unsuitable. The chord ab is also measured. A very suitable measure for the flatness of the nose is then

$$\beta = 100 (\text{arc} - \text{chord}) / \text{arc}.$$

This is adequate without any determination of the subtense.

But, if we desire to find the subtense, it is needful to choose a curve sufficiently elastic to represent the nasal bridge. The circle was no use, the parabola somewhat better, but finally, after drawing a good many nasal bridge sections, I concluded that the catenary gave the best result. If then

$$a = 100 \text{ subtense} / \text{chord},$$

it is needful to compute a for each value of β . Tables to find a given β were calculated by H. E. Soper and Dr Julia Bell†.

It must be admitted at once that like Mérejkowsky's simometer this method starts with the assumption of a symmetrical nasal bridge. But if the nasal bridge be skew, the increased length of arc will to some extent compensate in an increased subtense, which compensation is lacking in the case of the simometer.

It would, I think, be of actual practical value, especially as the racial value of the simotic index is becoming more generally acknowledged, to take a series of, say, one hundred skulls, and measure the chord ab , the arc afb and the minimum of the maximum subtenses, using

- (a) The cranial co-ordinatograph to give "the true value";
- (b) Mérejkowsky's simometer with the mid-leg as bisector‡;
- (c) Woo's simometer, with no attempt at personal appreciation of a median nasal ridge, and
- (d) The catenary method.

Such an experiment would enable us, with due regard to accuracy of result, to the time and to the labour involved, to select the practically best method and to discard the others.

* See *Biometrika*, Vol. xxv. pp. 217—253.

† Published originally in *Biometrika*, Vol. viii. pp. 316, 318 and Vol. ix. pp. 401—2; reissued *Tables for Statisticians and Biometricians*, Part I. pp. lvi—lvii, 62—64.

‡ It would not be difficult to devise a mechanically less crude instrument of the Mérejkowsky type than the present one.



The Wilkinson Head in Right Profile, showing the oak pole and the corroded tip of the iron prong, and the cincture marking the removal of the skull-cap to take out the brain. Note flowing moustache and hair on chin.

BIOMETRIKA

THE WILKINSON HEAD OF OLIVER CROMWELL
AND ITS RELATIONSHIP TO BUSTS, MASKS AND
PAINTED PORTRAITS.

BY KARL PEARSON, F.R.S. AND G. M. MORANT, D.Sc.

1. *Introductory.*

So much has been written about this Head, and the controversy has been so keen, that it might appear that there was nothing to be said on the topic which had not been said already. In other words that the authenticity of the Head must be ever left in that state of doubt in which historians and critics have enveloped it. Yet when one has studied the innumerable notes, letters, and newspaper articles one finds only a mass of contradictory *opinions*, repetition of various absurd myths about Cromwell's body, not one single trustworthy measurement or fitting of the head to any form of portrait; in fact the whole of the century of discussion is *vox et praeterea nihil*. Had the authors of the present paper merely wished to contribute surmises, criticisms of earlier surmises, vague statements that the Head was in their opinion like or unlike Cromwell's portraits, there would have been no excuse for this monograph. The essential difference between this and earlier discussions of the subject is (a) that the authors have no bias for or against the authenticity; (b) that they trust solely to measurements on the Head, and to its good or bad fit to portraits; and (c) what has been essential to their investigation, that two great privileges have been granted to them by the owner, Canon Horace Wilkinson, (i) to retain the Head adequately long in order to carry on the comparison with busts, masks and portraits, and (ii) to state freely what conclusions they have reached as a result of their investigations.

It is not possible to express sufficient gratitude to Canon Wilkinson not only for permission to take skiagrams and photographs of the Head in positions corresponding to the portraits, but for the great trust he has placed in the writers by allowing the Head to be in their custody for the time necessary to carry on a very laborious investigation, and also for his indulgence in prolonging that period as new data for comparison came unexpectedly to light.

The fact is that the Cromwell portrait material seems unlimited*, and even some of the best authenticated portraits appear to differ as much from one another as they do from the Head. Again in the case of the well-known portraits by Cooper, we have come across four or five independent miniatures claimed to be by that

* Thus *Notes and Queries*, Series I, Vol. v. 1852 mentions some ten miniatures of Cromwell, nearly all ascribed to Cooper, and only we think in a single case—that of the Sir Joshua Reynolds-Lord Crewe miniature—identical with one of the various Cooper miniatures reproduced in this memoir!

master, and no one seems hitherto to have settled which is the original, which are replicas by Cooper himself, and which may be copies by contemporary or later hands. What we have indicated of the pictures is, if anything, truer of the busts, while the death-masks, the life-mask, and the wax-masks form a bewildering medley of discordances. We may safely surmise that it would not be feasible for the same skull to have formed the framework of all these most varied distributions of external organs and flesh!

2. Remarks on the Fit of Photographs to the actual Skull of a Subject.

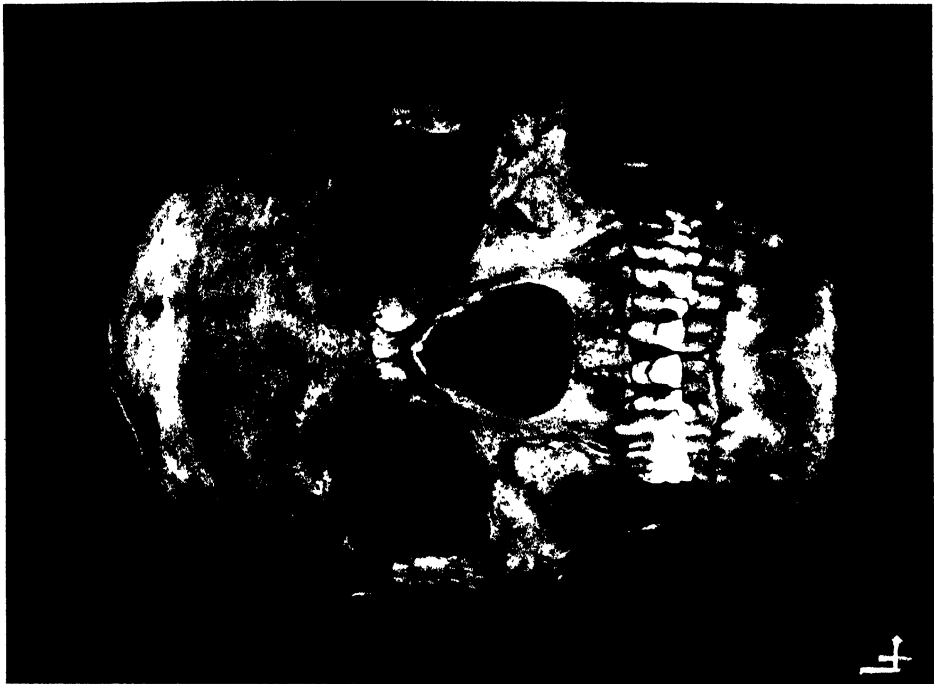
We are in the position to fit an actual skull to the photograph of its former owner. In this case the body of an Egyptian criminal after electrocution was brought into the dissecting room at Cairo, and a photograph was taken of the head. The skull was afterwards preserved, and it thus became possible to compare the two. We owe this remarkable opportunity to the courtesy of Dr D. E. Derry. Now it is true that a skiagram may give some idea of the skull inside the living flesh, but that is hardly the problem with which we have to deal on the present occasion. We have a portrait, in this case a photograph, without any statement as to scale. We have a skull actual scale. We have first to photograph the skull in the exact attitude of the subject of the photograph and then alter the scale of one or the other until a reasonable fit is obtained. The task is an arduous one and a considerable number of skull photographs and reductions may be needful, before a moderately satisfactory fit can be achieved.

Plate I shows the photograph of the criminal and a selected photograph of his skull (not the one actually fitted to the head). Plate II provides the superposition of traced drawings of the photographs of subject and of his skull, the latter having been placed in like aspect and modified in scale to fit the former. We can hardly hope for anything better than, or indeed as good as this when we attempt to fit a skull to an artist's portrait, even if that portrait is certainly that of the former owner of the skull. It is desirable that the reader should pay special attention to one or two points in Plate II. The flesh of the chin projects considerably below the lower border of the mandible, some 5·5 mm. on the sketch, corresponding to 9·5—10 mm. in life size. Without allowance for this the lowest point of the chin in a portrait is not a suitable point to base comparative measurements on. Further the reader will note from the sketch of the skull how much the fleshy borders of the cheeks in a full or nearly full face portrait protrude beyond the cranial boundaries, here those of the condyles. The best portions for comparison seem to be the nasal bridge, the orbits, the nose itself and the teeth, indicating the lip line. The best points on the mandible are probably those midway between the mandibular angle and the gnathion*, for here the flesh appears to approach most closely to the border of the mandible.

But great as are really the practical difficulties of fitting skulls to portraits, we find new problems arising when we have not the simple skull, but an embalmed head; in this case the skin may be drawn tight down to the cranial bones in places, the skin over the orbits falls in and the eyelids more or less close, while the lips

* Lowest point on lower mandibular border in "mid-sagittal plane."

Pearson and Morant: *The Cromwell Head*



Photograph of Egyptian Criminal and of his Skull.



Drawing of the Skull of Egyptian Criminal fitted to the Drawing of his Face. To show the manner in which the real Skull fits its actual Face.

and nostrils may be grievously distorted. If the reader will examine the photograph of the embalmed head of Jeremy Bentham—the “Auto-Icon”—in Vol. III of this Journal (p. 393), he will be conscious of some of the effects of embalming, and the rough outline of the embalmed head superposed on Bentham’s portrait (*Ibid.* Vol. XVIII. p. 256) will indicate how much the flesh draws in as a result of embalment. The embalmed head will continue to diverge widely from a portrait at the chin and cheeks. Now one of the most noteworthy features of the Head which forms the subject of this study is the very thin coating of the dry leather-like flesh over the skull. It may well be doubted if the flesh on the cranium has at a maximum more than a third of its natural thickness. These points must be clearly borne in mind, and the reader must not expect that the sole task of the present writers has been to place the Head against the portrait and say whether they do or do not correspond. An embalmed head is not like a life-mask; some of its measurements, not all, should correspond with those of a life-size portrait, but the selection must be carefully made. Nothing could differ in appearance more from the portrait of a man than his embalmed head, and we would lay no stress whatever on the opinions expressed by artists of however great a reputation* that this Head is or is not Cromwell’s.

SECTION I.

3. *History of the Wilkinson Head, and some Evidence from the Head itself.*

The Head, practically in the same state as it is now, has certainly been known to exist since 1787, and with a very high degree of probability since 1710. If the Head is a forgery—and it would certainly be a surprisingly clever one—the preparation of it must have taken place considerably before 1787 or even 1773 and with

* We feel unable to be influenced by the statements that Flaxman and Reynolds were firmly convinced the Head was that of Cromwell, or that such historians as Samuel R. Gardiner or Frederick Harrison, however safe as critics, are in this case reliable judges of mere appearances. Nor can we give any more credit to the contrary judgment of Carlyle:

There does not seem the slightest sound basis for any of the pretended *Heads* of Oliver. The one at present in vogue was visited the other day by a friend of mine: it has hair, flesh and beard, a written history bearing evidence that it was procured for 100 l. (I think of bad debt) about 50 years ago:—it now appears to have once had resinous unguents, or embalming substances in it, and to have stood on a spike:—likely enough the head of some decapitated man of distinction: But by the size of the face, by the very width of the jaw bone were there no other proof, it has not any claim to be Oliver’s head. A professional sculptor, about a year ago, gave me the same report of it: “a very much smaller face than Oliver’s, quite another face.” The story told, of a high wind, a sentinel, etc. is identical with what your old neighbour heard, long since, of the Oliver Head in the shape of a Scull. In short the whole affair appears to be fraudulent moonshine, an element not pleasant even to glance into, especially in a case like Oliver’s.

I remain always

Yours with sincere thanks,

T. CARLYLE.

5 Cheyne Row, Chelsea. 21st Feb. 1849.

(From *Notes and Queries*, Series X, Vol. xi. p. 453.)

The impetuosity of judgment in Carlyle is well illustrated by this letter in which he speaks of a head which he had not seen, still less investigated, as “fraudulent moonshine.” He does not even trouble to ask himself what other “man of distinction” was “likely enough” to have had his skull-cap removed *before* embalment, to have been decapitated *after* embalment, and then to have had his head placed on a spiked oak pole!

high probability before 1710. The evidence against its being forged may increase the puzzle of determining how the Head came into existence if it be not Oliver's, but clearly cannot be accepted as positive evidence that it is Oliver's.

Let us note one or two particulars as to the Head. First it has been embalmed, there is no questioning this. How has it been embalmed? There is a little pamphlet printed by Thomas Hardy, London, 1639, entitled: *The Charitable Physitian shewing the manner to Embalme a dead Corps*. By **Philbert Guilbert** Esquire, Doctor Regent in the Faculty of Physicke at **Paris**. Translated into English. By I. W.

Ten years after the publication of this book, 1649, Charles I was executed. His body was embalmed and buried with small ceremony by a few friends in St George's Chapel at Windsor*. Now the little pamphlet to which we have referred is clearly part of a larger work, for it is paged 143 to 173, and of these pages only 143—150 are concerned with "The manner to Embalme a dead Corps." The rest deals with preserves, balmes, oyntments, etc. We are not concerned here with the rules as to the extraction of various organs, or of the blood. There is no "Corps" to test how closely the special rules of this treatise were followed in this case. But we come on p. 144 to the following paragraph:

The head or Cranium shall be sawed in two, as you doe in an Anatomie, and the braines and parts shall be put into the vessell with the bowells, together with the blood that hath been drawne out of the three bellies; that is the head, brest, and belly inferiour, and put them altogether in a barrell, and hoope it round, to be buried or put into the ground, but if they desire to carry them far, or to keep them you may embalme them as followeth.

Two points are to be noted in this direction. The removal of the cap of the skull in order to take out the brain must have been a common feature of 17th century embalmmnt. Further, there is a barrel of *ejecta* to be disposed off†.

We may cite still further evidence.

M. Dionis, "Chief Chirurgion to the late Dauphiness and to present Dutchess of Burgundy," writes in his *Course of Chirurgical Operations, demonstrated in the Royal Garden at Paris*, and translated into English, London, 1710, a very full account of embalmmnt, especially in the case of royal personages. We must first note that in those days the "three venters or cavities" refer to the brain case, the breast and the belly. Dionis then describes how the operator (who *must* be a surgeon, *not* an apothecary!) is to lay the *ejecta* into the *Leaden Barrel A* with layers of powder:

Stratum super stratum, till he has laid into the Barrel all the parts which were contained in Head, Breast and Belly, except the Heart, which he separates, and puts to soak in Spirit of Wine, till he has finished the whole Body, when he embalms that in particular. ...The three Venters or Cavities being thus evacuated, we are to wash them with Spirit of Wine, which is in the Bottle C, before we fill them up, which done, we begin with the Head, filling up the Skull with powder and

* Wood's *Athenae Oxonienses*, Second Edition, 1721, Vol. II. Col. 708, or, better, Sir Thomas Herbert's *Memoirs of the Last Two Years of the Reign of Charles I*, 1815.

† Such barrels are mentioned in the Treasurer's accounts for the embalmmnt of the body of Henry Stewart, Lord Darnley: see *Biometrika*, Vol. XX^B. p. 87.

tow mixed together; and having got in as much as it will contain, we put it again into its place; and before we sow the Hairy Scalp over it, we put betwixt them some of the finer or Balsamic Powder which is in the Vessel *D* [p. 493].

Although these references to the contents of the head are sufficient to indicate that the skull-cap was taken off and sewn on again after filling the brain-box, it will be said there is no statement as to the manner of taking the cap off. But there is no statement either of how to open the "breast" or "belly." The reason is perfectly clear, it lies in the fact that the preceding section, pp. 483—490, is entitled: *Opening of a Dead Corps*, and describes how to open the Head, Breast and the Belly. The instruments for taking off the skull-cap, the frame or fret saw, *C*, the levator *D*, which follows in the track of the saw and the pincer-like instrument *E* are fully described by the plate, and in the text we find their uses.

There can be no doubt that in Europe in the 17th and early 18th centuries the removal of the skull-cap was usual in all major and particularly in royal embalmments. The heart was specially enclosed in a heart-shaped leaden case, sealed by the plumber, and placed by the surgeon on the top of the coffin. The disposal of the leaden vessel containing the embalmed *ejecta* is not discussed beyond the statement that it must be sealed by the plumber.

The Head we are about to study has had its skull-cap removed "as you doe in an Anatomie." It has been raised as an objection to the Wilkinson Head that the head of Charles I did not have its skull-cap removed* and therefore that the removal of the skull-cap was *not* the customary method of removing the brain in those days†.

The case of Charles I's embalmment is, however, exceptional, and is not likely to have followed the usual routine. Halford‡ tells us that when Charles' coffin was opened in 1813, the head was found separated from the body by a very clean cut through the middle of the fourth cervical vertebra. The hair, after being cleaned, was of a "beautiful dark brown colour" (p. 9), and the beard a redder brown than

* Charles I's remains were embalmed and the following account is given in Anthony Wood's *Athenae Oxonienses*, Second Edition, 1721, Vol. II. Col. 703:

Mr Herbert during this time was at the door leading to the scaffold much lamenting, and the Bishop coming from the scaffold with the Royal Corps, which was immediately coffin'd and covered with a velvet pall, he and Mr Herbert went with it to the back-stairs to have it embalm'd; and Mr Herbert, after the body had been deposited, meeting with the Lord Fairfax the General, that person asked him *How the King did!* whereupon Herbert being somewhat astonished at that question, told him the *King was beheaded*, at which he seemed much surpriz'd: see more in the said General Fairfax in the *Fasti* following, among the creations of Doctors of Civil Law, under the year 1649. [Where there is nothing further to our point.] The Royal Corps being embalmed and well coffin'd, and all afterwards wrapt up in lead and covered with a new velvet pall, it was removed to St. James's where was great pressing by all sorts of people to see the King, a doleful spectacle, but few had leave to enter or behold it.

A rather lengthy account follows of the removal to Windsor and the burial of the coffin in the Royal Chapel of St George on February 9th, 1649. See also Sir Thomas Herbert's own account in *Memoirs of the Last Two Years of the Reign of Charles I*, London, 1815.

† *Notes and Queries*, Vol. 150, pp. 210—212, 318—319, 353—354, 407—408, 444—445, and Vol. 151, pp. 12—13, 47—50, 119—120, 154—156, 194.

‡ Sir Henry Halford, M.D.: *An Account of what appeared on opening the Coffin of King Charles I*, London, 1813.

the hair. One eye remained and lasted a few minutes after being exposed to the air, but the likeness to the portraits of Charles was unmistakable. It will be clear from this that the brain had not been extracted by the foramen magnum, or by removal of the skull-cap. On the other hand "the cartilage of the nose was gone" (p. 8), and in a not very lengthy or scientific examination it may have been that the possibility that the brain had been extracted through the nose was overlooked. Halford remarks:

I am aware that some of the softer parts of the human body, and particularly the brain, undergo in the course of time, a decomposition and will melt. A liquid therefore might be found after long interment, where solids only had been buried: but the weight of the head, in this instance, gave no suspicion that the brain had lost its substance; and no moisture appeared in any other part of the coffin as far as we could see [the coffin was only opened at the top] excepting at the back part of the head and neck [ftn. p. 9].

The whole head was covered with an unctuous substance and much liquid which "gave a greenish red tinge to paper and to linen." The spectators thought the liquid "in which the head rested" might be blood. This seems unlikely as the decapitation alone*, to say nothing of the embalmers' practice, would be unlikely to leave enough blood in the head for the latter "to rest" in it as a liquid after some 165 years. It is more probable that the liquid was largely a preservative, spirits of wine and unguents, possibly stained by hair pigment, and prevented from evaporating by the enclosure in a sealed lead coffin. As for the weight of the head, we may note that when the brain was removed the brain-box was usually stuffed with tow and unguents so that weight would hardly be a test of non-removal. Nasal withdrawal of the brain as a possibility does not seem to have occurred to Halford, or he would have mentioned it.

However, whether Charles' brain was removed or not does not really concern us; the circumstances of his embalmmment were exceptional, and cannot be cited as evidence against the process followed by the embalmer of the Wilkinson Head. The fair comparison is one between James I's and Cromwell's embalmmments.

In the case of the Wilkinson Head the foramen magnum is closed as in Charles' case by vertebrae and the spike which pierces the Head has been forced between the two branches of the mandible; it passes anteriorly to the foramen, and protrudes through the right parietal. There are needle holes round the leather-like skin which indicate that the skull-cap had once been stitched on again. The Wilkinson Head is not a head which has been at the time of embalming separate from the body. The skull-cap has been replaced and stitched on; although the sawing off of the cap has been somewhat crudely carried out. The threads which once re-united the skull and its cap have long perished, probably by age and weathering, or may have been broken by the force with which the spike was thrust through the cranium†.

* Numerous persons before the embalmmment dipped their kerchiefs in the blood as mementos of the slaughtered monarch.

† The force was so great that besides breaking out the square piece of the skull-cap, through which the spike passes, it has fractured the cap from the spike hole to its right border: see our Plate XXXVI (d).

Before we continue our remarks we would observe that it is conceivable that the disposal of the barrel of *ejecta* may have led to the various stories of Cromwell's body being sunk in the Thames, buried near London or even carried down into the country. To bury his heart on the field of Naseby would have had sentimental value! But whatever may have been done with the *ejecta*, it is quite clear that none of the stories to which we have referred has any sufficient basis whatever, we have only the wild rumours circulated by the Puritans *after* the desecration of his body. Now is the removal of the skull-cap any argument against the Head being that of Cromwell? Is it an argument against or in favour of an eighteenth-century fraud? Let us look first into the history of Cromwell's embalmment.

Cromwell died at 3 p.m. on September 3, 1658. There appear to have been two functions, an autopsy and an embalmment, but the matter is not clear. One of Cromwell's physicians, George Bate, published in London, 1661*, London, 1663 and in Amsterdam, 1663 a work entitled *Elenchus motuum nuperorum in Anglia*. In the *Pars secunda*, London, 1663 p. 417, Amsterdam p. 273, under the date of September 3, 1658, we have after the statement of Cromwell's death the following paragraph:

Dissecto cadavore, in *Animalibus* partibus vasa cerebri justo pleniora videbantur; in *Vitalibus* pulmones aliquantisper inflammati; sed in *Naturalibus* fons† mali comparuit; *Liene*, licet ad adspectum sano, intus tamen tabo instar amurcae referto. Corpus etsi exenteratum, aromate repletum, ceratisque sextuplicibus involutum, loculo primum plumbeo, dein ligneo fortique includeretur, obstacula tamen universa perrumpente fermento, totas perflavit aedes adeò tetrà mephiti, ut ante solennes exsequias terrae mandari necessarium fuerit‡.

This description of what happened at the autopsy by no means accords with an elaborate embalmment, such as the Wilkinson Head appears to have been subjected to. We think it probable that what occurred on September 3 as described by Bate was really only the autopsy and a preliminary attempt to enclose the corpse. The more so as *Mercurius Politicus* of September 4—the day *after* the autopsy—states:

This afternoon the Physicians and Chirurgians appointed by Order of the Council to embowel and embalm the Body of his late Highness and fill the same with sweet odours performed their duty.

It seems scarcely possible that, Cromwell dying at three o'clock in the afternoon,

* Of the London edition of 1661 (? 1660), we have only seen the *Pars prima*, and are uncertain if the *Pars secunda* ever appeared. Howarth in *The Head of Oliver Cromwell* (p. 8, LXVIII. p. 218; see footnote *, p. 279) says he will give the words in Bate's own Latin. We think he must have meant in his, Howarth's, own Latin!

† There is a sidenote: *Liene prae caeteris partibus dissecti cadaveris malè affecto*.

‡ An English translation of Bate's *Elenchus* by A. Lovell appeared in London, 1685. On p. 236 of the Second Part we read:

His Body being opened; in the *Animal* parts the Vessels of the Brain seemed to be overcharged; in the Vitals the Lungs a little inflamed; but in the *Natural*, the source of the distemper appeared; the *Spleen*, though sound to the Eye, being filled with matter like to the Lees of Oyl. Nor was that Incongruous to the Disease that for a long time he had been subject unto, seeing that for at least thirty years he had at times heavily complained of Hypochondriacal indispositions. Though his Bowels were taken out, and his Body filled with Spices, wrapped in a fourfold [*sic*] Cerecloth, but put first into a Coffin of Lead, and then into a Wooden one, yet it purged and wrought through all, so that there was a necessity of interring it before the Solemnity of his Funerals.

the Lords of the Council could give their order for embalmmnt, that the fitting herbs, oils, etc., could be collected and necessary preparation made for a regular embalmmnt immediately following the autopsy*. Indeed *Mercurius Politicus* states that the order was given and the embalmmnt took place on the day after the death. The autopsy following the death immediately may have been due to a suspicion of poison. Now the autopsy says that the vessels of the brain were overcharged, and to have observed this would almost certainly have required the brain to be exposed, in other words there is small doubt that Cromwell's skull-cap was sawn off. But there is another fact which confirms this surmise; Cromwell's brain-weight appears in the anatomical text-books in association with those of Cuvier, Byron and others, but of a scarcely possible magnitude, namely 6½ pounds. It is true we do not know what pound the measurer was using, but if we take 12 ozs. to the lb. and 28·349 grammes to the oz. Cromwell's brain would have weighed over 2126 grammes. Perhaps the fluid with which it was overcharged was weighed also. Anyhow there appears small doubt that the brain was weighed and this would involve the removal of the skull-cap. It is curious that in the account of the autopsy no mention is made of Cromwell's brain-weight, although the condition of the brain is reported. Up to 1912 the anatomists dealing with the weight of Cromwell's brain had been unable to trace the history of the statement back beyond the year 1702, when the work *Anabaptisticum et Enthusiasticum Pantheon, und geistliches Rüsthaus wider die alten Quäker und neuen Frey-Geister* appeared, and describes briefly the autopsy of Cromwell with the words:

die Eingeweide ziemlich wohl bestellet, die Leber aber angesteckt, und das Gehirn sechs und ein viertheil Pf. schwer befunden worden [S. 12 of 40th section]†.

* Howarth (*loc. cit.* p. 8) also quotes the account of the autopsy given in Bate's *Elenchus*, but he confuses the Lords of the Council order for the embalmmnt of September 4th with the autopsy of September 3rd. He actually states that we have an account of the performance of the order in the work of Dr George Bate, M.D., whom he rightly states was Cromwell's own physician and wrongly assumes to be an unimpeachable authority. Bate he states gives us what Bate himself calls "the account of the embalming of the body." We can find no such words employed in any of the editions of Bate, and the use of the words between quotation marks is the more remarkable as Bate wrote in Latin! As for Bate being an "unimpeachable authority," we may cite what Norman Moore says of him in *D.N.B.*, namely that his work contains nothing "to make its author respected among contemporary politicians or valuable to subsequent historians." We have made some search to find the actual order of the Council, but although the index for the volume of the Privy Council for 1658 is in existence in the Record Office, the volume itself which would give the names of "the Physicians and Chirurgians appointed" is missing. We should, had it been found, have discovered whether Bate was concerned with the embalmmnt. In reply to certain questions made at the Bodleian Library, the keeper of the Western MSS. informs us that in the Rawlinson MS. which contains extracts from the Privy Council proceedings there are no extracts from the year 1658. The handwriting of the MS. is of the 17th century and the last extracts are of date 1670, so that it would appear that the volume was missing at that date. In the *Third Report of the Historical Manuscripts Commission*, p. 198, we find in the possession of the Marquess of Bath at Longleat, Co. Wilts, "Folio. Journal of the Council of Richard Cromwell from 3 Sept. 1658 to 22 March, 1659." The very thing we have been searching for! It may have come to Longleat with Whitelock's papers. Unfortunately Lord Bath has been unable to find this most important historical document. It would have been invaluable not only for our present purposes, but for the light it would throw on the Second Protectorate.

† The section has a separate title, "Der verschmitzte Weltmann und scheinheilige Tyrann Oliver Cromwell, u.s.w."

From this work the statement passes into the *Neues Magazin für Aerzte*, Bd. IV, S. 570, *Anecdoten*, 1782, from whence it was taken by Soemmering, who doubted its trustworthy character, in which he was followed by Rudolph Wagner and other anatomists. Now the *Anabaptisticum et Enthusiasticum Pantheon* is a work which makes no appeal to original sources of information; it is distinctly a *Sammelwerk*, and the senior author of this paper, dealing with Cromwell's Head in 1912, asked Dr Julia Bell to push the statement farther back if possible. After much arduous search she was successful in carrying the statement back at least 35 years to a work published in 1667. The date to be given to the information may even be nine years earlier. In Merian's *Theatrum Europaeum*, Bd. VIII, 1658—60, S. 970, we read:

Unterdessen machten sich die Anatomisten an den Verstorbenen Protector Oliviers entseelten Corper, und befanden die Interiora, oder das Eingeweid, ziemlich wol bestellt, aber die Leber angesteckt und in Haupt ein Gehirn von sechs und einen vierthel Pfunds*.

Bd. VIII of this glorified newspaper was published in Frankfurt in 1667. Again it would appear as if somewhere a fuller account of the autopsy had been published than in Bate's Amsterdam *Elenchus* of 1663. The search for an earlier *English* account of Cromwell's brain-weight, although much literature has been examined, has so far been without success. Still having regard to the statement of the *Elenchus* as to the vessels of the brain being overcharged, and to this early announcement of Cromwell's brain-weight, as well as to another fact, namely that in certain of the death-masks the face appears with a bandage across the forehead, we can hardly doubt that Cromwell's skull-cap was taken off at the autopsy†. Now there is no mention as far as we can find of the brain-weighing in any English publication; if therefore the Wilkinson Head were a fraud, in what manner did the contrivers manage to ascertain that a free skull-cap, with thread holes where the cap had been re-attached to the skull, would be needful to give the semblance of truth to the fraud? They must have obtained—not be it noted a skull—but a fresh head for embalming, and would almost certainly have proceeded with more art than by crude axe-blows to separate the head from its body.

There is another point also about the decapitation; it may be said almost with certainty that the Head was separated from the body *after* embalmment. In other words the forgers—if such existed—must have embalmed a large portion, if not the whole of the fresh corpse, and then proceeded to chop off the head with an axe in the crudest possible manner; the first blows of the axe did not strike the vertebral column and failed of their object; later blows several centimetres lower finally severed the vertebral column leaving four vertebrae attached to the skull. There was before 1799 no published record of the Tyburn treatment of Cromwell's corpse stating that the executioner made a failure in his first blows, so that the assumed

* It is noteworthy that in the *Medico-Chirurgical Review* of 1825 (p. 164) we read in a note on the Dissection of Lord Byron, "the cerebrum and cerebellum weighed six medicinal pounds." In neither case do we know exactly what these pounds stand for. Some readers may concur in believing Byron to have had a diseased or "overcharged" brain.

† The description of Cromwell's brain and its weighing are conclusive arguments against the Oxford and other reputed skulls of Cromwell where the skull-cap has not been removed.

forgers would have no reason whatever for mimicking such a procedure, even if they did take an axe to assist the decapitation. The evidence that we have not a head which has been embalmed after decapitation lies in the fact that the false blow of the axe cuts the embalmed skin and leaves embalmed skin on either side*, which before embalming would have shrunk back on either side. It looks as if a wedge of embalmed flesh had been cut out by the blows. There is another point to be regarded; the Head, though still smelling of the preservatives, shows evidence of long weathering; the hair is very sparse, pressed flat to the scalp. To judge by the effigies (see our Plates VII and VIII), Cromwell had long hair at the time of his death, and to judge by his pictures it was originally brown. Forgers would almost certainly have endeavoured to imitate this. The hair on the Wilkinson Head is macroscopically of a pale yellowish, reddish hue.

Microscopically Dr G. M. Duncan reports that the pigment in the hair is well preserved; there is a good central core of pigment which is slightly irregular in parts, but of a dark brownish black colour. The sheath of the hair contains numerous small pigment granules, and seems to have a diffuse reddish or dark yellow tint. From comparison with other hair, it bore most resemblance to a rather dark brown hair with a reddish tint in it. Too much stress cannot be laid on any deduction as to the exact original colour, but it would seem to have been a fairly dark hair.

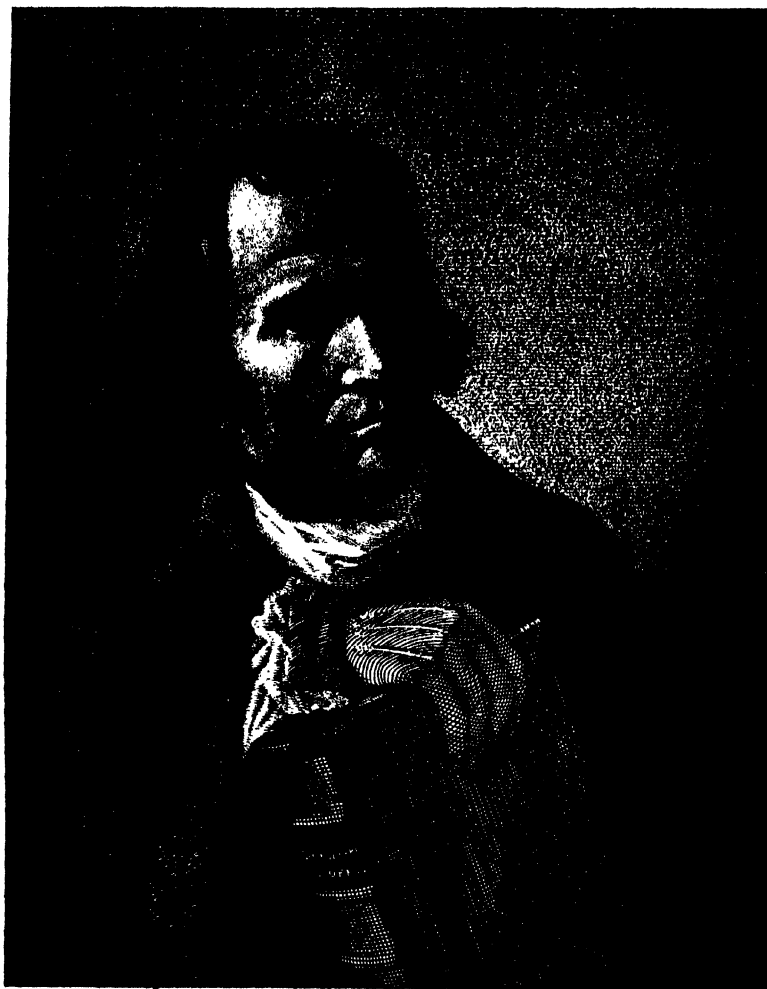
Forgers must indeed have been clever to reproduce such hair, even if they had been acquainted with the manner in which hair may change its character with exposure.

4. Story of the Head as possessed by Samuel Russell.

Apart from the incredible skill and historical knowledge which must have been involved in producing such a counterfeit†, we may ask whether anything in the history of the Head justifies the hypothesis that it was a fraudulent production. Curiously enough there are two men associated with the history of the Head, who were persons of singular mechanical and artistic power, but we may reasonably doubt whether they had the historical knowledge requisite to carry out the fraud. The first of these men was James Cox, who was a jeweller of the City of London living in Shoe Lane. He was undoubtedly a skilful artificer, and produced numerous gold or gilded ornaments bedecked with jewels, genuine or artificial, in a truly baroque (if not jazz) style. He is said to have possessed a "museum," but it is not clear that this contained curiosities like the Wilkinson Head. He appears to have sold these gaudy and gigantic gewgaws to Eastern potentates, but in 1773 in a time of depression, when he was overstocked with them, he obtained an act of parliament to allow him to dispose of over fifty of them by a lottery. The values he put on his gigantic gewgaws ran from £5000 downwards. To boom his lottery he published in 1774 a work entitled: "A descriptive inventory of the pieces of mechanism and jewellery, compriz'd in the schedule annexed to an Act of Parliament for enabling Mr J. C. to dispose of his museum by way of lottery" (4° London).

* Actually the Sainthill Manuscript shows that the executioner took *eight* blows of the axe to cleave through the sixfold cerecloth and neck of Cromwell. See p. 314 below.

† There is a further question of what it would profit the forgers to produce it. Du Puy, most probably the first owner, fixed its value at some sixty guineas; the time and labour required to create the artefact would hardly be compensated by that sum.



Painted and Engraved by J. B. Smith, Engraver of the Magazine of London & Commerce

JOHN CRANCH,

Born at Kingsbridge in Devonshire, 12th of Oct. 1751 Aged 44,

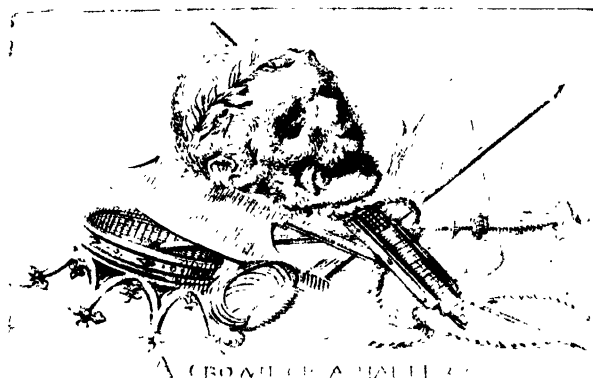
*Editor of the American Society of Arts & Sciences, Painter of an unique Picture of
the Death of Chatterton, & Author of the Economy of Testaments, &c.*

*Engraved from a Picture, which, with the Death of Chatterton, is in the possession of
Sir James Winter, Bart. Prov. F. S. A.*

*Note, Engraving obtained in private the Engraver's Office of Great Britain

Published & Engraved by N. Smith Rembrandt's Head, May's Buildings, St. Martin's Lane

"An impecunious person of poetic nature."



NARRATIVE

RELATING TO

THE REAL EMBLEMED HEAD

OLIVER CROMWELL

NOW EXHIBITING

IN MUSEUM OF THE OLD ROYAL SOCIETY

Title-page of John Cranch's account of the Cromwell Head
with an Engraving of it by him, 1799.

The descriptive inventory shows *nothing* in this "museum" by way of natural curiosities, we find only clockmakers' and jewellers' work of giant size. There is nothing to suggest in this inventory that his "museum" ever included natural and other curiosities such as the "museums" or "cabinets" of gentlemen of the century 1650—1750 contained. Now Cox must have been a very good artificer, but he does not seem to have purchased our embalmed Head until *thirteen* years after the sale of his "museum" of gewgaws. If we are to believe Cox forged the Head, then he also must have forged the deed by which Samuel Russell transferred it to him. This has every appearance of genuineness, and its authenticity was in 1911 attested by the authorities of the British Museum, Manuscript Department*.

There is a further point which tells much against any idea of fraud. What could induce the forgers, if they existed, to destroy the lips, break down the nose, and shake out the teeth, for, excepting two molars, these have been lost *since* death? All these characters would have been of service, if properly treated, to give greater resemblance to Cromwell. They are in our case, however, quite compatible with a head which has been subjected to very hard usage.

There is a second person connected with the Head, who appears to us to have had some of the ability needful for a successful attempt to produce such a head. This person was Mr John Cranch (1751—1821) (see Plate III). Cranch had some historical and antiquarian knowledge†; he was also an artist. He had a very good opinion of himself, and was capable of a good deal of advertisement. After the Head passed from the possession of James Cox to the three men who proposed to make of it a paying exhibition, it was Cranch who drew up the advertisement of the Head for the show made of it at No. 5 Mead Court, Old Bond Street, in 1799. He also prepared the narrative concerning it, with an engraving representing the Head, recognizably like what it is to-day (see Plate IV)‡. But Cranch came so late into the history of the Head that at most he could have done little but colour its history. It was Cranch who first published the tradition that this Head of Cromwell, blown off the end of Westminster Hall, came into the possession of a member of the Russell family from whom it descended to Samuel Russell the elder who sold it to Cox. Cranch's account of the Head is published in the *Narrative relating to the Real Embalmed Head of Oliver Cromwell, now exhibiting in Mead Court, in Old Bond Street, 1799*. Cranch does not state where he heard the tradition; it differs somewhat from that provided by Mr Josiah Henry Wilkinson§, and it is at certain points close to the 1720 tale about Cromwell's reputed skull formerly in the Ashmolean at Oxford. It differs again from that provided by Dr Elliston. It is worth while comparing the three accounts.

* Sir Henry H. Howarth, "The Embalmed Head of Oliver Cromwell," *Royal Archaeological Institute*, 1911, p. 12 (Vol. LXVIII, p. 242). We quote this paper in the first case from the offprint, where the text is otherwise arranged than in the journal, and has received the last corrections of the author.

† See for example his letter to J. T. Smith published in the latter's *Antiquities of the City of Westminster*, p. 257, 1807.

‡ This engraving is *reversed*, so that what is the left side in the actual Head has become the right in the sketch. The reader must bear this in mind when comparing Cranch's sketch with our photographs of the Head.

§ In a paper preserved by Canon Wilkinson, the present owner of the Head, and published by the Royal Archaeological Institute before Howarth's paper (*loc. cit.*).

We will take the last-mentioned first, because it antedates by at least twelve years the appearance of the showmen and their agent Cranch on the scene.

(i) *Dr Elliston's Narrative**. There seems no reason for questioning this account. It runs:

"Some years since a comedian went to Sidney Sussex College, Cambridge, and produced to the Master of the College Dr Elliston the head of Oliver Cromwell, which he conceived the Master might be disposed to purchase as Oliver was of that College. The account he gave of his having this head in his possession was: Oliver after being buried in more than regal stile was taken up at the restoration and his head with those of Bradshaw and Ireton were cut off and fixed on spikes in Palace yard, and the bodies gibbeted at Tyburn. After the heads had been spiked some time, a high wind one night broke the pole of that on which Oliver's was fixed. A soldier going by early in the morning took it up with the broken spike in it, and carried it home. The head being missed and it being imagined to have been taken away by some of Oliver's party, a considerable reward was offered for the discovery of the person who had it in his possession: this frightened the soldier, and made him conceal the circumstance of his having any knowledge of it; at his death it remained with his family. The comedian married the grand daughter of this soldier, and as Dr Elliston humourously observed had Oliver's head to her portion.

Dr Elliston imagining it might create some prejudice against him to have bought the head declined treating with the man for it. Some time after it fell into the hands of Cox, who had the famous mechanical exhibition and jewellery Museum at Spring Gardens, and who is supposed at this period to have it in his possession.

When Cox was known to have it, the great number of persons who intruded themselves upon him for permission to see it occasioned him to change his residence, and keep his removal a secret, and he will now show it only to persons who go with particular recommendations from friends.

Some time after Dr Elliston's interview with the comedian he happened to dine with Dr Powell the master of St John's when a Mr G. Astby, a fellow and president of St John's was present, and who possessing a vein of ridicule Dr Elliston scarcely cared to mention the incident in his presence, but it was however introduced as a topic of conversation. Mr Astby talked of it in his usual strain of treatment of all subjects which he thought he could render absurd, when Dr Powell observed he remembered hearing his father say the head of Oliver Cromwell was in possession of a person whose name he then mentioned, as well also as the street, and part of the town where it was to be seen, but the name and place Dr Elliston does not remember, but he added that Dr Powell's observations confirmed the relation given by the comedian.

Now Dr William Elliston (not Ellison) was Master of Sidney Sussex, 1760 to 1807, and William Samuel Powell Master of St John's from 1765 to 1775. It is

* Howarth (*loc. cit.* p. 17, LXVIII. p. 247) attributes this account to Mark North (*sic*), the author of *The Protectoral House of Cromwell*, writing in 1799. We have sought in vain for any publication of Mark Noble of this date. There is little doubt that Howarth got the statement from a manuscript addition to Mr Wilkinson's copy of Cranch's *Narrative*. This addition is entitled, "Head of Oliver Cromwell," and runs as we have reproduced it. It ends with the following words:

This was lent me, with permission to copy, by the Rev. Thomas Weeks Dalby, vicar of West Farleigh in Kent, and of Chippenham in Wiltshire. He received it from a clergyman in Cambridge.

MARK NOBLE.

1799.

This is important from another aspect, as it indicates that Noble in 1799 thought "the pickled head displayed for a shew" (*The Protectoral House*, Vol. I. p. 291) after all might be worth inquiring into.

The statement bears much evidence of being authentic, and reads like a letter to a newspaper. Whether it was copied by Cranch after 1799 on to his copy of the *Narrative*, which seems unlikely, or by Mr Josiah H. Wilkinson, on to the *Narrative*, which seems more probable, we cannot say. It is reproduced in *Notes and Queries*, Series I, Vol. XII. July—December 1855, p. 75, from the *Additional Manuscript 6806*, fol. 84, British Museum, dated April 18, 1818.

clear therefore that the comedian's interview with Dr Elliston could not have been later than 1775 and possibly earlier. The narrative does not say that the comedian was Samuel Russell, but it does state that the comedian had obtained the Head by marriage. There is no reference to any connection with the Cromwell family. The comedian was clearly impecunious in or before 1775, and it is noteworthy that among the papers delivered to Mr Josiah H. Wilkinson by Cox was a statement that Richard Southgate the numismatist, who became an assistant librarian to the British Museum in 1784, had studied the skull in 1775, and then said: "Gentlemen, you may be assured this is the real head of Oliver Cromwell." At the same time we have a paper from John Kirk, the medallist, who is stated to have died in or about 1778, which reads:

The head shown to me for Oliver Cromwell's, I verily believe to be his real head, as I have carefully examined it with the coin; and think the outline of the face exactly corresponds with it so far as remains. The nostril, which is still to be seen, inclines downwards, as it does in the coin; the cheekbone seems to be as it is engraved; and the colour of the hair is the same as [in] one well copied from an original painting by Cooper in his time.

Bedford Street, Covent Garden, 1775.

JOHN KIRK.

Now these statements are not reproduced to strengthen the evidence for the Head being Cromwell's. We believe, and shall indicate the ground of our belief later, that but little can be learnt from the medals and coins. The statements are provided to indicate that the comedian was exhibiting the Head five years before he comes into touch with Cox, and twelve before Cranch appears on the scene*. The Elliston episode is indicative of the comedian's desire to sell twelve years before he sold to Cox, and the opinions of Southgate and Kirk were probably collected to strengthen his position with possible customers. But why did the impecunious comedian—who can hardly have been other than Samuel Russell of Keppel Street, the vendor of the Head to Cox—seek out the Master of Sidney Sussex and offer to sell the Head to him? Would a strolling actor of drunken habits know of Sidney Sussex being Oliver Cromwell's College? We think there is a closer link between Elliston and Russell than this. Once upon a time there was a farmer at Cadgrave near Orford on the south-east coast of Suffolk. He had two sons, one of whom became Master of Sidney Sussex, and the other a watchmaker in Bloomsbury. Robert Elliston the watchmaker was a man of sorry habits, and his son Robert William Elliston, born in 1774, was educated by his Cambridge uncle and spent much of his holidays in Cambridge. At nineteen he ran away from home and started a stage career at Bath. He became an actor of much repute, if like his father of drunken and ill-regulated habits. At the same time we find another actor

* In *Notes and Queries*, Series I, Vol. xi. Jan.—June 1855, p. 496, we find a writer vaguely citing "a newspaper cutting September 1786," which runs as follows: "The curious head of Cromwell, which Sir Joshua Reynolds has had the good fortune to procure, is to be shown to his Majesty [George III]. How much would Charles the First have valued the man who would have brought him Cromwell's head." If this was the Wilkinson Head, it was in Sir Joshua's possession the year before Russell sold it to Cox. The possession may have been only a temporary deposit by Russell. It is noteworthy that the paragraph speaks of the head, not the skull, of Cromwell. Sir Joshua was interested in Cromwelliana (his wife was connected with the Cromwells), and the Marquis of Crewe's miniature of Cromwell was once owned by the great painter. This newspaper paragraph is the only printed evidence we have come across that Reynolds accepted the genuineness of the Wilkinson Head.

of fame, Samuel Thomas Russell, born in or about 1769. The latter is in touch with Elliston, and indeed is manager to him, when he is lessee of the Surrey Theatre. Now we are not concerned here with these young men, who were children at the time when the events we are describing occurred. But the father of Samuel Thomas Russell was a certain Samuel Russell, an impecunious "country comedian," i.e. a man who went about playing in provincial towns. We are told that Samuel Thomas first played at Margate, where his father, who was in the company, had done so well in the part of Jerry Sneak in Foote's *Mayor of Garratt* that the Prince of Wales recommended him to King of Drury Lane, who it is said engaged his son by a trick instead. It is extremely likely that their common habits brought Samuel Russell and Robert Elliston together, and what more likely than that Elliston would advise Russell to apply to his brother at Sidney Sussex in the first place and, knowing of Cox and possibly employed by him, to turn to Cox in the second place when the comedian wished to sell the Head? However this may be, it is clear that the Elliston narrative carries the Wilkinson Head back to 1775* at least, and out of the possibility of fraud on the part of either of those ingenious men, Cox or Cranch.

We will now turn to Cranch's account in the work already cited.

(ii) *Mr John Cranch's Narrative.*

The tradition respecting the head of Oliver Cromwell is, that being on a stormy night, in the latter end of the reign of Charles, or James the Second, blown off from the top of Westminster Hall, it was taken up by one of those many persons whom the flagitious conduct of these monarchs had by that time converted to a less unfavourable opinion of Cromwell. By this person it was soon after presented to one of the Russell family; and in the possession of one branch of that family, it remained many years, until the last possessor of it, of that name (Mr Samuel Russell) sold it to James Cox, esquire [*sic*], formerly proprietor of the celebrated museum which bore his name. The head was first seen by Mr Cox about the year 1780, when exhibited, as well as he recollects not far from Claremarket†. On that occasion Mr Cox first became acquainted with Russell, who then possessed the head as his property; and who being in indigent circumstances, requested Mr Cox, to assist him with money for his support; which partly from humanity, and partly (he confesses) with a view to the acquisition, sometime or other, of so great a curiosity, Mr Cox from time to time did, till the 30 April 1787; when, in consideration of £100, which had been advanced, and a considerable sum then paid down, Russell, by a legal deed‡ transferred the head to Mr Cox. That gentleman having since retired from business and from town, has lately sold it to the present proprietors.

* We may say 1773, if we credit the Additional MS. 6306, that Sir Joseph Banks had refused to see the Head 40 years earlier, i.e. earlier than 1813. Cranch came to London probably about 1770.

† The watchmaker Elliston had moved from Orange Street, Red Lion Square, to Charles Street, Long Acre, not very far from Clare Market. Russell is also said to have exhibited the Head in Butcher's Row, a block of houses once running from the east end of St Clement Danes in front of the present Courts of Justice, and dividing the Strand into two narrow passages. Butcher's Row is close to Clare Market.

‡ The deed of assignment, which appears in due order and genuine, is in the possession of Canon Wilkinson, and is given below:

Deed of Assignment of Oliver Cromwell's Head to James Cox from Samuel Russell.

KNOW ALL MEN by these Presents That I Samuel Russell of Keppel Street in the Parish of Saint Saviours in the County of Surry—as well for and in Consideration of the sum of One Hundred & one Pounds heretofore advanced to me by James Cox of Shoe Lane in the City of London Jeweller as for and in Consideration of the further Sum of Seventeen Pounds making together the Sum of One Hundred and eighteen Pounds to me in Hand paid by the said James Cox at and before the Sealing and delivery

Mr Cox, whose laudable attention to curious subjects is well known, says that having acquired a legal title to the head, though at the expense of so much money and of seven years' patience in waiting for Russell to make up his mind to part with it (which he did at last with infinite reluctance) he, from that time, purposely concealed it, even from his own family, in order to prevent the trouble of those incessant applications which he conceived would be made for a sight of it, in case it should be publicly known that he possessed so extraordinary and interesting a curiosity. And so very tenacious was he of this resolution that he even denied the request of the late Mr Alderman Wilkes with whom he was well acquainted, and who being at the Globe Tavern in Fleet Street, with a party of friends, sent a deputation to Mr Cox, expressly for that purpose.

It may be noted that one of the ears is wanting; and that there have been some other slight mutilations. This is accounted for by another of the family traditions which is that when the Protector's relations and admirers were occasionally admitted to see the head, they took those opportunities to pilfer such small parts as could best be come at, or were least likely to be missed. The ear is said to have been taken away by one of the Russells of Fordham.

The Russell Family of Cambridgeshire, appears to have been allied to that of Cromwell by no less than three distinct marriages, within the space of twenty years, viz.:

1st. The marriage of the Protector Oliver's youngest daughter Frances, the widow of Rob. Rich with Colonel Sir John Russell, Bart. of Chippenham, distinguished for his gallantry at the battle of Marston Moor.

2nd. The marriage of the Protector Oliver's son, the great and good Henry Cromwell, (deputy of Ireland) with Elizabeth the sister of the said Sir John Russell, and

3rd. The marriage of Elizabeth, the daughter of the said Henry Cromwell, with William Russell, esquire, of Fordham, by whom she had seven sons and six daughters, and after having nearly ruined her fortunes by extravagant expences, died in London, 1711.

From this last mentioned alliance, Samuel Russell is said to have been descended. At the time of his treaty with Mr Cox he resided in Keppel Street, Southwark; was afterward servant to a broker; and since thought to be living near Spital Fields; but in an obscurity which has hitherto eluded every endeavour to find him out. The greatest probability is that he is dead*.

of these Presents. (The Receipt of which said several Sums of Money I the said Samuel Russell do hereby acknowledge and thereof and therefrom and of and from the same respectively and every part thereof do acquit release and discharge the said James Cox his Executors and Administrators for ever by these Presents)—Have bargained and Sold released granted and confirmed and by these Presents Do Bargain and Sell release grant and confirm unto the said James Cox All that Scull or Head supposed to be the Scull or Head of Oliver Cromwell To have and to hold the said Scull or Head unto and to the only Use and Behoof of the said James Cox his Executors Administrators and Assigns absolutely for ever free from and without any interruption or Disturbance whatsoever of from or by me the said Samuel Russell or any other Person or Persons whomsoever And I the said Samuel Russell for myself my Executors and Administrators do by these Presents Covenant and promise that I the said Samuel Russell shall and will Warrant and for ever Defend the said Scull or Head unto the said James Cox his Executors Administrators and Assigns against me the said Samuel Russell my Executors and Administrators and against all and every other Person and Persons whomsoever, And I the said Samuel Russell have put the said James Cox in full Possession of the said Scull or Head by delivering him the same at the time of the Sealing and delivery hereof In Witness whereof I the said Samuel Russell hath hereunto set my Hand and Seal the Thirtieth day of April in the Year of our Lord one thousand seven hundred and eighty seven.

Sealed and Delivered and Livery and Seisin of the
said Scull or Head given to the said James Cox by
the said Samuel Russell delivering Seizin thereof } SAML. RUSSELL.
to the said James Cox in the Presence of

T MAGNIAC.

* Mr Josiah H. Wilkinson appears at a later date to have met Cranch in Bath, whom he describes as an old gentleman, who superintended the exhibition of the Head in Bond Street, and from him heard several particulars as to the Head, one at least is of importance. It runs:

He knew Samuel Russell and had seen him perform on Covent Garden Theatre, and he believed that, he was as often drunk as sober.—He appeared once before the house in Bond Street, where the

The history of the head from the period when it was first deposited with the Russell family, to the time of its coming into the possession of Samuel Russell is not, for the present, attempted to be more particularly given. It has been hinted that, by some concealment, or other indirect practise, the title of some of Samuel Russell's predecessors to the property of this head, was not quite regular. The remoteness of such a transaction would render it wholly immaterial whether this report were true or false, on any other account than this: that it is certainly no mean evidence of the authenticity and value of a thing, that it is worth the risque of stealing or fraudulently concealing it.

Now this *Narrative* prepared by Cranch with the view of advertising the Head at the time of its exhibition in Bond Street as a profitable show* must be read with caution. Cranch wished to make the most of the history of the Head as evidence for its genuineness. There is no reference here to the soldier who is said to have picked up the Head, nor to the marriage of the granddaughter to the comedian. It is, however, hinted that the Head had been acquired by illegal means, and had come into the possession of the Russell family, where "it remained for many years" until it descended to the impecunious Samuel.

Cranch goes so far as reciting the three Cromwell-Russell marriages of the Protector Oliver's time. Clearly the second link, that in which Henry Cromwell married Elizabeth Russell, is not to the point as the descendants would be Cromwell in name.

The first link, that of the Protector's daughter Frances Cromwell, with Sir John Russell, 3rd Bart. of Chippenham, offered some possibility, but we have drawn up a full pedigree of these Russells on the basis of information given by Mark Noble† and James Waylen‡, and there appears to have been no opportunity for a descendant of Lady Frances Russell, née Cromwell, to have acquired the Head and handed it down to Samuel. The name and the baronetcy came to an end with Sir George Russell, the 10th Bart., who died childless in 1804. He was the grandson of Charles Russell, who married the heiress of Chequers and carried the Russell Cromwelliana to that famous house, where they are still to be seen. With the death of Sir George the Russell estates descended to Greenhills and Franklands, connected with Frances Cromwell through the female line. Had the Head come into the hands of the Chippenham Russells there is little doubt that it would never have been made a show of, and would to-day be at Chequers.

The third Russell marriage was that of Elizabeth Cromwell, daughter of Henry Cromwell, to William Russell, a grandson of Sir William Russell, 1st Bart. This

Head was exhibited, and drew a mob round him by declaring that the embalm'd head of his great ancestor was his property, and that he had been "juggled out of it" by Cox, who had taken advantage of his poverty, had arrested him, and had clandestinely got the head from him, when he, Russell, was in a state of inebriation. The mob threatened to pull the house down. *Memorandum* of J. H. W. in Canon Horace Wilkinson's possession.

At some time therefore Cranch had found Russell.

* The show seems to have been a failure, for the explanation given for the papers concerning the Head, even the Russell-Cox deed, remaining in the hands of Cranch and his descendants is that the showmen who purchased the skull from Cox had never paid Cranch for his labours in regard to the *Narrative* and the Mead Court exhibition!

† *The Protectoral House of Cromwell*, 8rd Edition, 1787, Vol. II. pp. 408—417, 449—458.

‡ *The House of Cromwell*, New Edition, 1897, pp. 48—52, 127—145.

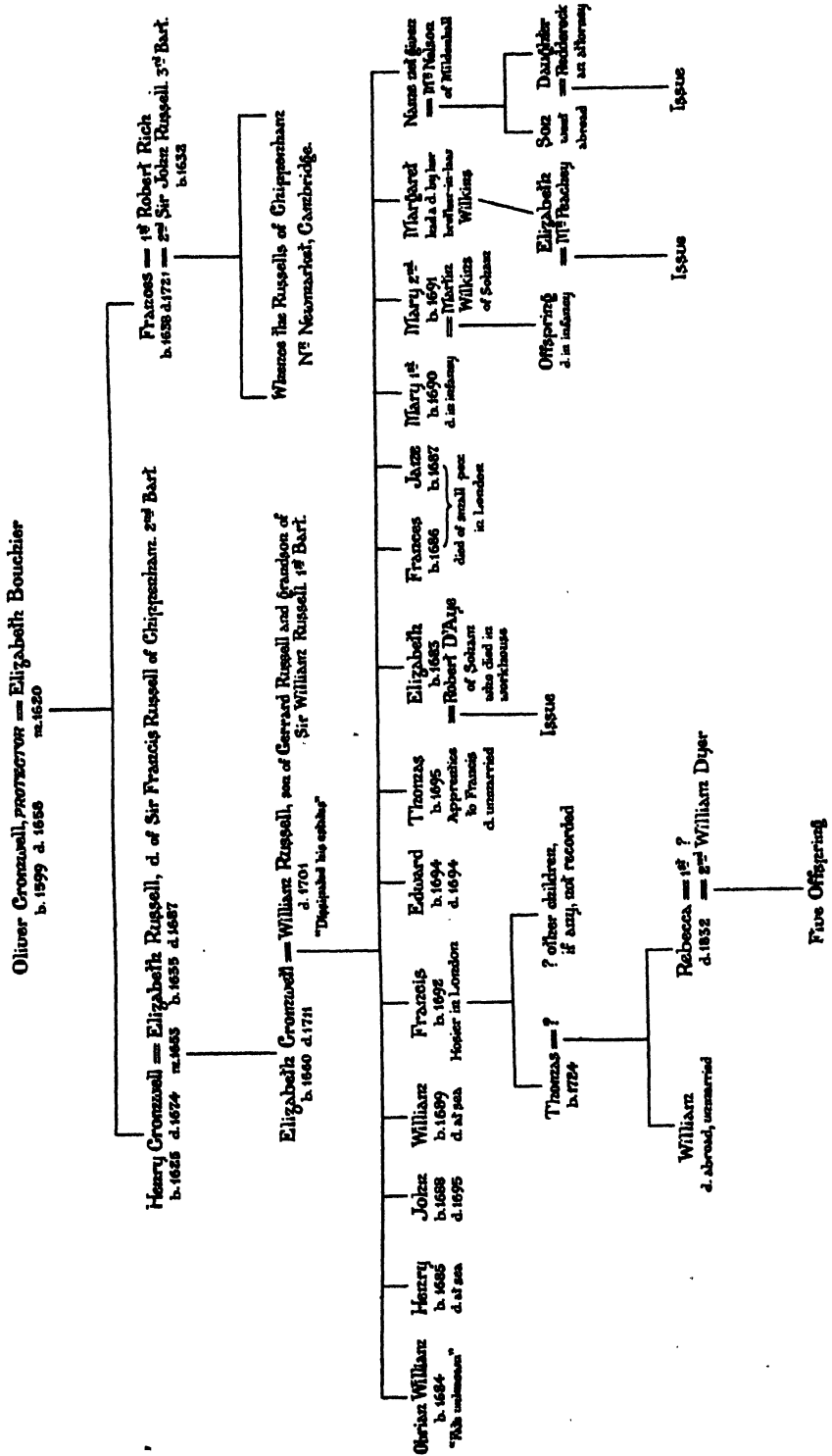
branch of the family was very unfortunate; possibly the fact that Henry Cromwell remained for many years in exile and impecunious added to the misfortunes of his daughter. Elizabeth Cromwell's husband, William Russell, lived extravagantly and, dying in 1701, left his widow, who appears to have been also a spendthrift, his debts and eleven out of fourteen * children. Some time after she with her children fled from the debts and the Fordham estate secretly by night to London. The family pedigree is given on p. 286. Considering her children, the eldest son Obrian† William, b. 1684, would be 17 years old when his father died, and grown up when his mother fled to London. Noble says he was brought up to no profession. Of him, all we know is recorded in the words "fate unknown." It is improbable that he led a settled married life, or something would have been known of him. Still we cannot say that he was not the grandfather of Samuel Russell. The second son Henry died at sea, as did the fourth son William, presumably both young and without offspring. The third son John died between six and seven years of age; the sixth son Edward died as an infant. We are left with two sons: Francis, b. 1691—2, who became a hosier in London, and Thomas, b. 1695, who was apprenticed to Francis, and died unmarried.

Now of Francis only one son, Thomas, b. 1724, is recorded, but *we are not told that he had no other offspring*. This Thomas had two children: William, who died abroad unmarried, and Rebecca (d. 1832), who married and had offspring by her second husband William Dyer, with which offspring—as not bearing the name of Russell—we are not concerned. Thus if Samuel Russell was a descendant of the Russells of Fordham and at the same time of Cromwellian descent, he must either have sprung from the Obrian William Russell of "unknown fate," or from some other child of Francis the hosier than Thomas. There is one other possibility. Of the seven daughters of Elizabeth (Cromwell) Russell, several of whom married and of whom some died in extreme poverty, one may have had an illegitimate son who took his mother's name. We may briefly consider these daughters.

(i) Elizabeth, b. 1683, married Robert D'Aye of Soham, who died in the work-house; they had offspring. (ii) Frances, b. 1686, and (iii) Jane, b. 1687—8, died of small-pox when their mother came to London. (iv) Mary 1st, b. 1689—90, died as an infant. (v) Mary 2nd, b. 1690—1, married Martin Wilkins of Soham, and had two children who died as infants. (vi) Margaret had intercourse with her brother-in-law Wilkins, by whom she had an illegitimate daughter Elizabeth.

* Noble says 13 children but appears to record the names of 7 sons and 7 daughters. Cranch says 6 daughters.

† We have spelt this name as it occurs in the records, and at first sight the name in an English family and its spelling seem very unusual. On investigation, however, the origin of the name is clear. Sarah, daughter of Sir Francis Russell of Chippenham and widow of General Reynolds, was the second wife of Henry O'Brien, 7th Earl of Thomond, being married in 1660. This Sarah would therefore be sister of Elizabeth Russell and aunt of Elizabeth Cromwell (daughter of Henry Cromwell), great-aunt of the latter's eldest son, to whom doubtless she was godmother, giving her married name to her godchild. It is possible that some trace of Obrian William Russell might be found in the records of the Earls of Thomond. Sarah had four children of whom three married into distinguished English families.

PEDIGREE OF FORDHAM RUSSELLS.

This daughter was married to a Mr Peachey by whom she had offspring. Lastly, (vii) a daughter unnamed who married a Mr Nelson of Mildenhall and also had offspring. There seems no evidence here of any of these sisters having an illegitimate son to bear the name of Russell. It is, of course, possible there may have been an unrecorded case, but if there were, it is unlikely that the Head would have come into his possession*.

We have thus been wholly unable to trace any connection between Samuel Russell and the Russells of Fordham, grandchildren of Oliver Cromwell. If such connection existed the evidence for it depends solely on Cranch's statements, particularly that contained in the footnote on our p. 283. Cranch was especially desirous of linking up the Head with the Russells of Fordham in order to emphasise its genuineness at the Bond Street show. Hughes and Cranch tried to get more information from Cox, but the latter only replied in generalities. All Cox vouchsafed was that the Head "was blow'd down & picked up by a Republican & that in his family it was preserved & kept from the knowledge of the Public & has remained so nearly to the present time."

Cranch in a letter of January 15, 1799, to T. M. Hughes writes:

At all events, we must, if possible, find out Russell, and get what we can from him—primarily & principally, his relationship to Mark Noble's Russells; and then, all that he chuses to disclose of the history of the *descent of the subject*. A plausible course of *derivation* must be made out; and though that derivation in some instances may not have been perfectly consonant to the laws of *meum* and *tuum*, yet it may equally avail for our purpose—he equally curious and interesting, and go equally to the point of the *authenticity of the subject itself*, to shew that it got into the hands of some of the former owners by some indirect adventure. [Letter in the possession of Canon Wilkinson.]

It is clear that Cranch was only seeking for a "plausible," not necessarily an accurate "derivation."

Further we must admit that Cranch's story is not consistent with Elliston's report of what the comedian told him, namely, that he had got the Head through marriage with the granddaughter of the soldier who picked it up. According to Cranch it was picked up by a person, who owing to the conduct of Charles II and James II thought better of Cromwell than was customary, and this person presented it to a member of the Russell family through whom it descended to the impecunious Samuel. With the pedigree before us, we ask which member of the Russell family; and if, as we feel pretty confident, the Head was in possession of Du Puy in 1710, then we ask how did Samuel Russell get possession of it? By 1711 Elizabeth Cromwell was dead, and her husband, William Russell, had died ten years before. Her scattered and impecunious offspring were hardly in a position to buy a head which Du Puy valued at 60 guineas! We think Cranch's tale invented for the purposes of the exhibition, and that he possibly knew the real story is indicated by his remarks that whether it was originally obtained illegally was immaterial to the then ownership. Luckily while Cranch was clearly not a person whose words could be implicitly

* Our information is based on Noble, *op. cit.* 3rd Edition, Vol. II. pp. 440—453. Waylen (*loc. cit.*) has if anything less information than Noble.

trusted* there is too much good evidence of the existence of the Head when Cranch was a youth†, and to all accounts before he came in touch with it, to believe he fabricated the Head and all the documents concerned with it as well as his *Narrative* of its history.

We now turn to the third account of how the Head came into the possession of Samuel Russell. This is due to Mr Josiah Henry Wilkinson, who purchased the Head from the husband of the daughter of the last of the three proprietors, who had obtained the Head for £230 from Cox for the purposes of exhibition. Mr Wilkinson obtained the Head about 1814‡ and wrote his *Narrative* in 1827. He was able to come directly or by correspondence into touch with Cox, Hughes, Cranch and Noble and to hear more of the traditions concerning the source whence Russell obtained the Head than we can discover to-day.

(iii) *Mr Josiah Henry Wilkinson's Account of 1827§.*

The tradition respecting the Head of Oliver Cromwell is that being on a stormy night, in the latter end of the reign of James II, blown off from the top of Westminster Hall, it was taken up by a sentinel who was on his parade on the parapet||, and at whose feet it fell, and who perceiving what it was, placed it under his cloak till he went home; there he hid it in the spacious chimney of his room without acquainting his wife or daughter of the circumstance. Having concealed it for two or three days before he saw the placards which ordered any one possessing it to take it to a certain office, he was afraid to divulge the secret, till on his deathbed he discovered it to his wife and daughter. The latter being married, her husband looked out for the best market, and sold it to one of the Cambridgeshire Russells, through which family it descended privately, in the box in which it is now deposited, till it came into the possession of the late Samuel Russell, who being an indifferent comic actor of dissolute habits, and very needy, exhibited it at a place near Clare Market. [Here follows an account of Russell's sale of the Head to Cox.]

The source of this third account¶ of how Samuel Russell obtained the Head we do not know. If it came from Russell himself, directly or through Cox or Cranch, it differs essentially from Russell's tale to Elliston, wherein the skull came to him by marriage. It differs from Cranch's narrative wherein the Head was picked up

* He is described as an impecunious person of "poetic" nature!

† He was born 1751, and would accordingly be about 22 when Samuel Russell first tried to part with the Head.

‡ It would appear that in 1818 one of the purchasers from Cox was showing it occasionally in Fenchurch Street in the City, and in April of that year had offered it for sale to William Bullock of the Museum in Piccadilly. Liverpool Papers, British Museum. Further it was offered to Mr R. G. Russell, M.P. for Thirsk (1818—1831), somewhere about the same time, but was declined by him. A Mr Wilkinson of Fenchurch Street purchased in 1820 two copies of Caulfield's *High Court of Justice on Charles I.*

§ Manuscript in the possession of Canon Wilkinson.

|| The words "on the parapet" occur in Mr Josiah Henry Wilkinson's original manuscript, to which we have had access, but are omitted in the version of that account issued with Howarth's paper (*op. cit.* p. 4, Vol. LXVIII. p. 234).

¶ There is still another variant of the tale, which is provided in *Notes and Gleanings*, Vol. II. p. 56, Exeter, 1889. According to this version, a Mr F. Gale, lecturing before the Literary Society of Exeter, stated that a man, living in the suburbs of London, had once shown him the head of Oliver Cromwell, which had been exposed on Westminster Hall, and a sentry stationed to guard it. The Head was blown down in a dreadful storm, and the sentry trying to recover it, a looker-on offered him a shilling to allow him to take it away; which was accepted!

by one of the many persons beginning to judge more highly of Cromwell, and presented by him to one of the Russell family. Both say it descended to Samuel Russell from that family. Cranch connects the Head with the Fordham Russells; Mr Wilkinson states merely the Cambridgeshire Russells, which would include those of Chippenham. None of the three accounts is consistent with the Head being in 1710 in the possession of Du Puy. Even in 1685, it may be doubted whether the Fordham Russells would have been a "better market" than the Chippenham Russells, and it is difficult to believe that from the latter it could pass, apart from the rest of their Cromwelliana, to Samuel Russell. The Fordham Russells were scarcely in a position after 1710 to purchase the Head had Du Puy died and the *curiosa* of his Museum been dispersed. Finally we have no valid evidence that Samuel Russell the elder was a member of the Fordham Russell family. If we suppose that the Du Puy Head is *not* the Wilkinson Head but that the sentinel's son-in-law sold the latter Head to Cromwell's granddaughter Elizabeth or her husband, then we must suppose that when Elizabeth soon after 1701 fled by night in the "family coach" from Fordham and from her creditors to London, she carried the Head with her, and this leaves us with the difficult task of linking up Samuel Russell with one or other of her children. Such an account for his possession of the Head is diametrically opposed to what, according to Dr Elliston, the comedian told him, namely, that he had obtained the Head by marrying the sentinel's granddaughter. Among such discordant accounts, we can only consider where the greater probability lies, and this seems to us to be that the sentinel's son-in-law found his "best market" in Du Puy, a collector of curiosities; that later it came by some unexplained transactions into the hands of the comedian Samuel Russell, who gave various accounts of how he became possessed of it. Samuel may or may not have been a descendant of Cromwell; up to the present there is no verification of this statement, beyond the narrative of Cranch, which Mr Josiah H. Wilkinson probably followed.

It seems to result fairly certainly that the Head was not faked by Cox or Cranch, who were men of ability; still less likely by Russell. To say that it was not faked does not prove it the genuine head of Cromwell. It might be an embalmed head of another—Ireton, for example, although improbably, as he was far less carefully embalmed than Cromwell*.

5. *The Du Puy Head of Cromwell.*

It is fitting that we now turn to Du Puy's relation to the Head of Oliver Cromwell. Du Puy must have been a person of considerable wealth and some importance in the city of London or its environs in the early years of the 18th century. Notwithstanding our endeavours we have so far been unable to come across notices of him, although his collection of *curiosa* must in 1710 have had widespread fame, if it attracted a visitor to London at that date to go and see it as one of the sights of that city. We do not despair of further information as to Du Puy being forthcoming, when a more thorough investigation of the records is made than has been possible

* See Sainthill's account on our p. 813.

in our case*. All we have discovered is that he died in the parish of St Anne's, Westminster, in 1738, a bachelor and intestate. He is not in the alphabetical *Directory*...[for] the *Cities of London and Westminster and Borough of Southwark*, 1736. We reproduce what von Uffenbach tells us. Claudius Du Puy was by birth a French Swiss, and carried on the business of a calico-printer. Von Uffenbach reports him "ein besonders curioser Mann," but omits to tell us where he lived. He must, at any rate in 1710, have been a wealthy man for he had "vier Zimmer voller Curiositäten und meist Naturalien." It is clear that it was not a public show, but an enlarged private cabinet of *curiosa*, so common among the wealthy in the 17th and 18th centuries, of which the "museum" of the Royal Society in its early days was a good example. The contents of Du Puy's museum extended from humming birds to marine animals, from idols to waxworks, from musical instruments to strange footwear. Von Uffenbach gives a fairly lengthy account of all these wonders.

In the second room among coins and herbals we find a very important object:

Auch zeigte uns Herr du Puy als eines seiner curieusesten Sachen den Kopf von Cromwel, wie er mit der hölzernen Stange, so abgebrochen ist, herunter gefallen, wenn man anderst Herrn du Puy mit einem andern Todten-Kopfe nicht betrogen hat. Dann mir kommt es sehr verdächtig vor, dass ein Holz darinnen steckt, und der Kopf damit herunter gefallen seyn soll; indem man die Köpfe der Maleficanen nicht auf hölzerne, sondern auf eiserne Zapfen oder Stacheln zu stecken pfleget. Jedoch Herr du Puy versicherte, dass er sechzig Guineen davor haben können. Mich wunderte, dass denen Engelländern dieses monströse Haupt noch so lieb und Werth seyn könnte; jedoch es giebt dergleichen Köpfe noch viele in Engelland als dieser war, und es sollten noch viele seyn, die damit eben so oben hinaus wollten. Bei diesem Kopfe des Cromwels war auch ein Kopf von einer Mumie, die mir lieber hätte seyn sollen.

This visit of Zacharias Conrad von Uffenbach to Du Puy's collection took place on July 1st, 1710†. Thus if the head of Cromwell reported to be in his keeping was the Head sold by Russell to Cox and seen by Southgate in 1775, we can carry back its history a further 65 years, if we cannot account for the transfer of it from Du Puy's museum to the care of the bibulous comedian Samuel Russell. No one can read von Uffenbach's description of the contents of Du Puy's four rooms of oddities without perceiving that he was a wealthy collector of *curiosa*, and he himself stated that this was one of the most curious things among his *curiosa*. He obviously valued it highly, for he assured von Uffenbach that he could sell it for sixty guineas,

* There was a musician, Thomas Sanders Dupuis (1733—1796) whom the *D.N.B.* states was the son of John Dupuis, a member of a Huguenot family, who is said to have held some appointment at Court. Pepys in his *Diary*, August 5, 1667, writes: "After done with the Duke of York, and coming out through his dressing room, I there spied Signor Francisco, tuning his gittar and Monsieur de Puy with him, who did make him play to me which he did most admirably—so well that I was mightily troubled that all that pains should have been taken upon so bad an instrument." This appears to be the only reference to de Puy in the *Diary*. Wheatley in his index has: "Puy (Mons. de), servant to the Duke of York." He is probably the same man whom Narcissus Luttrell (*A Brief Historical Relation of State Affairs, September 1678 to April 1714*, 1859) states to have been the Duke of York's barber and who appears to have been connected with the plot against Sir Edmundsbury Godfrey (see Vol. I. pp. 85, 90—91). We are unable to state whether there existed any connection between this Monsieur de Puy, John Dupuis, who held some appointment at Court and von Uffenbach's Du Puy. We have found many Du Puy's, De Puy's, and Dupuis in the lists of Huguenots in England, but no Claudius Du Puy.

† *Merkwürdige Reisen durch Niedersachsen, Holland und Engelland* [1710]. Zweiter Theil, Frankfurt u. Leipzig, S. 522—529, especially S. 525.

and this is some evidence that he felt certain of its genuineness. We must note also that von Uffenbach terms it a *Kopf*, a *Todten-Köpfe*, and a *monströses Haupt*. It is almost impossible to believe von Uffenbach could use such expressions of a mere *Schädel*; a skull is not a monstrosity, but the Wilkinson Head of Cromwell to a man with small anthropological or historical instinct, who probably looked upon Cromwell as a mere rebel, and who noted its battered condition, might well appear a *monströses Haupt*. He would not be the first person to think it such!

Von Uffenbach expresses some doubt with regard to the genuineness of the Head, but upon a ground which is very satisfactory in the case of the present inquiry. He says a wooden staff is stuck into the Head but that it was customary to place the heads of malefactors on an iron spike or stake. This is of much interest because it is on the broken end of a rounded *pole*; it is true that an iron prong is attached by crude nails (see our Plates XXVII, XXXII and XXXIV) to this pole, but the pole is more conspicuous than the prong which is mostly concealed inside the brain-box, and the spike which pierces the skull-cap has largely rusted away. Now, where did von Uffenbach get his conception of the heads of malefactors being placed on iron stakes?

On June 21st, some ten days previously, our traveller and his friends had visited London Bridge*:

Nach dem gingen wir Londons-Bridge, oder die Brücke über die Temse zu sehen. Wir waren schon wohl zwanzig Schritt auf selbiger fortgegangen, ohne dass wir wussten, dass wir auf der Brücke waren, bis ich unsern Dolmetscher fragte, ob wir dann bald an der Brücke wären? Der es uns dann sagte, dass wir wirklich fast auf der Mitte der Temse wären. Man erkennt die Brücke daher nicht, weil sie auf beyden Seiten grosse und ansehnliche Häuser hat, da unten lauter Boutiquen sind....Auf dieser Brücke haben sonst die Köpfe von den Parricides, wie Patin dans ses Voyages, p. 167 sagt, das ist Cromwells und seiner Anhänger gesteckt, allein es ist davon nichts mehr zu sehen, als die Eisen, darauf sie gesteckt gewesen.

Cromwell's head was never on London Bridge, although those of some of the regicides may have been. It does not follow that because there were iron prongs for the heads of malefactors on London Bridge, these would be used wherever their heads were displayed. Indeed it seems highly improbable. Grosley, visiting London about 1770, saw on Temple Bar the heads of three of the Pretender's chief leaders, who had been captured in the Rebellion of 1745, and reports in his work *Londres* that they were on poles fifteen to twenty feet high set equally apart. It is hardly conceivable that these poles ("perches") could have been made of iron. Grosley's account runs as follows:

Au centre de Londres même, la superstition a un monument qu'elle ne perd point de vue & qui intéresse tous les Ordres de l'Etat. Ce sont trois perches, à la sommité desquelles furent fichées les têtes de trois des principaux seigneurs qui en 1746 ayant suivi le parti du prétendant, furent pris les armes à la main & exécutés comme criminels de haute trahison. Ces trois perches de quinze à vingt pieds de haut, sont plantées, à distances égales, sur le *Temple-bar*: porte dans le goût de l'ancienne poste de la conférence à Paris, & qui pare le vieux Londres du Strand. Les Anglois m'ont paru en général persuadés que la chute de chacune de ces trois têtes, doit être le signe & peut-être le signal de quelque révolution dans l'Etat. Ce préjugé populaire s'est accrédité par la chute de la tête du milieu qui, lors de la mort du dernier roi, s'est détachée de la perche qui la soutenoit. [Pierre Jean Grosley], *Londres*, Lausanne, 1771, T. II. pp. 9—10.

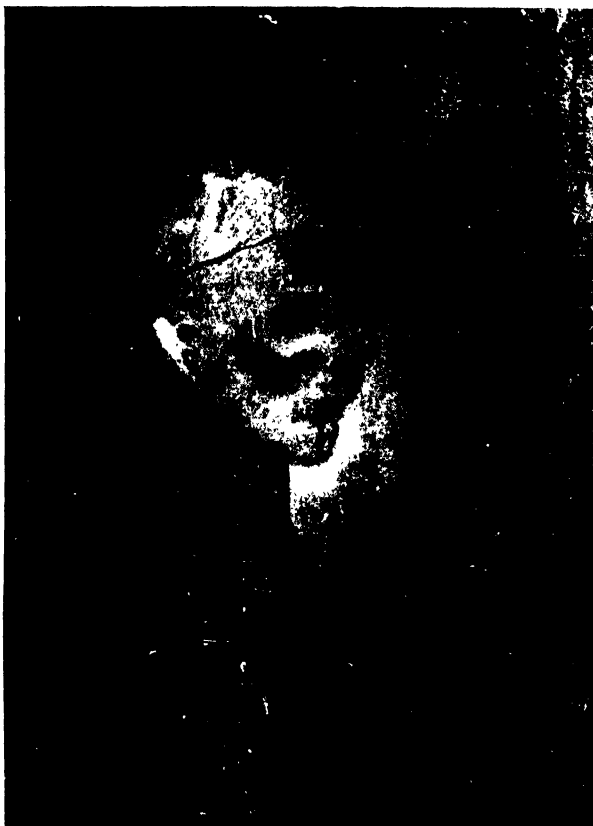
This extract is of much interest. Some of the heads were still up in 1770, thus such heads could remain in position for twenty-four years, but the head in the middle had fallen in 1760, that is after fourteen years, when George II died. It is accordingly quite possible that the heads of Cromwell, Bradshaw and Ireton would remain on Westminster Hall till 1685 or longer. Further, the fall of Cromwell's head at the time of James II's flight may well have given rise to the superstition of which Grosley speaks. When we come to consider the mounting of the regicides' heads on the south end of Westminster Hall, we must be careful to remember that there were low buildings on the west and south sides of that Hall, and that only a portion of these were public offices; they included the taverns popularly termed "Heaven" and "Hell," which will be discussed in the following section. In the next place the Hall had gable ends with a pinnacle at the roof-ridge, and possibly, but improbably, small pinnacles at the terminals, which are, however, not shown on the prints: see our Plates XV, XVII, XVIII and XXII. Between the ridge-pinnacle and the terminals there seem to have been short stone knobs giving the appearance of a balustrade in Queen Elizabeth's day, but they do not appear in later prints. To the ridge-pinnacle we may assume the pole for the head of Bradshaw was attached, while the poles for those of Cromwell and Ireton would be fixed to the stone abutments, or the terminal pinnacles if there were such. To render the heads more visible from Old Palace Yard, St James's Park and the river, we may reasonably assume poles* of considerable length and strength were used. This would be better attained by solid oak posts (such as we find a piece of in the case of the Wilkinson Head) than by small sectioned iron rods. If it be asked why was the southern gable of the Hall used in preference to the northern, we must recall the reason given for this choice, namely they were to be posted as nearly as possible above the place where in the Hall below the judges of the Court which condemned Charles I had sat†. For this reason the head of Bradshaw, the president of the Court, was placed in the centre above where he had sat, with Cromwell on his right and Ireton on his left. Cf. the Dutch Print on Plate XIV.

6. *Evidence that the Head has remained unchanged for more than a century.*

It is, perhaps, worth while, if hardly necessary, to indicate that the Head has remained in practically the same condition as it was when exhibited 134 years ago in Mead Court. This is evidenced both by the picture of the Head painted by Cranch and the engraving he provided for his *Narrative*: see our Plates IV and V. A further sepia drawing of the Head occurs in a grangerised Pennant's *Account of London* of 1790 in the Print Room of the British Museum. The drawing may date from the last quarter of the 18th century. There is no other record of the skull-cap being tied on with tapes, and it is singular that they were not removed for the occasion: see our Plate VI. A description provided in a letter of Maria Edgeworth's of March 9, 1822, shows the Head at that date in practically the same

* Their heads were set on poles on the top of Westminster Hall. *Gesta Britanorum*. Wharton's *Almanack*, 1688 (under 1668).

† Howarth (*loc. cit.* p. 10, Vol. LXVIII. p. 240) is wholly in error as to the position of Cromwell's head on Westminster Hall.



John Cranch's oil painting of the Head. The painting is in the possession of Canon Horace Wilkinson. Painted in 1799 or thereabouts.

Pearson and Morant: *The Cromwell Head*



Sepia drawing of the Wilkinson Head, from a grangerised copy of Pennant's *London*, 1790, in the Print Room, British Museum. Probably from the last quarter of the 18th century.

condition as to-day. To understand this letter we must point out that Maria Edgeworth was a friend of Mrs Ricardo, Mr Josiah H. Wilkinson's sister Fanny, and that she has just been describing a breakfast party at Mr Ricardo's house in London. At this party Mr Wilkinson, the first of that name to own the Head, was a guest and he had brought it for examination. It can hardly be expected that Miss Edgeworth, writing down later from memory what Mr Wilkinson said, would be wholly accurate, but the picture of the Head is true enough. It runs as follows:

But to go back to our breakfast and Mr Ricardo.—After the last at which Capt. Beaufort was with us, we saw—What do you think? Oliver Cromwell's head—not his picture—not his bust—nothing of stone or marble or plaister of Paris, but his real head, which is now in the possession of Mr Ricardo's brother in law (Mr Wilkinson)—He told us a story of an hour long explaining how it came into his possession. This head as he well observed is the only head on record which has after death been subject to the extremes of horror and infamy—It having been first embalmed and laid in satin State—Then dragged out of the coffin at the restoration—chopped from the body and stuck upon a pole before [*sic*] Westminster Hall, where it stood twenty five years—Till one stormy night the pole broke and down fell the head at the centinels feet who stumbled over it in the dark twice, thinking it a stone, then cursed and picked it up and found it was a head. Its travels and adventures from the centinel through various hands would be too long to tell—it came in short into the Russell family and to one who was poor and in debt and who yet loved this head so dearly that he never would sell it to Coxe of the Museum till Coxe got him deep in his debt arrested and threw him into jail—Then and not till the last extremity he gave it up for Liberty—Mr Wilkinson its present possessor doats upon it—a frightful skull it is—covered with its parched yellow skin like any other mummy and with its chestnut hair eyebrows and beard in glorious preservation—The head is still fastened to the inestimable broken bit of the original pole—all black and happily worm-eaten—By this bit of pole Mr and Mrs Ricardo and family by turns held up the head opposite the window while we stood in the window and the happy possessor lectured upon it compasses in hand—There is not at first view it must be owned any great likeness to picture or bust of Cromwell—but upon examination the proofs are satisfactory and agree perfectly with historic description—The nose is flattened as it should be when the body was laid on its face to have the head chopped off—There is a cut of the axe (as it should be) in the wrong place where the bungling executioner gave it before he could get it off—One ear has been torn off as it should be—And the plaister of Paris cast which was taken from Cromwell's face after death, being now produced all the measures of jaws and forehead agreed wonderfully and the likeness grew upon us every instant as we made proper allowances for want of flesh—muscles, eyes, etc. To complete Mr W.'s felicity there is the mark of a famous wart of Olivers just above the left eye brow on the skull precisely as in the cast. But then Captain B. objected or was not quite convinced that the whole face was not half an inch too short. Poor Mr Wilkinson's hand trembled so that I thought he never would have fixed either point of the compasses and he did brandish them about so afterwards when he was exemplifying that I expected they would have been in Fanny's eyes or my own and I backed and pulled back. Mrs Ricardo gave her staff to whom she listed—she could not bear the weight of Old Noll thro' the whole trial. Mr Ricardo gave up too when a bit of cotton-wool* was dragged from the nostrils—("Oh I cannot stand the cotton wool"). He delivered over the staff and went to the fire to comfort himself dragging up the skirts of his coat as men do in troubles great.

I was glad Capt'n Beaufort let the poor Mr Wilkinson off easy about the length of the face and we all joined in a chorus of conviction—He went off with his head and staff the happiest of connoisseurs—Moreover I suggested that for future convenience he might leave it fixed under a glass case—the broken staff to fit into a tube as candle in candle stick.—

* The material is flax lint, not cotton-wool, which would have been suspicious. See p. 340 below.

How much time and paper it takes to tell anything in writing!—Excuse tiresomeness! inevitable when I have not time to make things properly short.—

We owe a copy of this letter to the kindness of Professor H. E. Butler.

We will conclude this section of our memoir by remarking that it seems to us impossible that the Wilkinson Head was a fake. The bibulous actor Samuel Russell the Elder was not capable of preparing a head of this character; he was without all the knowledge of history and the art of embalming requisite for the purpose. The two men who might have some of the necessary ability—Cox and Cranch—come too late into the story. It is clear that Samuel Russell was trying to sell the Head as early as 1775, if not before that date.

In our minds there is small doubt that the Wilkinson Head is the “monströses Haupt” upon a wooden staff in the possession of Du Puy, a collector of *curiosa*, to whom anyone who had picked up the head of Cromwell would be likely to turn in order to sell it, possibly the more so as he was a foreigner, and not perhaps cognizant of any offence in the possession of it. How it passed from Du Puy to Samuel Russell, we cannot say. We have only Cox’s and Cranch’s statements that he was a descendant of the Fordham Russells, including Cranch’s account of what Russell cried out in Mead Court when he was drunk. If we could show that Samuel Russell was descended from Elizabeth Russell, Henry Cromwell’s daughter, there would be some reason for its being in Samuel’s possession. That extravagant lady might have purchased the Head from Du Puy in 1710 to 1711 or at a sale of his effects. But there is no light at present on any such relationship. We can only say that from 1710 to 1773—5 the Head, as far as we can ascertain, remained in obscurity*. Beyond the inherent superficial evidence in the attributes of the Head itself, which we shall shortly consider, we must ultimately turn to the evidence for or against it in the masks, busts and portraits of Cromwell to which later sections are devoted.

Before we can deal with possible contradictions in the inherent characters of the Head itself, and what we know of Cromwell’s head, we must study what is known of the latter, and briefly dismiss the numerous myths which have attached themselves to his remains.

SECTION II.

7. *History of Cromwell’s Remains.*

We may dismiss at once the various rumours and wild tales as to the disposal of Cromwell’s body. There exists of course the possibility that the *bodies* of any or all of the three, Cromwell, Bradshaw and Ireton, were dug up secretly from the

* If Du Puy retained the Head until his death, which took place in 1738, then for 37 years only is the fate of the Head uncertain. He may not have done so, as he died intestate, and no relative, but a creditor, appears as administrator. The entry in the General Register Office is as follows:

11th August, 1738. Dupuys, Claudius. On the eleventh day Administration of all and singular the Goods, Chattels, and Credits of Claudius Dupuys, late of the parish of St Anne, Westminster in the County of Middlesex, Bachelor decd. was granted to John de Normant, Principal Creditor of the said decd. he being first sworn duely to administer.

hole at Tyburn and secretly buried elsewhere*, but such possibility does not concern us, who have to deal solely with their heads. We must, however, turn for a moment to the disposal of the body of Charles I. Cromwell with certain restrictions† as to any ceremonial allowed the friends of Charles to bury his body in St George's Chapel, Windsor. Here it was laid in the vault of Henry VIII. Now Barkstead and possibly other roundheads spread a report that the body of Charles I was interchanged with that of Cromwell, and that it was the former that was hung at Tyburn. The whole tale is absurd. Sorbière writing in 1663 mentions a like tale as spread among the common folk, but terms it ridiculous‡. Pepys meeting with Jeremiah White writes on October 13, 1664:

When I told him of what I had found writ in a French book of one Monsieur Sorbière, that gives an account of his observations here in England; among other things he says, that it is reported that Cromwell did, in his life time, transpose many of the bodies of the Kings of England from one grave to another, and by that means it is not known certainly whether the head that is now set up upon a post be that of Cromwell, or of one of the Kings; Mr White tells me that he believes he never had so poor a low thought in him to trouble himself about it.

* The story of the Naseby Field burial attributed to the son of the regicide Barkstead (*Harleian Miscellany*, Vol. VII. p. 271, 1810) is idle, because it involves the concurrence of Cromwell himself. The burial of Cromwell's body at Newburgh Priory, the seat of the Fauconbergs in Yorkshire, is more probable as Mary Cromwell was a much loved daughter of the Protector. The truth of the burial could only be verified if the mass of stone in an upper chamber at Newburgh Priory were opened and a headless embalmed body found there. In that case it should be easy to test whether the Wilkinson Head fitted the body.

Philip Williams in his *History of the Protector's Remains* speaks of friends staying at Newburgh Priory (near Easingwold, Yorks), the seat of Sir George Wombwell, being shown on an upper floor at the end of a long and gloomy passage, a low and very strong old door, close upon the floor. The hostess said it had never been opened since the day the Protector's body went inside there. Her husband was a descendant of Viscount Fauconberg, the son-in-law of Oliver, and she said that the family had arranged that the body of Cromwell "should be secretly conveyed here, and here it had remained since the day it was enclosed within that door." It has been stated that Cromwell's heart is also at Newburgh Priory (*Notes and Queries*, Series X, Vol. XI. p. 452). Cf. Dionis' account of embalming on our p. 272.

† See Anthony à Wood's *Athenae Oxonienses*, 2nd Edition, 1721, Vol. II. Col. 703.

‡ M. Samuel de Sorbière: *Relation d'un Voyage en Angleterre, où sont touchées plusieurs choses, qui regardent l'estat des Sciences, et de la Religion, et autres matieres curieuses*, 1664. The permission to print the work was given to "Samuel Sorbiere Histographe de Roy" on October 2, 1660, and the journey to England must have been in the same year. An English translation was published in 1709. P. 143 speaks of the remains of the rebels which could be seen on London Bridge and on the towers of Westminster. Pp. 164—166 are those that concern us. We read:

Mais ce que je vis de plus remarquable, furent les superbes tombeaux de Richemont, de Bukingham, et de Henri VII qui vont au pair avec les nostres de S. Denis, si mesme ils ne les surpassent. Je ne dis rien d'un bruit ridicule qui courut à Londres; comme si Cromwel avoit donné ordre qu'on le mit secrettement dans quelqu'un de ces tombeaux. Ce que l'on s'est imaginé sur la précaution qu'il apportoit de son vivant, lors qu'il avoit une vingtaine de lits en autant de chambres, afin qu'on ne sceut jamais où il devoit coucher. Mais ce queis touche icy du corps du Protecteur qui a esté mis au gibet, et sa teste plantée à un poteau sur une tour à costé de la salle de Parlement, me fait souvenir d'une opinion qui court parmi le Peuple. Il y en a plusieurs qui croyent que la finesse de Cromwel est allée jusques à faire ouvrir quelques tombes des Rois qui sont dans la chapelle de Westminster, et à prendre soin qu'on fist une transposition.

It is interesting to note that Voltaire can hardly have known Sorbière at first hand, because he describes what Sorbière says as a dull scurrilous satire upon a nation of which the author knew nothing. Dr Welldon also could not have read what Sorbière wrote. See *The Nineteenth Century and After*, Vol. LVII. p. 943. Even what Sorbière writes about Cromwell changing his room is only an exaggeration of what Clarendon reports: see his *History of the Rebellion*, Vol. III. p. 504, 1704.

The rumour that Cromwell shuffled the royal bodies or that it was done by his relatives after his death, so that Charles I was hung in Cromwell's place at Tyburn is confuted not only by the realisation that it would be difficult to hang a beheaded man; but further by the fact that in 1813 the vault of Henry VIII in St George's Chapel, Windsor, was opened up officially. In it was found a plain leaden coffin with the words "King Charles, 1648" upon a scroll of lead surrounding it*. The coffin was laid on top of those of Henry VIII and Jane Seymour. The head was seen severed from the body when the coffin was opened, and the features were still recognisably those of Charles. There can be no doubt that Charles had remained where he was buried. A small mahogany coffin, laid on the pall covering King Charles', contained the body of an infant. This is known to have been a still-born child of Princess George of Denmark, afterwards Queen Anne†. Thus the rumour referred to by Sorbière and probably originating in the younger Barkstead should be for ever dismissed. It was only the lowest of the roundheads who could thus have endeavoured to score off the rabble of the cavaliers who demanded the desecration of the tombs of the dead, by thus soiling the character of the leader whose spirit they could not appreciate. Cromwell allowed the friends of Charles to bury him quietly in the company of kings. He had no thought of desecrating tombs, or that others would be so small-minded as to do so either. He clearly thought that his body would rest with those of his daughter and friends; and the suggestion that before dying he made arrangements for his secret burial is merely an invention by which one rabble hoped to outwit a second by setting the report of an infamy of their own party against the definitely achieved infamy of their opponents.

We must now turn to the actual funeral arrangements prepared for the Protector's body, and in considering these we shall find it needful to consider briefly the *last* royal obsequies in England, namely those of James I. We must remind the reader of one or two important facts. Cromwell died at 3 p.m. on September 3, 1658; immediately after this announcement in Bate's *Elenchus* follow the words: "Dissecto cadavere" and the account of the autopsy. Then follow the words about disembowelling, filling the body with sweet herbs, wrapping it in sixfold cere-cloths, next placing it in a lead and outer wooden coffins, and finally the remark that decay had so far set in that it was found necessary to bury it before the state funeral. We have already seen that it was not till the day following the death, September 4, that the physicians and surgeons appointed by the Council embalmed the body of the Protector. We do not even know that Bate was one of these; he may have had no experience of embalming. We do not even know that he took part in the dissection. But what we are certain of is that proper embalming is a much longer process than that described in the words following the account of the autopsy, so that a crude attempt may have been made at embalmment after the

* Called the "Girdle or Circumscription of Capital Letters in Lead put about the Coffin." Herbert gives a full account of the burial of Charles which tallies with what was seen on opening the vault in 1813: see *Memoirs of the Last Two Years of the Reign of King Charles I.* By Sir Thomas Herbert, 1815.

† See *An Account of what Appeared on Opening the Coffin of King Charles the First in the Vault of King Henry the Eighth in St George's Chapel at Windsor, on the First of April MDCCXIII.* By Sir Henry Hallford, Bart., F.R.S. and F.A.S., London, 1818. (4to of 19 pages.)

autopsy, afterwards rectified. Thus Bate may be speaking of what he had merely heard reported; especially the known fact of an earlier burial, may have led him to account for it by the statement in the *Elenchus**. Anyhow we shall see that what happened in Cromwell's case exactly followed the order of events in the obsequies of James I†. To these we now turn.

The account of James's autopsy and funeral ceremonies will be found in the following work by John Nichols, F.S.A.: *The Progresses, Processions and Magnificent Festivities of King James the First*, London, 1828. On p. 1037 of Vol. IV we have the account of the autopsy and embalmment. King James died on March 27, 1625, aged 59, the age of Cromwell at death. He was disembowelled on March 29 and this was accompanied by an examination of his organs:

his harte was found to be great but soft, his liver fresh as a younge man's; one of his kidneys very good, but the other shrunk so little as they could hardly find yt, wherein there was two stones; his lites and gall blacke, judged to proceed of melancolly; the semytur of his head soe stronge as that they could hardly breake it open with a chissell and a sawe and so full of braynes as they could not uppon the openninge keep them from spillinge, a great marke of his infinite judgement. His bowels were presently put into a leaden vessel and buryed; his body embalmed.

No one can read the account of Cromwell's autopsy without seeing in it the parallelism with that of James I. But what concerns us is that in the last account we have of the embalmment of an English sovereign we find the skull-cap was removed to take out the brain. Time, perhaps, did not permit of this in the case of Charles I, or the brain may have been removed through the nostrils. The relative haste of the embalmment is possibly indicated by the head not being stitched again to the neck before that undertaking. Thus all the discussion as to the removal of the skull-cap in royal embalmments covering pages of *Notes and Queries* is purely idle.

* Bate was writing after the Restoration, and when it seemed to please the cavalier wit to associate stench, whether of earthly or hellish origin, with Cromwell. Of the three roundhead leaders whose bodies were hung at Tyburn, Cromwell and Ireton (who had died in Ireland some weeks before his interment at Westminster) had been embalmed. Bradshaw was not embalmed and the effluvia at Tyburn was much more likely to have arisen from his body than from that of Cromwell. Still a cavalier present attributed it to the latter, and met with the coarse but apt retort of a roundhead acquaintance given in the *Somers Tracts*: A cavalier who witnessed the exhibition of Cromwell's body upon the gallows said to a roundhead acquaintance, "Methinks friend, thy old master stinks damnably." "True," answered the republican, "but you would have smelled worse had you been as near him when living as you are now."

Ja. Heath's *Flagellum*, 2nd Edition, 1663—which is thoroughly untrustworthy for historical purposes—after grossly exaggerating Bate's statement in the *Elenchus*, adds (p. 195) that Cromwell's corpse "raised such a noisome stink, that they were forced to bury him out of hand; but his name and memory stink worse." He then suggests (p. 199) for this reason only, that an empty coffin must have been taken to Somerset House on September 20th, Cromwell having been of necessity buried earlier. His body was buried on October 27th. All these tales seem traceable to the *Elenchus* and, as we have indicated, it is not really known whether Bate had anything to do with the actual and final embalmment. What we do know is that it was possible to take a mask of Cromwell's face some ten to fourteen days after his death.

† This correspondence was noted very early in the *Perfect Politician*, 1680, p. 280. Ludlow's statement (Ed. Firth, 1894, Vol. II. p. 47) that Mr Kinnesley, Master of the Wardrobe, recommended taking as a model the obsequies of Philip II of Spain seems only made to cast a charge of popery against that gentleman and the Somerset House arrangements.

We return now to James:

His Royall corps was forthwith embalmed, and with all due rytes apperteyning thereunto and being seared and wrapped in lead, was put into a sumptuous coffyn, which was filled up with odour and spices within, and covered without with purple velvett, the handles, nayles, and all other iron-worke about it being ryehly hatched with gold; and upon the King's brest was fixed an inscription in a plate of gold "Depositum invictissimi Principis Jacobi Primi, Magnae Britanniae, Francae et Hiberniae Regis, qui...."

Then follow the dates of his reign in Scotland and in England, his age and the years of his birth and death.

The parallelism with the proceedings in Cromwell's case is very close. We understand now why Serjeant Norfolk claimed as his fee for superintending the disinterment of Cromwell the corresponding plate on Cromwell's breast, believing it to be of gold, but it was only of copper double gilt and it may well be that James's was the same. Cromwell's plate is still in existence*, and a copy of the original reproduction of it by Dr Cromwell Mortimer, Sec. R.S., is in the possession of one of the authors of the present paper and is reproduced again, with the stonemason's receipt for digging up the bodies of Cromwell, Ireton and Bradshaw, on Plates IX and XI.

James died on March 27 at Theobalds, and was taken on the 29th to Denmark House (later Somerset House). The procession to Denmark House from Theobalds compares with that from Whitehall to Somerset House in Cromwell's case. The hearse was drawn by *six* goodly black horses; it arrived by *night* with torches burning, accompanied by the King's servants and the lords of the court. Now compare this with the following from the *Mercurius Politicus*, Sept. 20, 1658:

This *night*, Sept. 20, the corps of his Highness was removed hence in a private manner, being attended only by his own servants, viz. The Lord Chamberlain and the Comptroller of his Highness's household, the Gentlemen of his Bed Chamber, the Gentlemen of the Household, the Gentlemen of the Life Guard, the guard of Halberdiers, and many other officers and servants of his Highness. Two Heralds or Officers of Arms went next before the body, which being placed on a hearse drawn by *six* horses was conveyed to Somerset House, where it rests for some days more private, but afterwards will be exposed in state to public view. [The italics are ours.]

There is no sound reason for believing that the actual body of the Protector was not taken on this September 20th to Somerset House.

At Denmark House, precisely as in the case of Cromwell, four rooms seem to have been funereally decorated, i.e. hung with black cloth, black velvet, scutcheons, bannerets, taffety, etc., etc. These rooms were the Bed Chamber, the Privy Chamber, the "Presens," and the Guard Chamber. Nichols describes these hangings at length. Thus we read:

At Denmark House the body was carried into the lobby beyond the Privie Chamber there, which was prepared for that purpose, and a frame of boards lyke a large bedd, so made that the coffin was set even with the worke, and then that was covered with a fyne holland sheet conteyning forty ells, and a large pall of velvett blacke conteyning sixty nine yardes, which sheet

* It is said to be now in the possession of Earl de Grey.

Olivierus Protector Republicae
Angliae Scotiae et Hiberniae
Natus 25th Aprilis Anno 1599
Inauguratus 16th Decembris
1653 Mortuus 3rd Septembris
Anno 1658 hic Situs
est

This is a complete print of an inscription taken from a Copper plate gilt, found in a leaden casket
 lying on the breast of the corpse, when the grave & coffin of Oliver Cromwell in Henry VIII Chapel, were
 broken open by order of the government after the Restoration, in order to have his body brought up at
 Tyburn. The platy casket was prepared by Sir James Wolfe, then Serjeant at arms to the house
 of Commons, who left it to his only daughter & child, Mary, who married to Roger Gifford Esq.
 Counsellor at Law of Colchester, whose only daughter & child
 Mary, died of Smallpox near Hadden in Essex, she left it to his third wife, in whose possession it
 was until 1740. Oliver Cromwell mortuus M. D. An. Sane. May. 22. 1740.

Facsimile of Inscription on the gilded plate found on Cromwell's breast when he was exhumed.
 Copy with statement by Dr Cromwell Mortimer in the possession of Karl Pearson.

was turned up about a yard and sewed to the velvett. Six goodly, large and high silver candlesticks, which King Charles had bought when he was in Spaine, were placed about the Body and in them were put tapers of four foot in length of virgins' wax, which burned all night. A canopie was provided to hang over this bed; the Chamber was hanged with black velvet, and a majestie scutcheon over the King's head wrought upon cloth of gold. Ymediately a representation of his majestie was layd upon the said pall, over the Body, in his robes of estate and Royall diadem, and so it conteyned untill the Funerall, all Kinge James his servants removing from White-hall to Denmarke House and Kinge Charles his servants from St James to White-hall, the service contynued in all points as if his majestie had been lyveinge.

As the body was taken to "Denmarke House" two days after the death, it was probably placed in the "lobby" while the "bedd" was being prepared. The statement that the effigy was placed over the body, probably means that the coffin was under the bed, although the words "a large bedd, so made that the coffin was set even with the worke" are difficult of interpretation. Perhaps they only signify that the coffin was placed parallel to the sides of the "bedd." A glance at our Plate VII will indicate to the reader how Cromwell's "effigies" was placed on the "bedd," doubtless with his coffin beneath it. The following description is taken from the *Mercurius Politicus*, October 14—21:

A particular and exact relation how *Somerset House* is prepared for the effigies*, or representation of his late Highness, by particular order of the Lords of the Council, which was first showed publicly on Monday last.

The first room the people enter, was formerly the Presence Chamber, which is hung completely with black, and at the upper end a cloth of estate, with a chair of estate standing upon the Haut-place under the state. From thence you pass to a second large room, which was the Privy Chamber, all compleatly hung with black, and a cloth of estate at the upper end, having also a chair of estate upon the Haut-place, under the cloth of estate. The third room is a large withdrawing chamber, compleatly hung as the other with a black cloth, and a cloth of estate at the upper end, with a chair of estate as in the other rooms. All these three large rooms are completely furnished with escutcheons of his Highness's arms, crowned with the imperial crown, and upon the head of each cloth of estate is fixed a large majesty escutcheon fairly painted and gilt, upon taffety. The fourth room, where both the body and the effigies do lie compleatly hung with black velvet, the roof of the said room cieled also with velvet, and a large canopy or cloth of estate of black velvet fringed over the effigies; the effigies itself apparelled in a rich suit of uncut velvet, being robed first in a kirtle robe of purple velvet, laced with a rich gold lace, and furred with ermins, upon the kirtle is the royal large robe of the like purple velvet laced, and furred with ermins, with rich strings, and tassels of gold; his kirtle is girt with a rich embroidered belt, in which is a fair sword richly gilt, and hatched with gold, hanging by the side of the effigies; in the right hand is the golden scepter representing government; in his left hand is held the globe, representing principality; upon his head, the cap of regality of purple velvet, furred with ermins. Behind the head is a rich chair of estate of cloth of gold tissue; upon the cushion of the chair stands the imperial crown set with stones. The whole effigies lies upon a bed covered with a large pall of black velvet, under which is a fine Holland sheet upon six stools of cloth of gold tissue; by the sides of the bed of state lies a rich suit of compleat armour, representing his command as General; at the feet of the effigies stands his crest, as is usual in all

* "I made mention in my last of an effigies of wax made to represent his late Highnes. The charge of wayters is great &c. It is therefore not to continue above 14 days at the farthest. In the meantime black velvet is bought all London over to hang in Whitehall and Somerset House; and because men cannot mourn enough for the death of his Highnes, the stones and walls are taught to do it." Newsletter, Sept. 16, 1658, *Fifth Report, Historical MSS. Commission*, Vol. III. p. 144^a.

ancient monuments. This bed of state upon which the effigies so lies, is ascended unto by two ascents covered with the aforesaid pall of velvet, and the whole work is encompassed about with rails covered with velvet; at each corner is a square pillar or upright, covered with velvet; upon the tops of them are four beasts supporters of the imperial arms, bearing banners or streamers crowned; the pillars are decorated with trophies of military honor, carved and gilt. The pedestals of the pillars have shields and crown gilt, which make the whole work noble and compleat; within the rails stand eight great standards or candlesticks of silver, being almost five feet in height, with great tapers in them of virgin wax, three foot in length. Next to the candlesticks are set upright in sockets, the four great standards of his Highness's arms, the guidons, the great banners, and banrolls, all of taffety richly gilt and painted: the cloth of estate hath a Majesty scutcheon fixed at the head, and upon the velvet hangings on each side of the effigies is a Majesty scutcheon, and the whole room fully and completely furnished with taffety scutcheons. Much more might be enlarged of this solemn setting up, and shewing the effigies at present in *Somerset House*, where it is to remain in state until the funeral day, which is appointed to be on the 9. of November next.

Now here is a perfectly clear statement that the *body* of Cromwell and his effigy were in the fourth room, when *Mercurius Politicus* for the week October 14—21 appeared. The whole arrangement for Cromwell's lying-in-state is so similar to James's, even to the rooms occupied in the palace, that we may well assume that the coffin was under the "bed" as on the earlier occasion.

Heath in the *Flagellum*, giving the account in the *Mercurius Politicus* practically verbatim, when he comes to the fourth room and the words "where both the body and the effigies do lie," replaces them by "where the effigies lies*." He does this no doubt to support the cavalier doctrine that, as Cromwell stank in their nostrils allegorically, he must have done so physically and his body have been already putrescent at death.

The *Mercurius Politicus* gives us no clue as to when the coffin was removed from Somerset House and buried in Henry VII's Chapel at Westminster Abbey. But it was probably at the time when a standing effigy replaced that lying on the bed. We may note that there is no record of when James's body was actually interred.

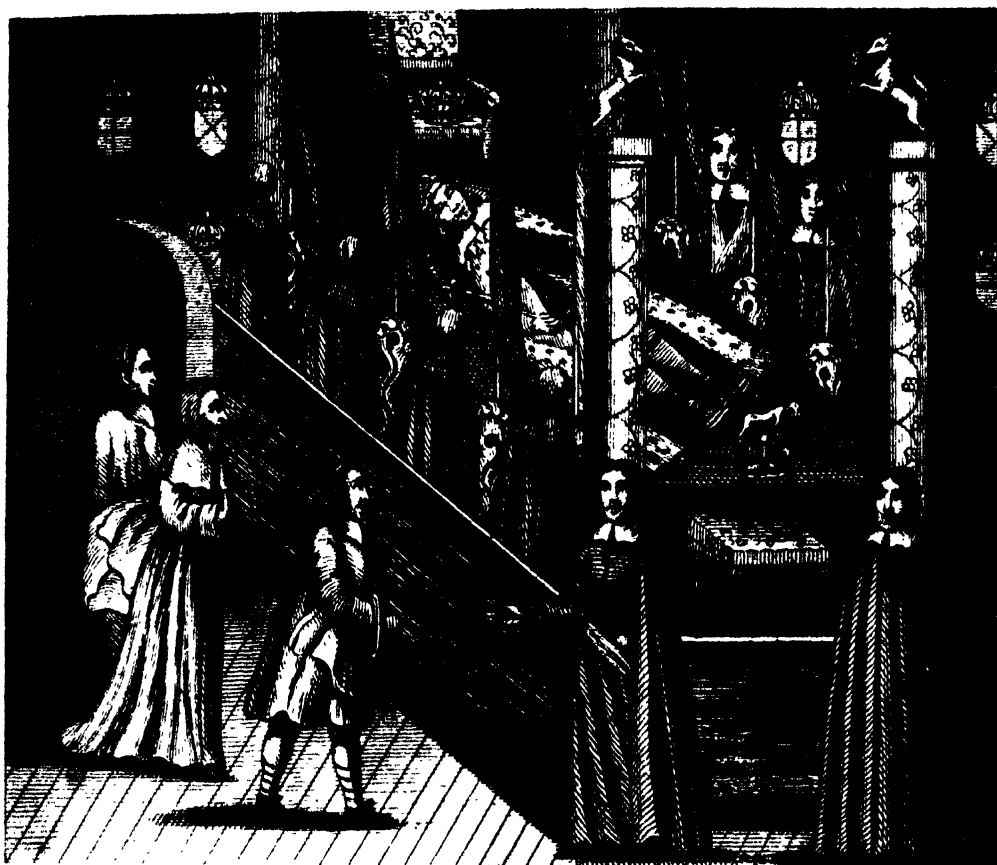
From James I's death on March 27 to his funeral procession on May 7 was almost exactly six weeks, and from Cromwell's death on September 3 to his funeral procession on November 23 was over eleven weeks. We know that Cromwell's funeral was at first announced for November 9†, somewhat over nine weeks from death, and afterwards postponed to the 23rd, but we do not know the reasons for this postponement. It may have been to allow time for the second part of the

* Carrington, *History of the Life and Death of His Serene Highness Oliver, late Lord Protector, 1659*, uses the words:

The fourth room, where both the corpse and effigies did lie, was completely hung with black velvet.

The Dutch translation of Carrington (Amsterdam, 1659) runs almost verbatim: "De vierde kamer in de welke beyde het Lichaem en de af-beelding waren, was kostelijck met swart Fluweel behangen."

† Newaletter of W. Smith to Mr John Langley: "It is said the funeral will be solemnized on the 5th day of November next" (Oct. 9, 1658) and later the same to the same: "His Highness's funeral is designed to be on the 9th of November" (Oct. 23, 1658). *Fifth Report, Historical MSS. Commission*, Vol. III. pp. 92 and 93.



The LORD PROTECTOR lying in State at Somerset House.

Engraved from the original print in the collection of JOHN TOWNLEY, Esq.

The original print must have been contemporary: see Plate VIII.



His HIGHNESS Effigies standing in state.

Engraved by J. C. Gadsden from the original Print

in the Collection of JOHN TOWNLEY Esq.

The original print must have been contemporary, for it appears, much reduced, as frontispiece to Carrington's *History of the Life and Death of... Oliver, late Lord Protector*, 1659, the year following Cromwell's death.

ceremonial at Somerset House, the *standing-in-state**. We have not found any record of James I's effigy standing-in-state. Carrington, after describing the lying-in-state room, continues:

After which, his late Highness's effigies was several days shown in another room, standing upon an ascent, under a rich cloth of state, vested in royal robes, having a scepter in one hand, and a globe in the other, a crown on his head, his armour lying by him, at a distance, and the banners, banrolls, and standards being placed round about him, together with the other ensigns of honour. The whole room which was spacious, being adorned in a majestic manner, and several of his late Highness's gentlemen attending about the effigies, bare-headed; in which manner the effigies continued until the solemnisation of the funerals.

On the day of the procession, after the guests had seen the standing effigy, the latter was *laid* "in an open chariot covered all over with black velvet" to be taken to the structure on which it was drawn to Westminster [see Prestwich's *Respublica*, pp. 172—176].

While the room of the lying-in-state had been in darkness, except for the tapers in the silver candlesticks, and all draped in black, the room of the standing-in-state was brilliantly lighted. Ludlow—whom we have no occasion for always believing—thus describes the second display of the effigy:

This scene of purgatory [i.e. that of the lying-in-state] continued till the first of November, which being the day preceding that commonly called All Souls, he was removed into the great hall of the said house and represented in effigy, standing on a bed of crimson velvet covered with a gown of the like coloured velvet, a scepter in his hand, and a crown on his head. That part of the hall wherein the bed stood was railed in, and the rails and ground within them covered with crimson velvet. Four or five hundred candles set in flat shining candlesticks were so placed round near the roof of the hall, that the light they gave seemed like the rays of the sun; by all which he was represented to be now in a state of glory. This folly and profusion so far provoked the people that they threw dirt in the night on his escutcheon† that was placed over the great gate of Somerset-house.

This account of Ludlow's is at least confirmed in respect of the candles by the print which represents the standing-in-state (see our Plate VIII), where the row of candles is visible at the top of the print, as well as the great silver candlesticks on either side the effigy. Carrington merely says of the body of Cromwell, after his account of the funeral procession:

The corpse having been some days before interred in Henry the Seventh's chapel in a vault purposely prepared for the same, over which a costly monument is preparing.

That the body of Cromwell was not taken to Westminster in the funeral procession seems as certain as that it was taken to Somerset House. In this respect Edward Burrough (1634—1662) may be cited, because he wrote a strong pamphlet against the funeral ceremony *without* the body.

* Ludlow finds the whole ceremonial popish, the lying-in-state corresponding to the period during which the subject would be in purgatory, and the standing-in-state to the ascent into heaven. Whether the origin of this succession of lying and standing had a precedent from Catholic royal obsequies (say, those of Philip II of Spain), we cannot say, but we feel certain that such interpretation was not in the minds of those who organised Cromwell's ceremonies, as Ludlow suggests. Ludlow is particularly bitter as to Mr Kinnesley, Master of the Wardrobe, who he says "was suspected of being inclined to popery." Kinnesley's requests for "cloths of estate" will be found in the *Calendar of State Papers*, 1658—59, pp. 91, 181—182. P. 143 notes various requests for mourning suits.

† This may have been the work of the cavaliers, or of the extreme puritans, scarcely of the "people" as a whole.

All this stir and costs and preparation for many weeks beforehand and such decking in mourning attire of great and noble men and all to accompany an image from one place to another....Whereby people are deceived who might look upon it to be the burial of Oliver, Protector, when as it was but an image made by hands and decked and trimmed in a vain manner, as if it had been for some poppet play. Which if it had indeed been his bones they had accompanied to the grave in such a manner, that had been less condemnable and I should not have had aught against it*.

Unfortunately neither Carrington nor the Quaker tells us the date at which Cromwell's body was removed from Somerset House to Henry VII's Chapel at Westminster.

A letter of Lady Hobart's is reproduced in the *Verney Memoirs*, Ed. 1925, Vol. III. pp. 422—425, but unfortunately without any date. She writes: "My lord protector's body was Bered last night 'at one o'clock very privittly and 'tis thought there will be [no] show at tall: the army dou bluster a letell: god send us pes for I dred a combuston." A newsletter to Berwick states that Cromwell's "corpse was removed on Wednesday night from Somerset House and carried to Westminster and there interred in a vault in Henry the Seventh's Chapel, a day will suddenly be appointed for celebrating the funeral, the charge wherof will not amount to above £80,000†." As the only statement here is that the private burial in Henry VII's chapel took place on a Wednesday, this may have occurred on October 20th or 27th. Since Ludlow says the effigy was erect on All Saints' day, and on view, the later date is probable. It could not be on the 13th because the lying-in-state opened on Monday the 18th and we are told by *Mercurius Politicus*, October 14—21, not only that the effigy was "shewed publicly on Monday last," which would be the 18th, but that in the fourth room "both the body and effigies do lie." Probably October 24—30 was a blank week as far as the public were concerned, and the erect effigy was publicly exhibited from November 1st to the day of the funeral. In this case the Wednesday, Oct. 27, of that week would be the date of the interment of Cromwell's body in Henry VII's Chapel. Sir Roger Burgoyne is equally vague as to dates. Writing on Nov. 11 he says:

We are all a whilst, no newes stirring, but that the Old Protector is now gott upon his leggs againe in Somerset House, but when he shall be translated to the rest of the Gods at Westminster I cannot tell. Pray do you come and see. (*Verney Letters*, Vol. III. p. 422.)

All we can conclude is that when the lying-in-state ended the coffin was removed to Henry VII's chapel at the Abbey and the standing-in-state began. There seems no reason to doubt Lady Hobart's‡ statement that the body of

* *A Testimony against a great Idolatry committed: and a true Mourning of the Lord's Servant—upon the occasion of that great stir about an image made and conveyed from one place to another, happening the 28 day of the ninth month.* By E. B. 1658.

† *Historical Manuscripts Commission. Calendar of the MSS. of the Corporation of Berwick-on-Tweed. Report on MSS.* Vol. I. 1901, p. 19. The cost for black cloth alone at the funeral was £6,929: see p. 12 of Appendix (Part III) to the *Eighth Report, Historical MSS. Commission*. "All our courtiers are preparing for the greatest funeral that has been seen in England. Yesterday his Highness [Richard] presented all the officers of the army with complete mourning." Newsletter, Oct. 9, 1658, *Ibid. Fifth Report*, Vol. IV. p. 98.

‡ Lady Hobart was presumably the wife of Sir John Hobart the third baronet of that name, and therefore Mary, the daughter of John Hampden the patriot, and first cousin once removed of the Protector, and second cousin of Richard Cromwell, and thus her news is likely to be trustworthy.

Cromwell was buried "privittly" in the Abbey before the funeral procession. It is quite possible that this burial before the funeral procession was customary in the case of royalty. It is not unreasonable to suppose that James I's body may have been interred privately before the public funeral. In reading the account of that funeral one is uncertain whether it is the body of James or his effigy of which the writer speaks. "It," whichever it may have been, was removed by stages from the bed in the Bed-Chamber into the Privy Chamber, thence into the Presence and finally into the Hall. We see here the use of the four funereally decorated chambers as gradual stages to the hearse. Mr Chamberlain wrote of this hearse to Sir Dudley Carleton: "the herse likewise being the fairest and best fashioned that hath been seen, wherein Inigo Jones, the Surveyor did his part*." The hearse† was a classical structure with eight pillars supporting a cupola, and eight sitting female figures on the column tops carrying banners. The cupola was bedecked with bannerets and there were also four draped female figures on separate pedestals. Under the cupola was the *effigy* lying on a bed with crown, orb and sceptre alongside. The hearse with the effigy was placed in the Chancel at the Abbey. We can find no statement of how long the hearse remained there. The procedure is exactly paralleled by that of Cromwell's funeral, and we understand why the like contrivance in Cromwell's case was called a "machine," "structure" or "monument."

We do not propose to compare the two processions at length. Nichols gives a full account of the mourners and followers, some 9000 in number. The procession started from Denmark House at 10 a.m., and the last mourners only reached the Abbey at 4 p.m., so that it had struck 5 before all had entered. One fact may be mentioned which indicates that the families of two regicides were of court-standing. Of the twelve bannerols one was carried by Sir Oliver Cromwell—the Protector's uncle and godfather—England impaling Woodville‡—and John Bradshawe, Roug-croix Pursyvant-at-Arms, bore another.

Cromwell's procession seems to have been at least once if not twice postponed§. In the *Calendar of State Papers* 1658—59 we find on October 7|| (1658) "His late Highness's funeral to be on 9 Nov." (p. 152) and on Nov. 16, "The funeral day for his late Highness to be 23 Nov." (p. 184). The reason for the delay we do not know, it may have been under-estimate of the time necessary for the preparations. Besides the structures in the rooms at Somerset House, there appears to have been not only a magnificent hearse, but, unlike James I's, the effigy was carried with the "hearse" to an independent structure in the Abbey chancel. The *Mercurius Politicus* under Nov. 23 gives an account of the procession. We learn how the effigy was carried on a hearse being vested in royal robes, a sceptre in one hand, a globe

* Nichols, *loc. cit.* p. 1049.

† Nichols gives a plate of the hearse.

‡ Edward IV married Elizabeth, daughter of Richard Woodville, Earl Rivers.

§ *Mercurius Politicus*, Oct. 7—14, referring to the Council under date Oct. 12 writes: "Consideration was also had about the appointing a day for the funeral solemnities of the body of his late Highness." This note is omitted by Stace in the *Cromwelliana*.

|| *Mercurius Politicus*, Oct. 14—21. The effigy "is to remain in State until the funeral day, which is appointed to be on the 9 of November next."

in the other, and a crown on the head, exactly as in the case of James I's effigy. The carriage on which the hearse was placed was drawn by six horses. All along the way from the Strand to Westminster the streets were railed, and behind the rails soldiers were placed. Then follows a long list of the persons of honour and dignity who attended: Judges, Ambassadors, Officers of State, chiefs of the Army and Navy, and so forth*; on either side of the carriage were borne the bannerols, twelve in number, and the Horse of Honour in very rich equipage, led by the Master of the Horse....

The whole ceremony was managed with very great state to Westminster; many thousands of people being spectators. At the west gate of the Abbey church, the hearse with the effigies thereon was taken off the carriage by those ten gentlemen, who removed it before†, who passing on to enter the church, the canopy of state was by the same persons borne over it again; and in this magnificent manner they carried it up to the east end of the Abbey, and placed it in that noble structure‡, which was raised there on purpose to receive it, where it is to remain for some time exposed to public view. This is the last ceremony of honour, and less could not be performed to the memory of him, to whom posterity will pay (when envy is laid asleep) more honour than we are able to express§.

* More important to us than the great officials in this immense cortège are the names below. Following the "musicians" of the court and preceding "His Highness's Butlers" we have:

Apothecaries, Mr Webb, Mr Phelps, Mr William Baghurst.

Chirurgions, Mr Fothergail, Mr Trapham, Mr Harris.

Later: *Officers of the Mint*, Mr Thomas Symond, chief graver....

Again: *Surveyor of the Works*, Mr Embree.

Keeper of the Wardrobe at Whitehall, Mr Clement Kinnesley [Ludlow's *bête noire*]....

Chaplains at Whitehall, Mr White [see our p. 295], Mr Sterry [Peter, see our p. 316], Mr Hooke, Mr Howe [John, preacher of toleration], Mr Lockyer [Nicholas, Provost of Eton], Mr Peters [the notorious Hugh].

Secretaries of the French and Latin Tongues, Mr Dradon, Mr Marvel [the famous Andrew], Mr Sterry [probably Nathaniel], Mr John Milton, Mr Hartlibbe, sen. [the well-known Samuel]. [Can "Mr Dradon" possibly be John Dryden, who started life as secretary to his cousin Sir Gilbert Pickering, Cromwell's Chamberlain?]

Doctors of Physic, Dr Clarke, Dr Goddard [Jonathan], Dr Projean, Dr Simcootts, Dr Bates, Dr Glisson, Dr Bathhurst [John].

From the *Lansdowne MSS.* 95, No. 2, cited in the Appendix, Vol. II. pp. 518 *et seq.* of the *Diary of Thomas Burton*, edited by J. T. Rutt, London, 1820. It will be seen that there was a plentiful choice of possible names to choose embalmers from!

† From Somerset House to the carriage.

‡ Heath in his *Flagellum* says that the "Picture," i.e. the effigy, was placed at the East end of the Abbey "in a most magnificent structure built in the same form as one before had been on the like occasion for James, but much more stately," p. 200.

§ John Evelyn, who watched the procession pass, terms it superb, with its imperial banners, achievements, heralds, guards, soldiers and innumerable mourners, and he was not an impartial witness! Largely owing to a dispute between the French and Spanish ambassadors for precedence, the procession did not reach the Abbey until after dusk. In a letter of M. de Bordeaux to Mazarin, we learn that:

There was not a single candle in Westminster Abbey to give light to the company, and conduct the hearse into a sort of chapelle ardente which had been prepared; there were consequently neither prayers, nor sermon, nor funeral oration, and after the trumpets had sounded for a short time every one withdrew in no particular order. (Guizot's *Richard Cromwell*, Vol. I. p. 288.)

The Genoese ambassador writing home describes how the effigy was placed at the altar:

On a royal couch under an edifice made expressly after the fashion of a pavilion, which cost more than £4000 sterling, where it will remain three months exposed to public view, and thence will be taken on to another edifice in the chapel of King Henry VII, and will be placed over the monument, under which is laid the body of the deceased, just as has always been the custom at the obsequies of the Kings and Princes of this nation. See J. E. C. Welldon, *The Nineteenth Century and After*, Vol. LVII. p. 925 (cited without reference to the source).

A few words must now be said as to these royal effigies. They were not crude work, but prepared by the best artists of the day. In the case of Cromwell, the wax face was moulded by Mr Thos. Symon, the designer of the Dunbar medal, and the most famous medallist of his day, and the body was carved in wood by Mr Phillips, carver to the House and surveyor*. The goodness of the wax masks of these effigies is of some importance for our present purpose, as they probably involved the taking of death-masks of the subject. Now we know that Cromwell's skull-cap was removed, and accordingly they must have been taken before or after this removal. If after the removal, the cincture should be visible unless it had been covered by a bandage, or signs of it disguised by the moulder. If before the removal, the mask should show no traces of the cincture, and in such a case the mask should represent Cromwell more nearly as he died. There are thus possible three types of death-masks of Cromwell:

(a) Before removal of the skull-cap.

(b) After removal of the skull-cap, showing the cincture wrapped over by a cloth.

(c) After removal of the skull-cap with the cincture worked out by the moulder or sculptor†.

The disadvantage of type (a) is that the mask may be taken so soon after death that the face has not reached the peaceful state which is usually attained somewhat later. The disadvantage of (b) is that the forehead is to a considerable extent lost, and the face may appear rounder than it actually was in life. The disadvantage of (c) is that even with a skilful sculptor the true portraiture of the forehead may be partially or wholly disguised. The bearing of these remarks will be more obvious when we come to our section on the death-masks of Cromwell. There appear to have been two effigies of the Protector, although this does not necessarily follow from the accounts of the ceremonies. Possibly one was used for the lying-in-state and the other for the standing-in-state. Unfortunately neither of these effigies has survived, for had either done so it would much have helped the present inquiry. The end of the effigies is marked by the following paragraphs in the public journals of 1660:

Mercurius Publicus (May 24—31) May 29. His Majesty was conducted to his Royal Palace at *Whitehall*, and the solemnity of this day was concluded by an infinite number of bonfires. And among the rest in *Westminster*, a very costly one was made, where the effigies of the old *Oliver Cromwell*, was set upon an high post, with the arms of the Commonwealth, which having been exposed there awhile to the public view, with torches lighted, that every one might take better notice of them, were burnt together.

Public Intelligencer June 18 to 25. This afternoon, June 14, there was exposed to view out of one of the windows of *Whitehall*, formerly the lodging of *Sir Henry Mildmay*, and now the jewel

* The MS. of Samuel Sainthill states that the effigy was "made to the life according to the best skill of the artist in that employed, viz. Mr Symon and a body of wood carved by Mr Phillips, carver to the House and surveyor." Probably the above Thomas Symond the graver to the mint (fn. *, p. 304).

† We have been informed by a sculptor, who has frequently taken death-masks, that much working up of the mask when it has been taken is very usual, if not general.

office, the effigies (which was made and shewn with so much pomp at *Somerset House*) in wax of *Oliver Cromwell*, lately so well known by the name of the Protector, with a cord about his neck, which was tied to one of the bars of the windows*.

We are not told what finally became of this effigy, but it is easy to imagine its fate in those days of riotous cavalier triumph†. It is possible that the effigy burnt at Westminster was one manufactured for the occasion, but it was more probably taken from Henry VII's Chapel in the Abbey, since the writer uses the definite article "the" with effigies. The other effigy is directly associated with Somerset House. It is thus likely that there were two effigies, one for the lying-in-state and one for the standing-in-state. In the case of the lying-in-state the eyes would be closed, and the legs lying together, the arms close to the body. In the standing-in-state the eyes would be open, the legs apart, and the stiff arms outstretched holding sceptre and globe. In the lying-in-state the effigy wore "the cap of regality of purple velvet, furred with ermins," and in the standing-in-state the effigy wore "the imperial crown set with stones." In either case it would have been feasible to use a death-mask with the cincture covered by a wrap, which would be hidden by "the cap of regality" or "the imperial crown." In the prints of the lying-in-state and the standing-in-state both come low on the forehead. If we have confidence in the *Mercurius Politicus* (Nov. 18—25) it was the standing effigy that was taken to Westminster on the hearse, and not the recumbent figure; for we read:

Somerset House Nov. 23. This being the day appointed for the solemn funeral‡ of the most serene and renowned *Oliver Lord Protector*; and all things being ready prepared, the effigies of his Highness standing under a rich cloth of state, having been beheld by those persons of honor and quality which came to attend it, was afterwards removed, and placed on a hearse, richly adorned, and set forth with escutcheons and other ornaments, the effigies itself being vested with royal robes, a scepter in one hand, a globe in the other, and a crown on the head. After it had been a while thus placed in the middle of the room, when the time came that it was to be removed into the carriage, it was carried on the hearse by ten of the gentlemen of his Highness forth into the court where a canopy of state, very rich, was borne over it by six other gentlemen of his Highness till it was brought and placed on the carriage;

It would therefore seem probable that the standing effigy was carried to Westminster and afterwards burnt there, while the lying effigy brought at some time to Whitehall from Somerset House was hung from the window and broken up. Thus two post-mortem portraits of Cromwell perished.

* Unfortunately Pepys was at Deal on May 29, and has no entry for June 14, so that we miss his characterisation of these occurrences.

† The MS. of Samuel Sainthill says that after hanging from the Whitehall window, it like the magnificent hearse was broken in pieces; see Noble, Edn. 1787, Vol. i. p. 280 footnote.

‡ Among the MSS. of Sir Edmund Lechmere is preserved an invitation to attend the funeral addressed to "Nicholas Lechmere Esq. Attorney to the Duchy in your morning gown and hood." It runs in print, bearing a seal of arms, a lion rampant:

You are desired to attend the funeral of the most serene and most renowned OLIVER, late Lord Protector, from *Somerset House*, on Tuesday, the 23rd of November instant, at eight of the clock in the morning at the farthest, and to bring with you this ticket; and that by Friday next you send to the Herald's Office, near *Pauls* the names of your servants that are to attend, in mourning, without which they are not to be admitted; and also to take notice that no coaches are to pass on that day in the streets between *Somerset House* and Westminster.

It is annotated: Received 18th Nov. 1658, N. Lechmere.

With regard to the fate of the hearse or structure we possess certain accounts. This was to remain six months in position. A satirical tract of May 26, 1659, entitled *Eighteen New Court Queries*, asks:

Whether the protector's cradles standing in Westminster Abbey in the same place where the High Altar, or Communion Table, formerly stood is not the setting up for one superstition where another superstition (as 'twas termed) was pulled down. And whether the effigies while it was there might not be called without any abuse of scripture the abomination of desolation in the Holy Place?

This was one day after Richard Cromwell's abdication on May 25, 1659. Another tract entitled *Twenty-seven Queries, relating to the General Good of the three Nations, which will neither please Mad-Men, nor displeuse Rational-Men*, dated 1659, appeared clearly after Richard's disappearance, as may be seen from Query XV. In Query VIII, there is reference to the "Great Engine" in Henry VII's Chapel. On June 7, 1659, the *Weekly Post* chronicles the taking down of the effigy, car, etc., and the sale of crown, sceptre, and other royal ornaments, after they had been broken up. The paragraph reads as if it referred to the "structure" in the chancel, not to the "engine" in Henry VII's Chapel*.

We have seen the remains of the Protector deposited in the vault of Henry VII's Chapel in the Abbey; they rested in the same chapel with the remains of his much-loved daughter Mrs Claypole, and of his son-in-law Henry Ireton, the husband of Bridget Cromwell; in the neighbourhood were the remains of Bradshaw, the president of the High Court of Justice which condemned Charles I, formerly a close colleague of Cromwell†.

Ireton died before Limerick on November 26, 1651, and his body reached Bristol on Dec. 17, where it was received with great ceremony. It was taken to London and lay-in-state in Somerset House, being finally buried in Henry VII's Chapel on Feb. 6, 1651—2. Such a long period before burial involved embalment. That Ireton's remains were embalmed is undoubted, although it was somewhat imperfectly carried out (see p. 313 below). The imperfection may not unreasonably be attributed to the fact that he is said to have died of the plague. Bradshaw died in London on the 22nd of November, 1659, and was given a state funeral some days later. His body was not embalmed.

8. *The Restoration and the Tyburn Affair.*

On May 25, 1660, Charles II landed in England. Ten days before, May 15, "The House ordered that John Bradshaw, deceased, late Serjeant-at-Law be one of those

* There are several documents in the archives of Westminster Abbey dealing with the "Herse." Thus: Petition of the watchers for payment; petition of the Surveyor to the Abbey to have the taking down of the Herse, and of the Officers of the Abbey for division of the proceeds of the sale of its parts. There is also a petition for payment for re-erecting pews removed for Herse. *Historical MSS. Commission Fourth Report*, p. 180.

† The fact that Cromwell allowed the bodies of his favourite daughter, his son-in-law, and his former chaplain to be buried in the Abbey is good evidence that he had no suspicion that the royalists would be restored, or if restored would be capable of desecrating tombs. Such evidence is fatal to the rumours of secret primary burials. The fact that the Lords of Council allowed President Bradshaw to be buried in Westminster Abbey as late as November, 1659, indicates that a year after Oliver's funeral there was still no fear that bodies buried in the Abbey were likely to be exhumed and foully handled.

that shall by Act of Parliament be attainted of high treason, for murdering of the late King's Majesty. The same order was made concerning *Oliver Cromwell, Henry Ireton* and *Thomas Pride*, deceased" (*Mercurius Publicus*, May 10—17). The "House" was the so-called Convention meeting on April 25, 1660. Meanwhile the tide of cavalier riotousness rose rapidly and had even mastered the Convention before its dissolution on December 29, 1660. Among its last resolutions one is thus reported in the *Mercurius Publicus* (November 29 to December 6) Dec. 6:

The Honorable House of Commons have dispatched the bill for preventing profane cursing and swearing. And while we speak of profanation, we cannot but acquaint you how the House in resentment of the honour of his Majesty and the nation have ordered, that the several bodies of *Oliver Cromwell, John Bradshaw, Henry Ireton* and *Thomas Pride* be taken out of their graves and drawn on an hurdle to *Tyburn*, where they are to be hanged and then buried under the gallows*.

The tone of the Journal does not here seem fully sympathetic to the proceeding, but it soon changes:

On Saturday (Dec. 8) the most honourable House of Peers concurred with the Commons in the order for the digging up the carcasses of *OLIVER CROMWELL, HENRY IRETON, JOHN BRADSHAW* and *THOMAS PRIDE*, and carrying them on an hurdle to *TYBURN*, where they are to be first hanged up in their coffins, and then buried under the gallows.—*Parl. Intel.* Dec. 3 to 10.

This day (Jan. 26) in pursuance of an order of Parliament, the carcasses of those two horrid regicides, *OLIVER CROMWELL* and *HENRY IRETON*, were dug up out of their graves†, which (with those of *JOHN BRADSHAW* and *THOMAS PRIDE*) are to be hanged up at *TYBURN*, and buried under the gallows.—*Merc. Pub.* Jan. 24 to 31.

Jan. 30. (we need say no more but name the day of the month) was doubly observed, not only by a solemn fast, sermons and prayers at every parish church, for the precious blood of our late pious Sovereign *KING CHARLES THE FIRST*, of ever glorious memory; but also by public dragging those odious carcasses of *OLIVER CROMWELL, HENRY IRETON, and JOHN BRADSHAW*, to *TYBURN*. On Monday night, *CROMWELL* and *IRETON*, in two several carts, were drawn to *HOLBORN* from *WESTMINSTER*, where they were dug up on Saturday last, and the next morning *BRADSHAW*. To-day they were drawn upon sledges to *TYBURN*; all the way (as before from *WESTMINSTER*) the universal outcry and curses of the people went along with them. When these three carcasses were at *TYBURN*, they were pulled out of their coffins, and hanged at the several angles of that triple tree, where they hung till the sun was set; after which, they were taken down, their heads cut off, and their loathsome trunks thrown into a deep hole under the gallows. The heads of those three notorious regicides, *OLIVER CROMWELL, JOHN BRADSHAW, and HENRY IRETON*, are set upon poles on the top of *WESTMINSTER HALL*, by the common hangman; *BRADSHAW* is placed in the middle (over that part where that monstrous High Court of Justice sat) *CROMWELL* and his son-in-law *IRETON*, on both sides of *BRADSHAW*.—*Merc. Pub.* Jan. 31 to Feb. 7.

There is a good deal that requires comment in these reports. It is clear, however, that what finally happened was that the coffins, not the uncoffined

* The writer of this paragraph seems as doubtful as Mr Pepys of the wisdom of this act. "December 4. This day the Parliament voted that the bodies of Oliver, Ireton, Bradshaw, &c should be taken up out of their graves in the Abbey, and drawn to the gallows, and there hanged and buried under it; which—methinks—do trouble me that a man of so great courage as he was, should have that dishonour, though otherwise he might deserve it enough." Besides the bodies of Cromwell, Ireton and Bradshaw, those of Admiral Blake, Pym, Mrs Claypole, and Dr Twisse were removed from the Abbey. The two latter found a resting place in St Margaret's Churchyard, the two former were thrown into a pit outside the Abbey doors. It was thus England allowed the royalists to treat one of its greatest sea-generals.

† For the mason's receipt for opening the graves, see our Plate XI.

carcasses, were drawn on sledges to Tyburn, where the bodies were "pulled out of their coffins" and hanged on the triple gallows. But much discussion has occurred as to the wording of the orders by the Lords and the Commons as to the exhumation.

It will be seen from the above that the "Honorable" House of Commons ordered the *bodies* of the regicides to be taken out of their graves, and drawn on a hurdle to Tyburn. The *Historical Manuscripts Commission Appendix to 7th Report*, p. 137, has in the House of Lords Calendar, 1660, December 16, Draft Order for the carcasses of Cromwell, Ireton, Bradshaw and Pride to be taken up, drawn upon a hurdle to Tyburn, and there *hanged up in their coffins* for some time, and afterwards buried under the gallows. Whatever the Lords may have done to modify the Commons' Order, it was accepted by the Commons.

The *Calendar of State Papers, Domestic Series, Charles II*, 1660—1661, has the following entries:

Dec. 7, 1660. The Commons have ordered the carcasses of Cromwell, Bradshaw, Ireton and Pride to be taken up, drawn on *sledges* to Tyburn, hanged and buried under the gallows [p. 406].

Dec. 10, 1660. Order by the Lords and Commons in Parliament that the carcasses of Oliver Cromwell, Henry Ireton, John Bradshaw, and Thos. Pride, wherever buried, be taken up, drawn on a hurdle to Tyburn, *hanged in their coffins* for some time, and then buried under the gallows; the Serjeant-at-Arms for Parliament is to see this performed by the common executioner, the Sheriffs of Middlesex, and Dean of Westminster to give the necessary directions therein [p. 408].

Cobbett's *Parliamentary History of England*, 1808, Vol. iv. A.D. 1660—1668, p. 158, provides the following account of the order:

Dec. 8, 1660. The Lords returned the Order sent up to them before, for taking up the bodies of Cromwell etc. with a small addition to it, which was agreed to. The Order as entered in both the Journals, stands thus, viz.: "Resolved, by the Lords and Commons assembled in parliament, That the carcasses of Oliver Cromwell, Henry Ireton, John Bradshaw, Tho. Pride, (whether buried in Westminster Abbey, or elsewhere) be, with all expedition, taken up, and drawn upon a hurdle to Tyburn, and there *hanged up in their coffins* for some time; and after that, buried under the said gallows; and that James Norfolk, esq. serjeant at arms, do take care that this Order be put in effectual execution by the common executioner for the county of Middlesex; and all such others, to whom it shall respectively appertain, who are required, in their several places, to conform to, and observe, this Order, with effect; and the sheriff of Middlesex is to give his assistance herein, as there shall be occasion; and the dean of Westminster is desired to give directions to his officers of the Abbey to be assistant in the execution of this Order*."

The italics are ours throughout these quotations. But the bodies were not *hanged up in their coffins*, but were drawn on sledges to Tyburn in their coffins and then taken out of their coffins and hanged on the triple gallows. Did the Sheriff of Middlesex exceed his orders, or has the common order of both Houses been wrongly read?

* This is identical with slight changes in capitals, italics etc. with the Order as it appears in the *Journals of the House of Lords* (Dec. 10, 1660), Vol. xi. p. 204. A newsletter of Dec. 6, 1660, runs: "The House of Commons voted yesterday the carcasses of Cromwell, Ireton, Bradshaw and Pride to be forthwith taken up and drawn upon a hurdle to Tiburne, there hung in coffins a while, and then buried under the gallows. It is intended to be moved that the hangman *pro tempore* may forever hereafter [be] give[n] Cromwell's arms; but that perhaps, being a wrong to others of the name better deserving, may not be granted." *Hist. MSS. Commission Fifth Report*, Vol. iv. p. 98.

The record is preserved in the Speaker's Library at the House of Commons and is reproduced in our Plate X by the kind permission of the Speaker.

Resolved

That the carcasses of Oliver Cromwell, Henry Ireton, John Bradshaw and Thomas Pride, whether buried in Westm^r. Abby or elsewhere be wth all Expedition taken up and drawn upon a hurdle to
in their coffins

Tyburn and there hang'd up for some time and after that buried under the said* gallows

If the circumflex over the first "l" of gallows was in the original draft, then there can be no doubt that the Lords' amendment was to the effect that the hanging up was to be in their coffins. On the other hand it may be that the circumflex is a more recent addition†. It does not, however, show in the reproduction any difference of ink. Failing the circumflex, the words "in their coffins" might possibly be inserted in the line above after "drawn." *The Parliamentary Intelligencer* (see above) introduced the Lords' amendment "in their coffins" after "hang'd up." It has been suggested that the Lords' addition had reference to the possibility that the "carcasses" were possibly not in a condition to be hung‡; it is just as likely that it was to prevent the bodies being destroyed or disfigured before the final scene at Tyburn. Anyhow the Sheriff interpreted the order as meaning that the bodies *not* in their coffins were to be hung on the gallows. We can understand that the *Calendar of State Papers* and Cobbett might misread the resolution but it is certain that *The Parliamentary Intelligencer* under December 8th, 1660, gave the resolution in the form "hanged up in their coffins."

The so-called Convention Parliament, which recalled Charles to England, and voted the exhumation on December 6th (modified by the Lords, December 8th), was dissolved on December 29th, and the new Parliament did not assemble till May 8th. A return may have been made of how the Order of Parliament was carried out by Serjeant Norfolk and the Sheriff of Middlesex, but no trace of it has so far come to light. Nor is there any reference to the matter in the Journal of the Privy Council for January 1660—1. It is possible therefore that, Parliament not sitting at the time, the ribald and revengeful Cavalier spirit did not hesitate to exceed the Order of the Houses, which in itself was base-minded enough. We have not been able to find the Sheriff's papers or accounts at the Middlesex County Hall.

The Mercurius Publicus states that on January 30th the bodies were pulled out of their coffins at Tyburn, which would indicate that the coffins were taken to Tyburn, whether open or not we do not know. Possibly some preparation of the bodies was made at the Red Lion in Holborn or in Westminster Abbey before the taking to Holborn, with a view to the proceedings at Tyburn.

* There has been no previous mention of gallows.

† The actual record was first reproduced in a very poor facsimile by Mr A. A. Taylor in *Pearson's Magazine*, January 1897, and he was of opinion that "in their coffins" should be inserted after "drawn." Such an opinion would hardly have been possible had he seen the circumflex. Hence either he judged from his facsimile, where no circumflex is clearly visible, or it did not then exist.

‡ This appears to have been the view of the Speaker in 1897, who might easily have settled the matter by pointing to the circumflex.

Pearson and Morant: *The Cromwell Head*

Dr. Dobson.

will inform that the carriage of Oliver Cromwell's bones
belonging to John & Bradshaw & Thomas Pride were
at Stonehouse in a Westons May or the winter but no all

Exposed in a box up and drawn upon a trolley to
Tyburn. The stone hanged off for some time after that
buried under the same ground. And that same

(Mortgage) say 5. 0. 0. sent out from attending
the mortgage of Commons has been sent back
or over bill put in spirituals Exemption
Mortgage

then sent out to the mortgage
Debit and credit. What is to say it is 6, 0, 0

Facsimile of the Resolution of the House of Commons for the exhumation of the bodies of the Regicides and their hanging on the gallows
at Tyburn, with the Lords' amendment: "in their coffins." From the Library of the Speaker, by his kind permission.



a Copy of the Mason's Receipt for taking up the Corpses.

*Received May 4th 1661, the sum of fifteen Shillings in full
of the Worshipfull Jurys^t Norfolk for taking up the corpses
of Cromwell, Jorton, & Brasaw From John Lewis
Cromwell Jorton Bradshaw.*

Facsimile of the Mason's Receipt for the money for digging up the bodies of Cromwell, Ireton and Bradshaw. From Dr Cromwell Mortimer, in the possession of Karl Pearson.

While the *Mercurius Publicus* only states that on "Monday night [Jan. 28th], Cromwell and Ireton in two several carts were drawn to Holborn from Westminster, where they were digged up on Saturday last [Jan. 26th], and the next morning Bradshaw [Jan. 29th]," White Kennett, although writing at a considerably later date, gives a more detailed account, which except in one particular seems to have verisimilitude*. After giving on his p. 326 the order of the Lords and Commons in the "hang'd up in their Coffins" form, he continues as follows on p. 327:

The Carcasses of Oliver C. and Henry Ireton... were taken on Saturday Jan. 26, 1660 [1661, new style] and on the Monday night following [Jan. 28] were drawn in two several Carts from Westminster to the Red Lyon in Holborn, where they continued all night. The Corps of Bradshaw buried in the Abbey Church Novem. 22 1659, was green and stank, therefore was not taken up 'till the Morning following [Jan. 29], and then was carried in a Cart to the Red Lyon†, and the Day following, being the Day of the Royal Martyrdom [Jan. 30] they were drawn to Tyburn in three sledges, where they were pull'd out of their Coffins and hang'd on the several Angles on the Gallows, where they hung 'till the next Day Sun-set ["next Day" is highly

* *A Register and Chronicle...*, Vol. 1. 1728.

† This stay at the Red Lion has given rise to various wild tales, such as the rumour that the bodies were interchanged with others during those two nights; it is hardly likely that the royalists would have left them unguarded or with untrustworthy guards. Even the verse composed about the stumpy obelisk in Red Lion Square (it was not inscribed on it)—

Obtusum obtusioris Ingenii Monumentum
Quid me respicias, Viator? Vade—

has been read to cover some mysterious reference to the burial of Cromwell at or near that site. (Are we to render it

"Blunt Monument of a blunter mind,
Why do you regard me, Passer-by? Go!")

To us it seems only a wag's fun at the designer of such an obelisk (see Pennant's *London*, 1790, p. 165), but others have thought even seriously that a Cromwell myth was connected with it! Among the older of these believers we have to note the Prestwichs. In 1787 appeared *Prestwich's Respublica or a Display of the Honors, Ceremonies, & Ensigns of the Common-Wealth under the Protectorship of Oliver Cromwell*. This work is said to be based on the papers of two ancestors, contemporaries of Cromwell, viz. Edmund Prestwich and John Prestwich, Fellow of All Souls. With the accounts of the arms of Cromwell and his supporters and of his funeral expenses we are not concerned; the latter appear to be extracted from MSS. now in the British Museum. But when we come to the relation of his death and funeral we find nothing new, only Bate's account of the autopsy and the *Mercurius Politicus* narrative of the lying-in-state with here and there a slight addition or change of wording to suit the views of the writer [or the editor, for we are never certain which is speaking]. But in the *Contents* under No. XVI we are told that we are to read about the armorial bearings, etc. of Cromwell and his supporters—

The same of those who signed the death-warrant of Charles Stuart, together with the secret where his Highness Oliver Lord Protector was interred, first made known to the Editor by the only remaining honorable person whose ancestor alone knew it.

The reader will observe that it is not the contemporary of Cromwell, but the *Editor* John Prestwich of 1787, who is here speaking! The great secret is not discussed in the book at any length, but on p. 149 under the section devoted to the arms of Cromwell we read:

His family of Huntingdonshire. His remains were privately interred in a small paddock near Holborn; in that very spot over which the obelisk is placed in Red-Lion-Square, Holborn—THE SECRET.
JOHN PRESTWICH.

There is no statement of when Cromwell's body was supposed to be interred in Red Lion Square. Was it immediately after his death, *pace* Bate, or after being stolen from his coffin when it rested in Red Lion Square? How could in either case only one man know about it, and why was only one "remaining honorable person" in possession of this great secret? The whole "secret," without any historical backbone to it, is absurd, although it appears to have satisfied John Prestwich and even some more modern writers.

improbable] at which time they were taken down, had their Heads cut off, and the Trunks thrown into a deep Hole under the Gallows, which serves for the monument of their Grave and Merit.

We have come across no other record which refers to the bodies being hung on the gallows one day and taken down on the next day at sunset.

The three diarists who touch on this great event, Townshend, Evelyn and Pepys, at least agree in the Tyburn incident being confined to one day.

In Townshend's *Diary**, Vol. I. p. 82, we read:

The carcases of those two horrid Regicides, Oliver Cromwell & Henry Ireton, were digged up in H. 7 Chapelt†, which with those of John Bradshaw (and Tom Pride) are to be hanged up at Tyburn & buried under the Gallows the 30th, was performed; and hung at the 3 Corners of the Gallows until Sunset. Then taken down, having their heads cut off and carcases thrown in a hole, being the day they murdered Charles the first 1648.

This does not convey anything new to us, and bears signs of not being written at one time.

Evelyn, under date of January 30, 1661, writes:

This day (O the stupendous and inscrutable judgements of God!) were the carcases of those arch-rebels, Cromwell, Bradshawe (the judge who condemned his Majesty), and Ireton (son-in-law to the Usurper), dragged out of their superb tombs in Westminster among the Kings, to Tyburn, and hanged on the gallows there from nine in the morning till six at night, and then buried under that fatal and ignominious monument in a deep pit, thousands of people who had seen them in all their pride being spectators.

Evelyn was probably not a spectator, and we could spare his moralising for a touch of the actual scene. What kept Pepys from seeing something of the proceedings? Was it his puritan upbringing? As a rule he never missed a great occasion, but on this he—went to church!

Jan. 30th (Fast day). The first time that this day hath been yet observed: and Mr. Mills made a most excellent sermon, upon "Lord forgive us our former iniquities"; speaking excellently of the justice of God in punishing men for the sins of their ancestors. To my Lady Batten's where my Wife and she are lately come back again from being abroad and seeing Cromwell, Ireton and Bradshaw hanged and buried at Tyburne.

Heath concludes his *Flagellum* (p. 200) with these words:

On the 30. day of January 1660 [old style], that day 12. years of his most nefarious parricide, his Carcass with *Bradshaws & Iretons*, having been digged out of their Graves, were carried to the *Red Lyon* in *Holborn*, and from thence drawn in Sledges to *Tyburn*, where they hanged from Ten of the Clock in the morning till Sun-set, with their Faces towards *Whitehal* and were then inhumed under the *Gallowes*, and his head set upon *Westminster-hall* to be the becoming Spectacle of his Treason, where on that *Pinnacle* and *Legal Advancement* it is fit to leave this Ambitious Wretch.

Here is, perhaps, one of the earliest mentions of the *Red Lyon Inn*. Otherwise the information is imperfect. There is one elsewhere unrecorded bit of news, i.e. that the corpses were hung to face Whitehall‡.

* Edited by Willis Bund for the Worcestershire Historical Society, 1915.

† Ruge's *Diurnal* after using much the same words adds: "Cromwell's vault having been opened the people crowded very much to see him."

‡ Ruge's *Diurnal*, with a like error as to the dates on which the several bodies were taken to the Red Lion Inn in Holborn, but with an important addition, runs as follows:

Jan. 30th was kept as a very solemn day of fasting and prayer. This morning the carcases of Cromwell, Ireton and Bradshaw (which the day before had been brought to the Red Lion Inn, Holborn),

We should have but a poor impression of the scene at Tyburn had we only these accounts to go by. Luckily there was another witness*, namely Sainthill, who was close to the corpses, and gives an account of what he saw, and a sketch as well, in a manuscript which in 1793 to 1799 was in the possession of Mr T. B. Colwich, of Farrington-house, near Exeter. It was consulted at that time by Noble† and Nash‡, but we are not aware what has now become of it. The whole manuscript would probably be worth printing if rediscovered. Noble must have read the passage, for he quotes half-a-dozen words from it, but it was probably thought by him too coarse for refined ears. Nash, as befits an editor of *Hudibras*, was luckily less nice, and we are grateful to him for the light his quotation throws on Cromwell's head. The account of Sainthill runs as follows:

The 30th of January, being that day twelve years from the death of the King, the odious carcasses of O. C., Major General Ireton, and Bradshaw were drawn in sledges to Tyburn, where they were hanged by the neck from morning till four in the afternoon. C. in a green-seare cloth, very fresh embalmed§; Ireton having been buried long, hung like a dried rat, yet corrupted about the fundament. Bradshaw in his winding sheet, the finger of his right hand and nose perished having wet the sheet through; the rest very perfect, in so much that I knew his face, when the hangman, after cutting it off, held it up; of his toes I had five or six in my hand, which the prentices had cut off. Their bodies were thrown into an hole under the gallows, in their seare

were drawn upon a sledge to Tyburn, and then taken out of their coffins, and in their shrouds hanged by the neck, until the going down of the sun. They were then cut down, their heads taken off, and their bodies buried in a grave under the gallows. The coffin in which was the Body of Cromwell was a very rich thing, very full of gilded hinges and nails.

The last sentence is to be compared with the statement on our p. 298, as to the "sumptuous coffyn" of James I, which was "covered without with purple velvett, the handles, nayles and all other iron-work about it being rychly hatched with gold."

* Another person who witnessed the hanging was Lauderdale, but he merely tells us that the only time he saw Ireton was when he was hanging on the gallows at Tyburn. *Scottish Historical Miscellany*, Vol. i. p. 250.

† *The Protectoral House of Cromwell*, Edn. 1787, Vol. i. p. 290, footnote: "The MS. was written by Sam. Sainthill, esq. a Spanish merchant; whilst in Spain he made many observations, particularly, upon the Escorial, nor did anything escape his observation in England during the civil wars; his Journal is too concise to be of service to the historian [*sic*]; but the smallness and the neat hand it is written in much enhance its value as does the elegance of the drawing of such things as most engaged his attention; there are besides several curious relations of things he had seen in various places, some of them worthy of remembrance."

‡ T. R. Nash's edition of Butler's *Hudibras*, 1793, republished in three large volumes. Vol. III. p. 378. Nash calls the writer of the Journal *Edward Sainthill* and Noble *Samuel Sainthill*.

§ The manner in which history is written is curious indeed! Howarth (*loc. cit.* p. 9) states on the basis of the above words that we have "the witness of one, who saw it then [i.e. at Tyburn] that it was extraordinarily preserved and that it was so carefully and elaborately embalmed that it was at that time very fresh in cere-cloth." Is this purely Howarth's own inference from "the very fresh embalmed" of Sainthill? No! There are intervening stages. We have Sainthill's actual words as above, then Noble (*loc. cit.* Vol. i. p. 290) referring to Sainthill's MS. writes: "The author, who was an eye-witness, says, the protector's [carcase] was in green cerecloth, very fresh embalmed." There is no distortion in this. Next we come to Mr Josiah Henry Wilkinson's narrative. He after referring to the state of preservation of the Head says it may be accounted for "by the fact related in Sainthill's manuscript, quoted in Noble's *Memoirs*, Vol. i.—Pag. 290 where the author mentions his 'being an eye witness of the state of the bodies, & that Cromwell's was so carefully and elaborately embalmed, that it was at that time, very fresh in green cere-cloth'." Although the sentence from Noble is quoted between inverted commas, the words "so carefully and elaborately embalmed" do not occur in either Sainthill's manuscript or Noble's work. Finally Howarth adds the words "extraordinarily preserved" as if they were in the original.

cloth and sheet. C. had eight cuts, Ireton four being seare cloths, and their heads were set up on the South-end of Westminster Hall.

Merian in his *Theatrum Europaeum*, Theil 9, i. 1660—62 (published in Frankfurt 1672) under the year 1660, S. 386—87 gives, besides the usual accounts, certain additional information. The bodies were taken to a "Wirtshaus" in Holborn, and exhibited there to all who wished to see them; they were drawn in *open* coffins to Tyburn. Cromwell was in green and Ireton in white cere-cloth, and Bradshaw, whose body was putrescent, enveloped in a coloured cloth (ein farbiges Tuch). As to the heads on Westminster Hall, Bradshaw's (as he was presiding judge) was placed in the centre, Cromwell's on his right hand and Ireton on the left.

Mr L. M. Anstey (*Notes and Queries*, May 5, 1934, Vol. 166, p. 317) kindly drew our attention to another witness, Peter Munday, of the scene at Tyburn; he gives an account of the proceedings in the last volume of his *Travels* now in preparation for the Hakluyt Society. In his Diary 30 Jan. 1660/1 he writes:

The bodies of Cromwell, Ireton and Bradshaw were drawn from Westminster on sleds to Tiburne, and there hanged on the three parts of the gallows, Cromwell and Ireton wrapped in searchcloth, supposed to be embalmed, but Bradshaw in a winding sheet. The body turned to putrefaction cast a most odious sent all the way it went. They hung four or five houres untill sunset, then cut downe, the heads severed from the bodies. The three bodies, all three, were tumbled together into one pit under the gallows.

This confirms the view we have already expressed, that the corpse of Bradshaw was the chief source of offence. For somewhat fanciful views of the scene at Tyburn, see our Plates XIII and XIV.

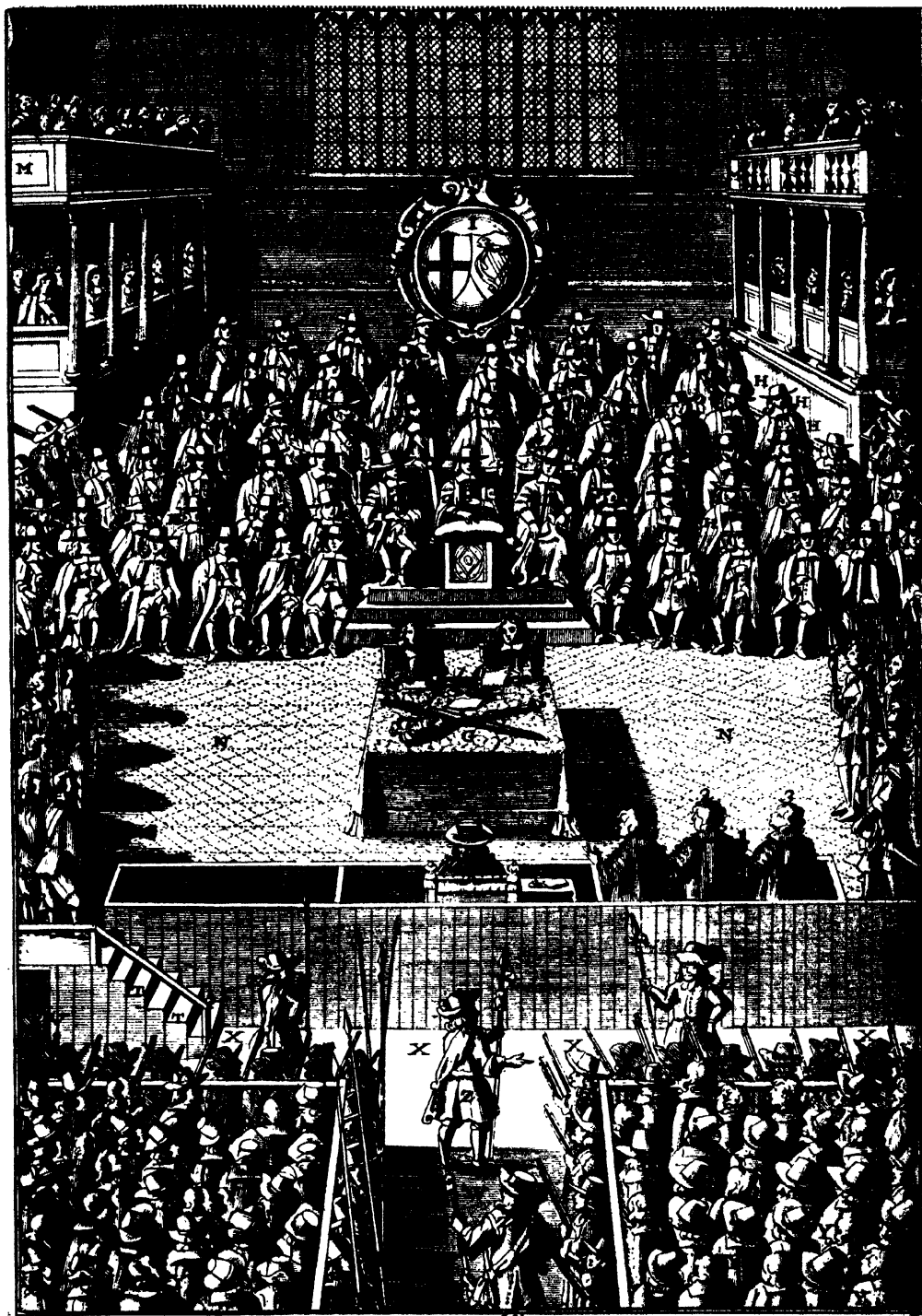
An unpleasant picture indeed of the ribaldry of the times, loosened as it was by the royalist revival. Sainthill made a drawing of the scene at Tyburn, with the bodies hanging and the grave underneath. Cromwell is represented like a mummy swathed in his sixfold cere-cloth, with no visible legs or feet.

There seems no reason to question that the head, which was cut off by eight blows of the axe owing to the thickness of the cere-cloth, was that of Oliver Cromwell, nor is it likely that the authority which superintended the proceedings at Tyburn would be so lax as to admit of any risk of losing the head before it was placed on the pole attached to the roof of Westminster Hall. The heads do not appear to have been placed on the Hall for several days*. It is stated by some that they were set up on February 6th, but Pepys saw them on February 5th†. We are told in the *Mercurius Publicus* exactly why the heads were placed on the south or farther

* Somewhere, but we cannot recollect where, we have read that it was customary to dip the heads of traitors in some tar preservative before exposing them. This statement if verified would be important in the present case.

† *Diary*, Feb. 5, 1660—61: "Into the Hall and there saw my Lord Treasurer (who was sworn today at the Exchequer, with a great company of Lords and persons of honour to attend him) go up to the Treasury Offices, and take possession thereof; and also saw the heads of Cromwell, Bradshaw and Ireton, set up at the further end of the Hall."

It is obvious that "set up at the further end of the Hall," following on the previous statement that Pepys went into the Hall, does not necessitate their being inside the Hall. It is possible that they were inside the Hall for a day or so, but their ultimate resting place was undoubtedly *outside* the far end.



Interior of Westminster Hall during the Trial of Charles I. Bradshaw presides, and Cromwell is on his right at K.



The Triple Gallows at Tyburn, with the pit for burial below the Gallows.

From a Dutch Print.



The scene at Tyburn. The building seen *under* and *above* the gallows appears to be Westminster Hall. Whether the building on which the heads are poled is supposed to be the South end of the Hall, we cannot say; it is more like the West side curtailed. From a Dutch Print.

(from the main entrance) end of the Hall. It was because at that end the judges of the High Court of Justice had sat with the President, Bradshaw, in the centre. Oliver Cromwell was on his right, if much towards the rear, where Henry Martin and he supported the Commonwealth escutcheon. See our Plate XII.

Thus the *Mercurius Publicus*, Jan. 31 to Feb. 7, 1660—1, writes:

The heads of those three notorious regicides, Oliver Cromwell, John Bradshaw and Henry Ireton are set upon poles on the top of Westminster Hall, by the common hangman; Bradshaw is placed in the middle (over that part where that monstrous High Court of Justice sat), Cromwell and his son-in-law Ireton on both sides of Bradshaw.

The important facts here are that the heads were "set upon poles," "on the top of Westminster Hall," and "over that part where" the High Court of Justice sat, i.e. at the south end: see Plate XII. It has been asserted that Cromwell *must* have been in the middle, and that the heads could not have been at the south end, as they would not have been visible because there were buildings there. The above extract explains why they were placed at the south end*. There certainly were buildings at the south end, but those who think they could have obscured the south gable of the Hall are dreaming of buildings like the Committee Rooms of the present House of Commons or of the older Houses of Parliament† destroyed in the fire of 1834. As a matter of fact, in the time of Charles II, the old prints show us that the south end of Westminster Hall was equally conspicuous from the park of St James and from the Lambeth side of the river: see our Plates XV—XVIII. Heads placed on 15 to 20 feet poles (as on Temple Bar: see our p. 291), would be conspicuous features not only from Old Palace Yard but from a wide surrounding area.

That there were certain buildings at the south end of the Hall as well as on the west side is clear from the statements we have about them as well as from the prints themselves, but what the arrangement of these buildings was is very obscure. Also it is important to know whether Cromwell's head was to the east or west of Bradshaw's. If Bradshaw's face was turned looking down the Hall, as it did in the High Court of Justice, then Cromwell's head, as its owner was placed on that occasion, would be on Bradshaw's right (Merian: see our p. 314) or on the eastern half of the gable, with Ireton on Bradshaw's left‡ or towards the south-west corner of the Hall. In the Dutch print (see our Plate XIV) Cromwell is certainly on Bradshaw's right, but the heads look as if they were poled on the west, not the south side of the Hall. If Bradshaw's head did not face down the Hall, but towards Old Palace Yard—a more reasonable aspect for spectators—then Cromwell, if on his right, would be on the western side of the southern gable of the Hall. Supposing Cromwell's and Ireton's

* For Pepys, living then in Axe Yard, to the west of King's Street at its northern limit, the "further end" of the Hall was the "south end." Axe Yard branched off to the west of King's Street, not very far from the Gate Tower at the north end of that street. If one passed southwards through that Gate, he had first on his right or west side Downing Street, then "Duffin's Ally" and next "Ax Yard." Cf. Strype's map facing p. 649 of Vol. II. of his 1755 Edition of Stow's *Survey*.

† Even in the case of the older Houses of Parliament Thos. H. Shepherd's drawing engraved by W. Deeble shows the southern gable of Westminster Hall visible above the then (1829) buildings from Old Palace Yard.

‡ We have seen no record of where Ireton sat in the High Court.

heads were not close up to Bradshaw's, fixed indeed to the same central pinnacle*—and the poled heads on a building seem to have been kept well apart (see our pp. 291—292)—then the question of whether Cromwell's head was east or west of Bradshaw's becomes of importance, having regard to where it would be blown down, especially if the heads of Cromwell and Ireton had their poles fixed near the springings of the roof†.

9. *The Taverns: "Heaven" and "Hell."*

Still the question remains unanswered whether Cromwell's head was to the east or west of the gable peak. Here Samuel Butler with his *Hudibras* may be of service‡. Can he really aid us? To understand Butler it is necessary to remark that Peter Sterry, who had been chaplain to Oliver, was a famous roundhead divine who preached the postponed sermon§ on the day following Cromwell's death. He had dreamed that Cromwell was on his way to Heaven. So he was, implied Butler, for:

Toss'd in a furious hurricane
Did Oliver give up his reign,
And was believ'd as well by Saints
As moral men and miscreants
To founder in the Stygian ferry,
Until he was retrieved by Sterry,

Who in a false erroneous dream
Mistook the new Jerusalem
Profanely for the apocryphal
False heaven at the end o' th' Hall,
Whither, it was decreed by Fate
His precious reliques to translate.

Part III, Canto II, 215.

Now "Heaven" was the name of a tavern adjacent to or some say under Westminster Hall. Hence, if Butler's poem is to be taken literally, the position of

* This is highly improbable, as it would have been difficult to place Harrison's head between those of Bradshaw and Cromwell. See our p. 326.

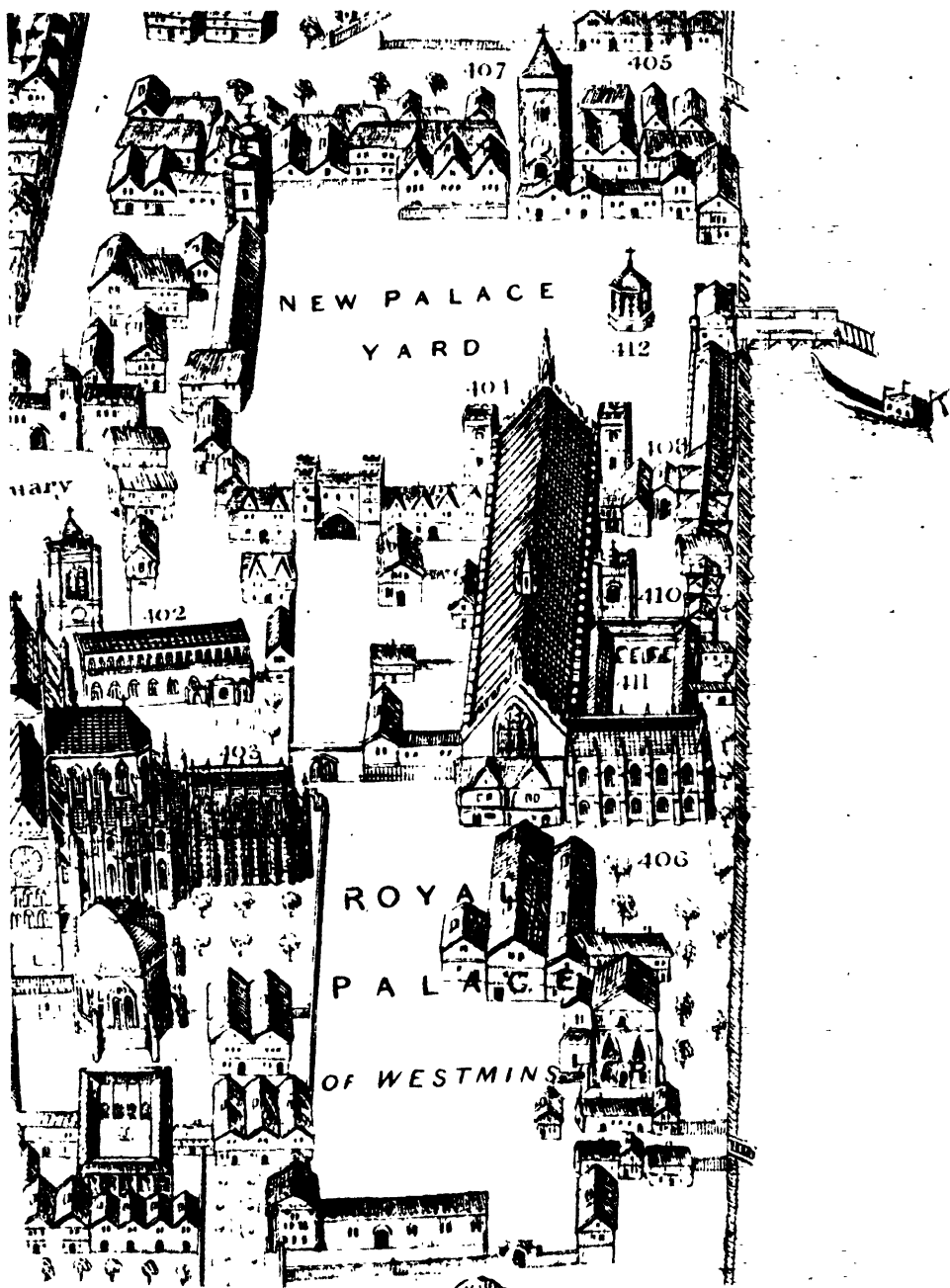
† Most of the views of Westminster Hall provide only a flat coping stone at the gable ends and this apparently close to the roof itself. On the other hand Hollar's engraving shows small vertical knobs (?) at intervals adorning (?) the gable copings at both ends. Morgan's shows them only at the north end. It does not appear that either coping or "knobs" would provide adequate support for poles, which might, however, have been fixed by brackets to the facing wall of the south gable. It is thus, perhaps, more likely that Cromwell's and Ireton's poles were at the springings of the roof.

‡ *Hudibras* came out in December, 1662. Pepys (Dec. 26th) bought a copy of *Hudibras* for 2s. 6d., but thought it so silly abuse that he sold it promptly for 18d.! The fact that Cromwell's head was on the Hall in 1662 does not add much to our knowledge of how long it stayed there.

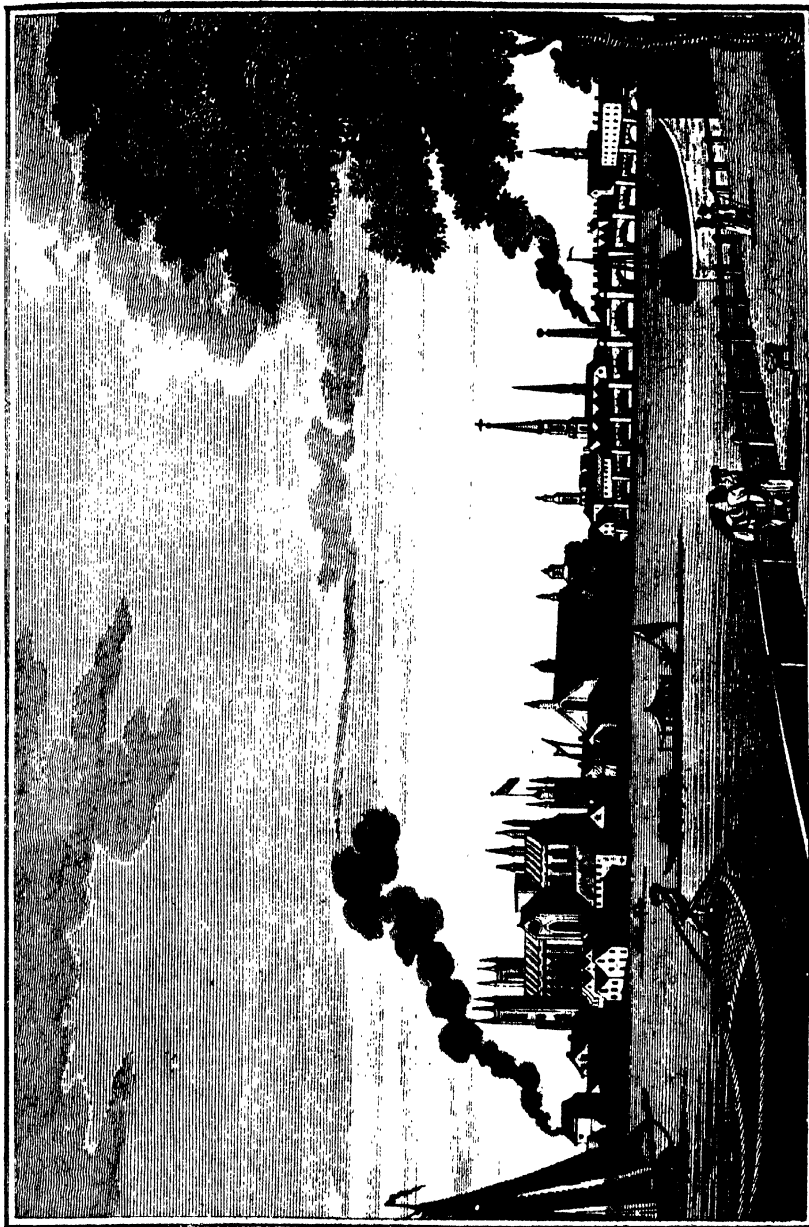
§ See our p. 303 for an explanation of the postponement. We have been unable to find that this sermon was ever published. Sterry was a mystic and a man of culture. He was among the chaplains assembled to pray for Cromwell in his last illness and, when the latter died, comforted them by saying that one who had been so useful in a mortal state would have the possibility of being more so now that he was translated. What appear to be his actual words are given in Morgan's *Phoenix Britannicus*, Vol. I. p. 154; Sterry holding forth his Bible said:

That if that were the Word of God, then as certainly that his Blessed Holy Spirit [the Protector's] was with Christ at the Right Hand of the *Father*; and if he be there, what may his Family and the People of God now expect from him? For if he were so useful and helpful, and so much Good influenced from him to them when he was here, in a Mortal State, how much more Influence will they have from him, now he is in Heaven? the Father, Son and Spirit, through him, bestowing Gifts and Graces &c. upon them.

Sterry, without probably recognising it, seems in that moment of trouble to have adopted the Catholic doctrine as to the intercession of saints!

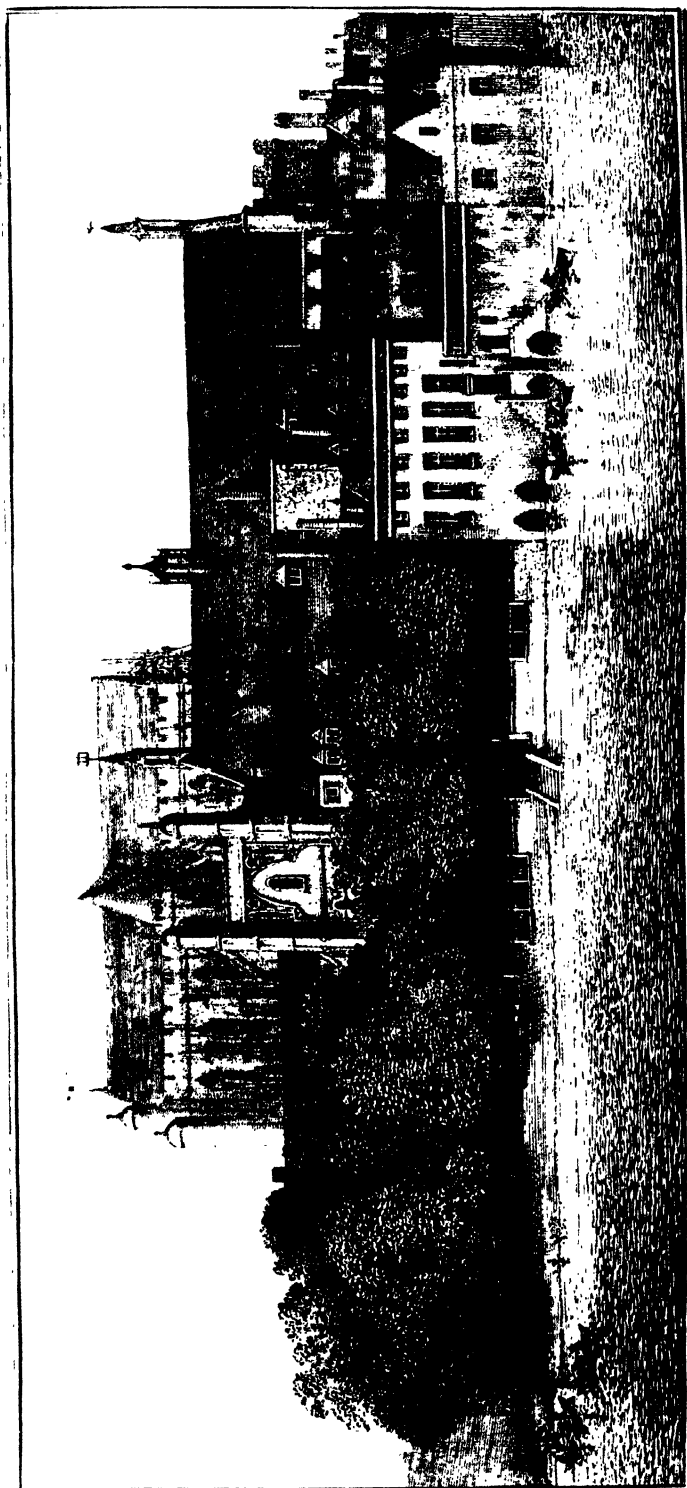
Pearson and Morant: *The Cromwell Head*

Plan of Westminster Hall and Palace in Tudor times, from William Newton's *London, Westminster and Southwark as in the Olden Times...* London, 1855. The plan shows jagged steps on both gables of the Hall, the Fish Yard to the West of the Hall, and the range of buildings at the South-West corner and others South of the Hall (the former may have included "Heaven").



View of the City of WESTMINSTER, &c.
taken near the Landing place at Lambeth Palace.

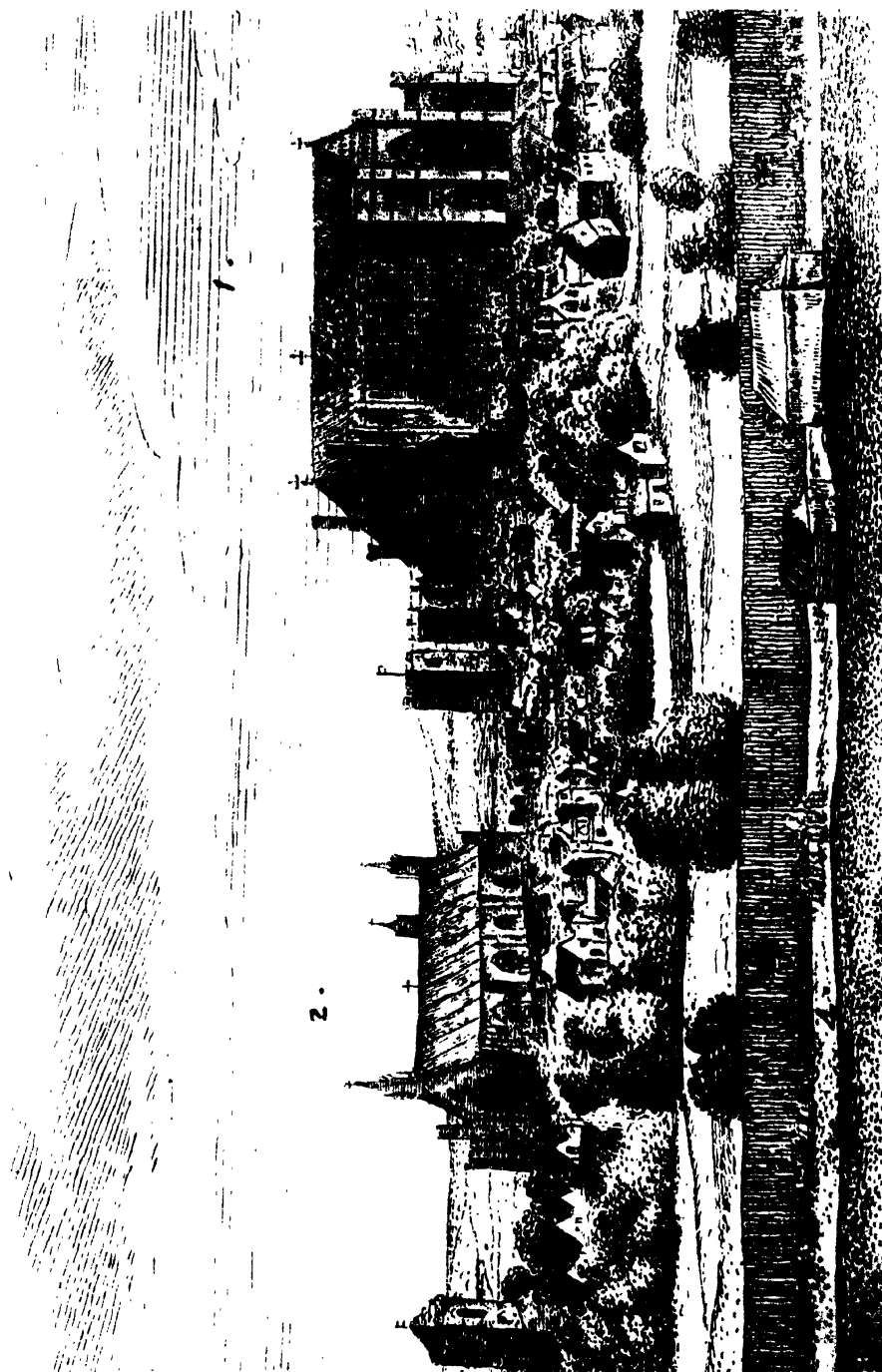
Print showing the buildings at South end of Westminster Hall, which would not hinder, however, the view from Lambeth and the river of heads poled at that end.



East View of the Palace of Westminster from the River Thames. The drawing is by J. G. Morant, and the engraving is by J. G. Morant. The drawing is signed 'J. G. Morant del.' and the engraving is signed 'J. G. Morant sc.' in the bottom right corner.

Print showing view of Westminster Hall from the River. The roofs of buildings at South-East corner would not screen heads poled on the South end of the Hall. Part of the parapet of the Hall, with windows opening on to it, is visible. Date of drawing circa 1722.

Pearson and Morant: *The Cromwell Head*



View of Westminster from St James's Park. Engraving by Hollar, circa 1660, of an ancient Drawing in the Collection of John Towneley, Esq. 1. is the Abbey, 2. the Hall. It is clear that heads poled on the South end of the Hall would be distinctly visible from the Park.

"Heaven" would determine that of Cromwell's head. Now the two taverns, "Heaven" and "Hell," are frequently referred to both in 17th and 18th century literature, and even before, and are linked up with the position of the Exchequer.

T. R. Nash in the notes to his splendid edition of *Hudibras** has the following statement on the above-cited passage.

Peter Sterry dreamed that Oliver was to be placed in Heaven, which he foolishly imagined to be the true and real Heaven above; but it happened to be the false carnal Heaven at the end of Westminster Hall.... There were at that time, two victualling-houses at the end of Westminster Hall, under the Exchequer, the one called "Heaven," the other "Hell"; near to the former Oliver's head was fixed, January 30, 1660 [?].

If we could accept Nash's statement†—and he had a good antiquarian spirit—"Heaven" was at the end of Westminster Hall, and that could only mean the south end where Cromwell's head was placed. Further, he tells us that both taverns, "Heaven" and "Hell," were "under the Exchequer." Thus the position of the Exchequer should fix the position of Cromwell's head. But did Nash mean the Exchequer Court or the Exchequer Office, and if the two were not together, under which were the taverns? Lord Braybrooke, in his edition of Pepys' *Diary* below the entry for January 28, 1659—1660: "And so I returned and went to Heaven, where Luellin and I dined," has the note:

A place of entertainment in Old Palace Yard, on the site of which the Committee Rooms of the House of Commons were erected some years ago. It is called in *Hudibras*, "False Heaven at the end of the Hall."

Further references to "Heaven" occur in Pepys' *Diary* for August 30 and November 12, 1660. If according to Nash "Heaven" were under the Exchequer, the Exchequer must have been at the south end of Westminster Hall. This is hardly admissible. At this end of the Hall was a passage leading to St Stephen's Chapel, i.e. the later House of Commons. It was crossed by an arch giving access from the "steps" at the south end inside the Hall to the *White Hall*, which projected southwards from the eastern half of the end of the Hall, but not contiguous with it (see our Plates XV, XVI and XXIII). In the White Hall were located at various periods the Courts of Wards, Liveries and Requests.

Those gentlemen who had been restrained in the Court of Wards were led through Westminster Hall by a strong guard, to that place under the Exchequer, commonly called Hell‡,

* *Hudibras* of Samuel Butler, in three volumes, with engravings by Hogarth, 1793. Vol. III. contains the Notes; see pp. 378—380.

† Zachary Grey (1688—1766) published an edition of *Hudibras* in 1744. In a note on the same passage of that work, he speaks of "Heaven" in the present tense. "His head [was] set up at the farther end of Westminster Hall near which place there is a house of entertainment, which is commonly known by the name of Heaven." The reader will note that Grey uses precisely Pepys' description of the south end of the Hall.

‡ "Hell" must in 1649 have been accessible from the Hall itself or possibly from the Courts. For we read that at the trial of Charles I:

all the narrow avenues to the Hall were either stopped up with masonry or strongly guarded, and it was expressly ordered "that all Back doors from the House called *Hell* be stopt up during the King's Tryal." *A True Copy of the Journal of the High Court of Justice for the Tryal of K. Charles I.* London, 1684, p. 19.

where they might eat and drink at their own costs what they pleased*. (Nash in Notes to *Hudibras*, Vol. III. p. 378.)

John Stow describes the position of the several courts in relation to the Hall. We have already seen that the Court of Wards and Requests was in the White Hall accessible from a passage running behind the Court of King's Bench which was on the steps in the south-east corner of the Hall. On the steps in the south-west corner sat the Chancellor and his assistant judges. The Court of Common Pleas still in the Hall itself was near the entrance in the north-west angle (see Plate XXI). Now we come to the Exchequer. Here we will quote Stow exactly:

Within the port, or entry into the hall, on either side are ascendings up into large chambers, without the hall adjoining thereunto, wherein certain courts be kept, namely on the right hand is the court of the Exchequer, a place of account for the revenues of the crown; the hearers of the account have auditors under them; but they which are chief for accounts of the prince, are called barons of the Exchequer, whereof one is called the chief baron of the Exchequer. The greatest officer of all is called the high treasurer. (1st Edn. 1603, p. 387.)

Stow continues speaking of the moneys paid into the Exchequer. It would thus appear as if in his day there was no distinction between Exchequer Court and Exchequer Office. On the east side in Stow's day was above the stairs the Court of the Duchy of Lancaster, the Office of the receipt of the Queen's revenues, and the Star Chamber. Stow tells us that

In the Palace Yard were anciently Pales [Enclosures] in which were two messes [?commensal groups] the one called *Paradise* and the other called the *Constabulary*; Both of which were granted to *John*, Duke of *Bedford*, 13 Henry VI.

Clearly they must have been feeding places of some profit. If *Paradise* was the original of the later "Heaven," it is fairly certain that the latter had a migratory career.

Again we are told that

Under the Hall are certain subterraneous appartments, which are called one *Paradise* and another *Hell*; Consisting of Tenements Houses and Mansions, which with other Tenements and Lands were held in King Edward the Sixth's Days by one William Fryes. (Strype's *Stow's Survey*.)

It is difficult to think nowadays of Houses and Mansions in cellars under Westminster Hall, but "mansion" in the 16th or 17th century signified little more than an "apartment." We do not know how far south these "subterraneous appartments" extended, but it seems at any rate probable that they were attached to and accessible from houses built-up against the west side of the Hall.

Old plans of Westminster Palace seem to show that at a later date than Stow (1593), the Exchequer Office was on the east side of the Hall, close to the north end. See our Plate XIX.

Edward Hatton in his *New View of London*, 1708, Vol. II. p. 658, tells us that the officers for the receipt of the Exchequer were in the *Lower Exchequer* near Westminster Hall. We might from this draw the conclusion that the Exchequer Office was under the Exchequer Court, and below the Office the two taverns

* The gentlemen were obviously prisoners who were confined for contempt of court owing probably to their treatment of the King's wards.



View from New Palace Yard of the North End of Westminster Hall. Coffee Houses built up against front on right, then Court of Exchequer Buildings, with "Hell" and "Paradise" beneath them. To left, door into Talley Court and Receipt of Exchequer; further to left, doorway to Speaker's Court.



A View of New Palace Yard, Westminster. The view is from the Palace Yard, looking down the street towards the entrance of the Exchequer.

This view is of some interest, because it seems to indicate in the wall or palisade in front of the Tudor buildings (Court of Exchequer) just above the two-horse coach in the foreground, the entrance to a cellar, from which two persons are coming out. Was this one of the entrances to the taverns under the Exchequer?

"Hell" and "Heaven." But if the Office were on the east side of Westminster Hall, at the north end, this would be wholly out of accord with Butler.

In 1755 Strype tells us that

The Street [St Margaret's Lane] between the two *Palace Yards* is now rebuilding, and *Hell* contiguous to *Westminster Hall* is soon to be rebuilt. (Strype's *Stow's Survey*, Vol. II. p. 641^b.)

This would certainly place "Hell" on the west side of Westminster Hall, and probably under the Exchequer Court, not under the Office if that was east of the Hall.

Sir Francis Palgrave in the Introduction to his work* refers to no less than *five* treasuries of the Exchequer, of which apparently only one concerns us:

A third Treasury was adjoining to the "Court of the Receipt of the Exchequer" commonly called the Tally Court in the Old Palace of Westminster.

The "Tally Court" was on the east side of the Hall: see Smith's plan, Plate XXIII.

Fuller in his *Worthies* has two entries which throw light on the matter. They run†:

PROVERBS.

As sure as Exchequer pay.)

All know, that the *Exchequer* was formerly the *Treasury* of the Kings of *England*, kept in this City, the *pleading part* on the one side, and the *paying part* on the other side of *Westminster-hall*. This proverb was in the prime thereof, in the reign of Queen *Elizabeth*, who maintained her *Exchequer* to the heigh, that her *Exchequer* might maintain her. The *pay* thereof was sure *inwards*, nothing being *remitted* which was due *there* to the *Queen*; and sure *outwards*, nothing being *detained* which was due *thence* from the *Queen*, full and speedy *payment* being made thereof. This Proverb began to be crost about the end of the reign of King *James*, when the credit of the *Exchequer* began to decay, and no wonder if the *streams* issuing thence were shallow, when the *fountain* to feed them was so low, the revenues of the *Crown* being much abated.

There is no redemption from Hell.)

There is a place partly under, partly by the *Exchequer Court* commonly called *Hell*; I could wish it had another name, seeing it is *ill jesting with edge tools*, especially with such as are sharpened by Scripture. I am informed that formerly this place was appointed a prison for the Kings debtors, who never were freed thence, untill they had paid their uttermost due demanded of them: If so, it was no *Hell* but might be termed *Purgatory* according to the Popish erroneous perswasion. But since this Proverb is applyed to moneys paid into the *Exchequer*, which thence are irrecoverable, upon what plea or pretence whatsoever.

These extracts show us first that the Court was on one side and the paying part on the other of Westminster Hall, and that "Hell" was under and "partly by" the Exchequer Court, i.e. on the west side towards the north.

Does Fuller mean by the "paying part" only the Office which paid out money? Was the *paying in* done at the actual Court of the Exchequer? If not, the proverb "There is no redemption from Hell" seems rather meaningless, as "Hell" was on the opposite side of Westminster Hall to the Exchequer Office in Tally Court. It would appear that the paying in was connected with some office on the west side of the Hall, if not the Court of Exchequer itself. Such an interpretation might be in accordance with Hatton's statement about the officers of the "Lower Exchequer."

* *The Antient Kalendars and Inventories of the Treasury of His Majesty's Exchequer*. London, 1836. Vol. I. p. cxxviii.

† London, 1662, pp. 579—580.

Yet this is not in accordance with Palgrave's statement that the Court of *Receipt* of the Exchequer was in Tally Court, i.e. on the east side of the Hall. Perhaps emphasis must be laid on Fuller's use of the word *formerly*. In the Commonwealth or in Charles II's time the paying-in office may have been removed to the west side. What is clear, however, is that "Hell" was certainly on the west side, and, according to Smith's plan (see our Plate XXIII), toward the north entry. But where was "Heaven"? Had the Dutch print been correct, and the heads been on the west side of the Hall, Cromwell's head would have been immediately above the Exchequer Court, and presumably above "Heaven." Smith's plan (see Plate XXIII) puts "Paradise" under the Exchequer Court, and rather suggests that it was north of "Hell," but it is not clear that "Paradise" can be identified with "Heaven," in tavern nomenclature. If we could implicitly trust Smith's plan, then "Heaven" stood in Old Palace Yard, between the apse of Henry VII's Chapel and the White Hall, and "Purgatory" between "Heaven" and the White Hall. See our Plates XV and XXIII. Unluckily the positions are only vaguely marked. They are, however, at the south end of the Hall. But we do not know whether Smith had more knowledge than we have, as he gives no authorities for his localisation; the latter is inconsistent with "Heaven" being under the Exchequer or "abutting" on Westminster Hall.

Reference to "Heaven" and "Hell" occurs in Act V of Ben Jonson's *Alchemist*, and his editor Gifford in 1816 thus annotates the passage:

Heaven and Hell were two mean ale-houses* abutting on Westminster Hall. Whalley [who edited Ben Jonson 1756] says that they were standing in his remembrance. They are mentioned, together with a third house, called Purgatory, in a grant, which I have read, dated in the first year of Henry VII. (Vol. iv. p. 174.)

While Pepys dined in "Heaven" (see above, p. 317), he took his "morning draught" in "Hell" (see the *Diary*, Nov. 27, 1660). "Hell" is connected with one very famous event in English history, namely Pride's Purge: the forty-one excepted members were locked up in "Hell" for the night:

Of whose names Mr Hugh Peters came to take a list; and then conveyed them into their great Victuallinge house near Westminster Hall, called "Hell" where they kept them all night without any beds†.

This quotation indicates that "Hell" was of considerable size. Hugh Peters brings us to another reference, this time to "Heaven," namely, during his trial on October 13, 1660, on the charge of "compassing and imagining the death of the King" [Charles I.]. A certain Mr Beavor was sworn and asked, "What do you know of Peters?"

Beavor. My Lord, and Gentlemen of the Jury,—Upon a day that was appointed for a fast for those that sat there as a Parliament, I went to Westminster to find out some company to dine with me, and having walked about an hour in Westminster Hall, and finding none of my friends to dine with me, I went to that place called Heaven, and dined there; after I had dined I passed through St Margaret's Churchyard to go home again (I lay in the Strand)...

* This description by no means accords with those of earlier writers. Peacham says very good meat was cooked in "Hell" in term time (1667), and even in 1782 we are told of the remarkable places in Westminster, that one was "Hell" very much frequented by lawyers. It probably had a reputation comparable with that of the "Cook" on the north side of Fleet Street near Temple Bar up to 1886.

† Dugdale, *A Short View of the Late Troubles in England*, 1681, p. 368.

and then the witness went into that church hearing that Hugh Peters was preaching—as he told the Court out of curiosity, not out of any manner of devotion;—and finally he records what Peters said.

But it is clear that none of these references—and there are probably many others—indicate as clearly as we might wish the exact position of “Hell,” still less of “Heaven.” To sum up: If “Heaven” was under the Exchequer Court, much poetic licence must be allowed to Butler; and Nash’s statements, if applied to the “farther end” of the Hall, are in error*. If “Heaven” stood on the site of the Committee Rooms of the House of Commons†, it was across the passage to St Stephen’s, and as far as we are aware there is no record of the Exchequer Court or Office being there, though there appears to have been a time after the removal of the Courts from the White Hall in which that building was used as a store for records. But south of St Stephen’s passage “Heaven” would not have been entirely, or in part, under Westminster Hall. This position would justify, however, Butler’s description of its relation to Cromwell’s head.

Finally, if we suppose “Heaven” as at the south-west corner of the Hall, partly occupying a cellar under the south end of the Hall, then Butler’s statement would be correct, but “Heaven” could not then be identified with “Paradise,” under the Exchequer Court; and we have no knowledge that the Exchequer Office, or its stores, extended right along the west side to the south-west corner of the Hall. All we can conclude safely is that if “Heaven” was somewhere on the west side or south-west corner of the Hall, then according to Butler and Nash Cromwell’s head was poled on the west of the gable end—possibly at the south-west corner—definitely not on the east or river side, where Ireton’s head must have been impaled. It is very unfortunate that the heads were not on the west side of the Hall with Cromwell’s over the Exchequer Court, for then all would have been straightforward, but the statements of Pepys and of Butler, to say nothing of the newspapers of that day, seem to make the “farther” or southern “end” of the Hall essential. Smith’s plan separates “Heaven” from “Paradise,” placing the former vaguely at some distance away to the south-west of the Hall and the latter under the Exchequer Court. This gets over some difficulties, but Smith, as we have said, does not give his authorities‡. We incline to the belief that the heads faced down the Hall, so that Cromwell’s was at the south-east corner. In this case Butler must claim the poetic licence he stands in need of so often!

Before we leave the Exchequer and the neighbourhood of Westminster Hall we must turn to another point. In the region round the Hall were not only the national records, and the records of the Courts of Justice, but also the money

* The Exchequer was not at the end of the Hall, the south end at which the heads were poled.

† Lord Braybrooke’s statement.

‡ Indeed his plan almost contradicts his own statement, *Antiquities of the City of Westminster*, 1807, p. 69:

On the West side of Westminster Hall was in the year 1781 a fishyard, and an house in which the King’s fishmonger resided, but the range of buildings called Heaven, the fishyard, and the fishmonger’s house were removed to make room for the above mentioned Committee Rooms and the staircase there; the only passages to the House of Commons having, till then, been through Westminster Hall, through Waghorn’s coffee house at the south end of the Court of Requests, and through a coffee house, afterwards a private house, and now converted to other uses on the West side of the Court of Requests. [See our Plates XV and XXIII.]

received from the revenue. Although the care of these was not so ample as it is nowadays, still, as we shall see, the need for it was recognised from the earliest times. This is illustrated by Pepys in his *Diary*. Thus he tells us on September 6th, 1666, how at the time of the Great Fire he had got his own goods removed, and he, going further westward, the Fire of London progressing, found the people of Westminster had become alarmed, and

so to Westminster, thinking to shift myself, being all in dirt from top to bottom; but could not there find any place to buy a shirt or a pair of gloves*, Westminster Hall being full of peoples' goods, those in Westminster, having removed all their goods, and the Exchequer money put into vessels to carry to Nonsuch†; but to the Swan, and there was trimmed: and then to White Hall; but saw nobody; and so home.

Further, in 1667 on June 14 Pepys, after writing about the scandal of the Dutch raid into the Thames, and there having been no money to fit out the big ships, or even pay the seamen, remarks:

It gives great matter of talk that it is said there is at this hour, in the Exchequer, as much money as is ready to break down the floor. This arises, I believe, from Sir G. Downing's late talk of the greatness of the sum lying there of people's money that they would not fetch away, which he showed me and a great many others.

It is clear from these entries of Pepys that at times there were large sums of money in the Exchequer Office, not only belonging to the Crown, but to other people, who during crises may have thought it safer to leave it there than to keep it at home or to entrust it to a goldsmith. The Bank of England was not founded till 1694, and therefore the Government stored its own monies. Now what is the purpose of this digression? To meet the type of critic who finds the tale of a sentinel picking up the Head wholly improbable. It may not be the true history of the Wilkinson Head, but there is nothing in the story itself improbable. For example, such a critic writes‡:

How did it happen that a sentinel was posted at the back, or south end, of Westminster Hall? What was the man's name? And as Westminster Hall was prolonged at the south end by another building, the Court of Wards§, and moreover the west side was surrounded by other buildings, what a prodigiously strong wind it must have been to make the head clear all these.

The answer to the first question is that sentinels would be posted about the Exchequer Office for the same reason that a file of soldiers as guard is now sent

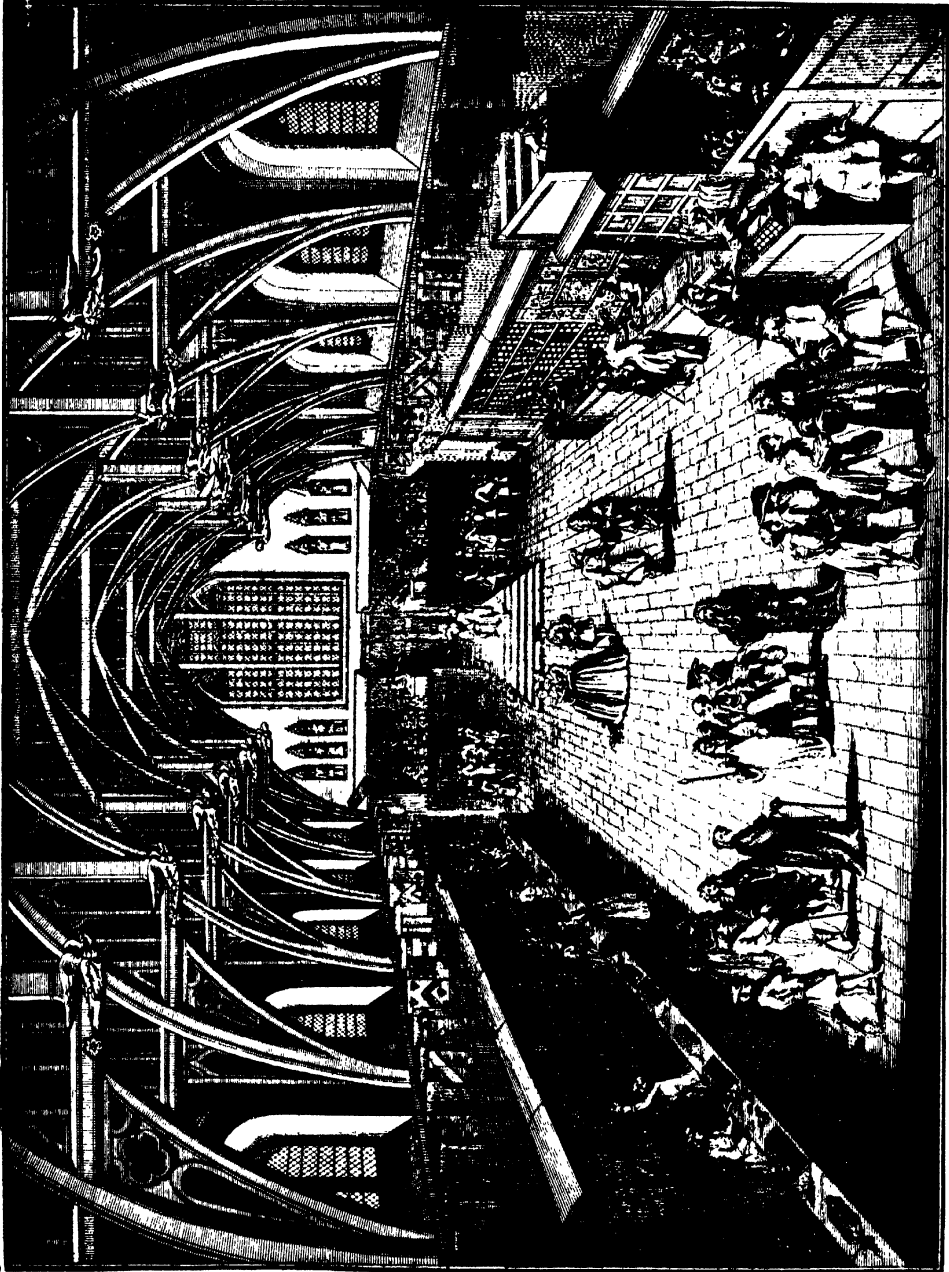
* There were stalls in the Hall for the sale of haberdashery, etc.; these are several times referred to by Pepys; see our Plate XXI.

† Nonsuch Palace, near Ewell; the money would probably be taken to Kingston and thence by road to Nonsuch.

‡ *Notes and Queries*, Vol. CLI. July—December, 1926, pp. 12—13.

§ The writer does not say what is his authority for the Court of Wards "prolonging" the south end of Westminster Hall. According to Stow the Court of Wards was in the building called *The White Hall* being a great Chamber accessible from the upper end of Westminster Hall behind the King's Bench. The White Hall adjoined the eastern half of the south end of Westminster Hall, but did not abut against that Hall; although there was access to it from the Hall, there were low buildings between the White Hall and Westminster Hall. The Court of Wards was abolished in the twelfth year of Charles II's reign, his Majesty receiving for it in perpetuity half the "excise on beer, and ale, coffee and other outlandish drinks." (Nov. 27, 1660.) The White Hall was then used as the Court of Requests and Liveries, and, after the Painted Chamber was destroyed, as the House of Lords.

Pearson and Morant: *The Cromwell Head*



Interior of Westminster Hall in Pepys' day. Court of King's Bench to left, Court of Chancery to right of steps,
Court of Common Pleas in right-hand corner of print. Stalls for sale of books and hosiery, etc.

nightly to the Bank of England*. The answer to the second question is today immaterial; if the tale had said that the sentinel's name was Brown, Smith or Robinson, it would in nowise enable us to unfold the dark portion of the story. Supposing the Head was picked up by a sentinel and he or his descendants sold it secretly, say, to Du Puy, they would certainly not desire their name to be spread abroad. As to the last criticism of the story, consisting in the prodigious wind needful to blow the Head over buildings down to the ground, we ask, Does the story involve blowing over buildings down to the ground? It is clear from Pepys' *Diary* that there were in certain London buildings flat roofs or "leads"†, which may or may not have been little more than broad lead gutters. It will be remembered that there is at Oxford a *skull* once attributed to Cromwell but impossibly his. The Memorandum Book of the Ashmolean Museum contains the following statement entered in the year 1720:

In the year 1672 [?] Oliver's skull [?] was blown off the north [?] end of Westminster Hall down into the *leads* of the same, and taken from thence by Mr John Moore then in the Old Petts. Sometime after this he gave it to Mr Warner, apothecary, living in King Street, Westminster. Mr Warner sold it for 20 broad pieces of gold to Humphrey Dove, Esq. then deputy paymaster to the treasurer of the chamber, but had been secretary to Fines‡ when keeper of the seals to Oliver. This skull was taken out of Mr Dove's iron chest at his death in 1687 by his daughter Mrs Mary Fiske of Westminster with which family it hath remained until given to Mr E. Smalterall.

Westminster, October 10, 1720.

Could there have been an interchange of the real head with some other skull?

* In *Angliae Metropolis, or The Present State of London* (First Written by the late Ingenious Tho. Delanne, Gent, and continued to the present year by a careful hand, 1690) we read on p. 130:

There are two *Ushers* whose Office is to secure the Exchequer by Day and Night, and all the Avenues leading to the same and to furnish all Necessaries, as Books, Paper, etc.

This would be an arduous and continuous job for two men! The Ushers of the Exchequer were officers of some importance whose business was to superintend the watches and generally to provide for the service of the Court and Offices. The need for protection for the Palace and in particular for the Exchequer was recognised in very early times. The office of Usher of the Exchequer was hereditary. The "*Officium Hostiarii de Scaccario*" was held by Roger del Exchequer in the 52nd year of Henry III's reign. He had lands at Eston in Oxfordshire by the service of keeping the doors of the King's Exchequer, and he could appoint deputies to undertake or assist in his duties. We find that with the land the office could pass to coheirresses, who would divide the revenues of the land and present suitable persons as Ushers for the approval of the Barons of the Exchequer. It was the Usher's duty to keep the Exchequer safely, and to take charge of the doors and avenues to it, so that the King's records laid up there might be in safety. He had the further duty of transmitting writs of summonses, which were issued by the Exchequer for the King's debts, i.e. to cause these writs to be delivered to the respective sheriffs to whom they were directed. An order of the fifth year of Edward II's reign runs (translation):

And that the said Ushers should have their own men, for whom they shall be ready to answer, lying every night in the Exchequer till the morrow of the close of Easter next, to wit, until the Court returns, and should not let them bring any fire into the Palace, either in a candle or any other way within the same period, but that they should with all possible diligence give their attention to the keeping of the said Exchequer and the Kings records therein.

These facts are extracted from Thomas Madox, *History and Antiquities of the Exchequer of the Kings of England*, 2nd Ed., 1769, Vol. II. pp. 275—276 and footnote (a), p. 278.

There need be no cause of surprise if in the reigns of Charles II and James II guards were placed at night in the neighbourhood of the Exchequer or about the Palace of Westminster.

† See A. Bryant's *Samuel Pepys, The Man in the Making*, 1933, pp. 138, 182, which suggests at least a flat roof!

‡ Query Lord John Fienes, one of Cromwell's lords.

Now this statement, as Howarth observes*, is very circumstantial; it gives the names of a number of persons, some of whom might possibly be alive in 1720, but it is clearly not an accurate account, for there are three most improbable statements in the first line which we have marked with notes of interrogation. But it is to be remarked that the skull did not fall to the ground but onto or into the "leads"†, which may mean the leaded gutters of the roof of Westminster Hall, or possibly even a flat roof to some adjacent buildings‡. At any rate this account of the fall of a head on Westminster Hall by "being blown off" does not necessarily signify that it would be blown to the ground, a descent which would certainly break in pieces any skull, and would probably badly fracture an embalmed head, however leather-like the condition of the flesh. Our view of Westminster Hall from Lambeth (see Plate XVI) shows the low buildings at the south end of the Hall, which may have been the "Old Petts." Our Plate XXII shows several "leads" near the south end of the Hall.

We have searched the Rate-Books of Westminster for the years 1665, 1673 and 1682 by the kind permission of the City Clerk.

In 1682 there was a John Moore in Spade Alley and another in St Peter's Street, both rated at low values. No trace was found of "Old Petts." In King's Street there was a house termed the Apothecary's, but no name was given of the occupier. There was an Edward Warner, who lived in Milbank and a John Warner who lived in Castle Street. There was no trace of Humphrey Dove or Mrs Mary Fiske. We are unable, therefore, to confirm from the Rate-Books of either St Margaret's or St Anne's Parish the existence of the individuals mentioned in the Memorandum Book of the Ashmolean Museum. The Oxford legend remains unconfirmed. We may state that the Rate-Books do not mention by name the taverns "Heaven" or "Hell."

It is, however, difficult to believe that Cromwell's head came down from Westminster Hall in 1672. Charles Patin, visiting England in 1671, asserts that the three heads were still on the Hall. Speaking of the "exécrables parricides de la majesté," whose heads were on London Bridge, he continues:

Celles de leurs chefs Cromwel, Ireton son gendre et Bradshaw, sont sur ce grand édifice qu'on appelle le Parlement, à la vue de toute la ville. On ne sauroit les regarder sans paslir, et sans s'imaginer qu'elles vont jeter ces paroles épouvantables : Peuples, l'éternité n'expiera pas notre attentat, apprenez à nôtre exemple que la vie des Rois est inviolable. (*Relations historiques et curieuses de Voyages en Allemagne, Angleterre, Hollande etc. etc.* Second Edition, à Lyon 1676, p. 188.)

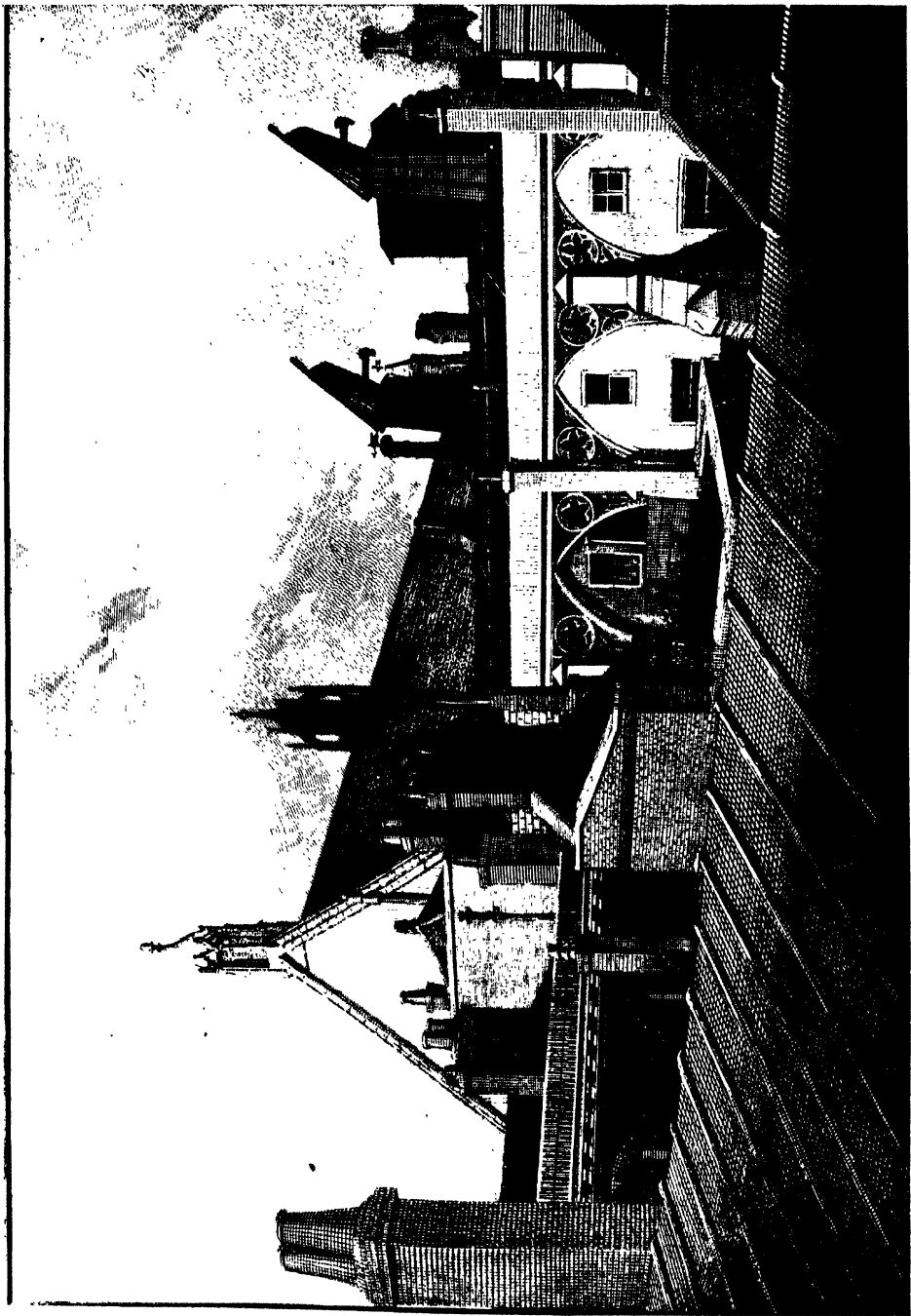
* "The Head of Oliver Cromwell," *Royal Archaeological Institute*, April 5, 1911, p. 10.

† At the trial of Charles I guards were placed upon the "leads" of Westminster Hall, and at the windows looking into the Hall; see our Plate XVII. The order runs: "That, there be guards set upon the Leads and other places that have Windows to look into the Hall." *A True Copy of the Journal of the High Court of Justice for the Tryal of K. Charles I.* London, 1684.

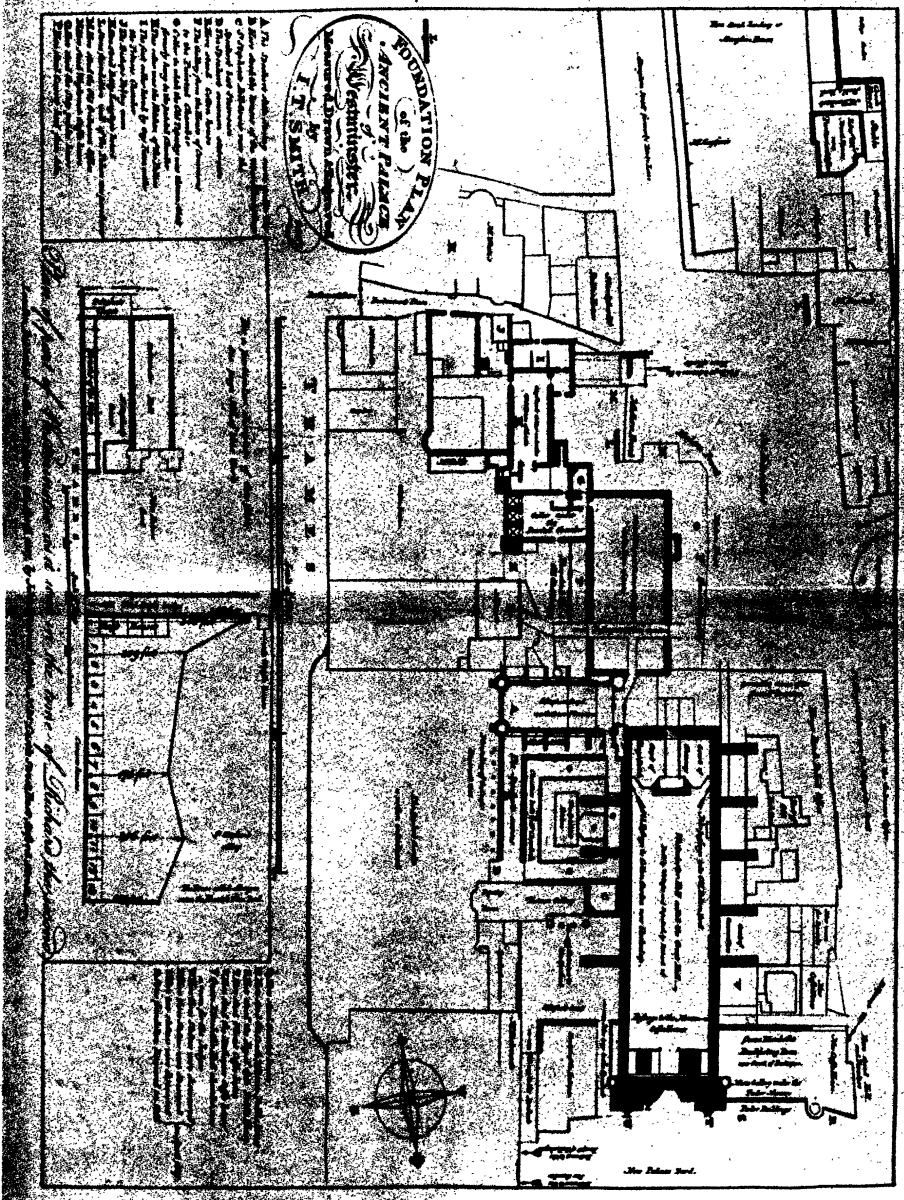
‡ It may be remarked that Mr Josiah Henry Wilkinson in his version of the tradition (see our p. 288) states that the sentinel at whose feet the head fell "was on his parade on the parapet." We do not know what authority he had for this, but Wilkinson had met Cranch who had known Russell. There is among the Wilkinson papers the following copy of an Entry in *Mr Cranch's Diary*:

Bath, 1819, July 26: Waited on Mr Wilkinson (from London) at Mr Clutterbuck's at Widocombe on his desiring to confer with me about the head of Oliver Cromwell when I related to him what I knew and understood of that subject, with which information he was highly gratified.

Pearson and Morant: *The Cromwell Head*



Leads of St Stephen's, of the White Hall, and of other buildings at the South end of Westminster Hall, on to any of which the Cromwell Head may have been blown down, without falling to the ground. (J. T. Smith, 1806.)



J. T. Smith's Plan of the Palace of Westminster, showing the positions in his opinion of the taverns, "Heaven," "Hell," "Paradise" and "Purgatory." [Were the taverns and coffee-houses enrolling 81 Stephen's the source of the inebriety of 18th century politicians, or were such houses there to satisfy their alcoholic cravings?]

The passage occurs near the end of the *Troisième Relation* which is dated "De Strasbourg en October 1671."

The next reference to the head of Cromwell shows that it was on Westminster Hall up till 1681, for in *Heraclitus ridens* No. 24 for July 12, 1681, we read:

Whether old Noll's head did not take it for an affront to be thrown down from the pinnacle of Honour, which is repairing at the South End of Westminster Hall, and whether he would not be better than a witch who could tell who shall be his successor in that high preferment?

Attention was we believe first drawn to this passage by J. B. Williams. In an article by J. G. Muddiman* some seven years later it is also cited, but the consequences drawn from these words in a satirical journal seem to us quite unwarranted. Muddiman asserts that Cromwell's head was not blown down at all, but "was thrown down by workmen in the day-time" and that "we may take it for granted that the street boys of the times left very little remaining of the bones to which the real head must have been reduced by 1681."

J. B. Williams, in his paper "What became of Cromwell's Remains?"†, uses earlier the same words:

The head, therefore, was thrown down by a workman in the day-time—not blown down in a storm at night—and this happened twenty years after it had been set up, not twenty-five. It was probably swept into the Thames with the rubbish from the pinnacle‡.

It may well be questioned whether such tales—considering that in 1681 Charles II was still reigning—as that "the street boys of the times left very little remaining of the bones to which the real head must have been reduced by 1681," or that the head "was probably swept into the Thames with the rubbish from the pinnacle"—a fairly long "sweep," by the by, considering the intervening buildings—are more credible than the story that a sentinel picked it up when it was blown down! It is not in the least likely that Thomas Flatman, poet, miniature painter and satirical writer, witnessed a workman throwing down "old Noll's head," and he does not say with Muddiman that the pinnacle was "rebuilt," but under repair. Before any repair was undertaken the Surveyor of the House would have to report its necessity, and the first question which in those days would arise would be the preservation of the heads of the miscreant regicides. If the heads were in position in July, 1681, they would undoubtedly be restored when the repairs of the pinnacle or parapets were completed. Flatman's "thrown down by a workman" is as figurative as if we stated that "Cromwell dragged Charles I from his throne." The head of Cromwell in the reign of Charles II was far too valuable a symbol of "the stupendous and inscrutable judgements of God" to be lightly swept with rubbish into the Thames.

* *Notes and Queries*, Vol. CL. 1926, pp. 210—212. This paper as well as many other papers in the same Journal contains valuable references, but the conclusions drawn from them are rarely helpful.

† *The Month*, Vol. cxxxiv. 1919, pp. 108—118.

‡ We are inclined to believe that this "pinnacle" is as allegorical as other words in Flatman's paragraph. The only pinnacle on the south end of Westminster Hall in any views of the Hall in the time of Charles I which we have seen is that at the top of the gable, and to that as the central figure Bradshaw's head was fixed. Hence either Flatman was speaking throughout metaphorically, or else he was confusing Bradshaw's skull with Cromwell's head. This he might easily do 21 years after the actual polling of the heads.

And the view that the heads would be restored to the repaired gable is confirmed by what Echard tells us concerning Sir Thomas Armstrong. The latter was tried for complicity in the Rye House Plot. *The London Gazette*, June 19—June 23, 1684, runs:

London, June 20. This day Sir Thomas Armstrong was executed, being drawn on a sledge to Tyburn, and there hanged and quartered.

He [Sir Thomas Armstrong] was executed at *Tyburn* on *Friday* the 20th of *June*; which was observed by some to be the same Day of the same month...in which the five Jesuits were executed ...just five years before. His Head was set up on Westminster-Hall between those of Cromwell and Bradshaw; one of his quarters upon Temple Bar*, two others at Aldersgate and Aldgate, and the fourth was said to be sent down to Stafford, for which Town he had been a Burgess in Parliament†.

Echard's account is at least partially confirmed by a statement of Henry Muddiman in his *Journal*, August 28, 1684:

The head of Sir Thomas Armstrong is by order removed from the further end [note Pepys' exact phrase] of Westminster Hall, where it was first placed, and put where Harrison's head stood, over the entrance to the Hall from the New Palace Yard, where every one may see it as they go into the Hall.

Harrison's head, and that of Axtel [? Cooke] are said‡ to have been placed on the east side of Westminster Hall, so that if Armstrong's replaced Harrison's the latter must have been on the north-east corner of the Hall.

If the above version of the story of Cromwell's head be correct, it was still on its pole on Westminster Hall in 1684, some five months before the death of Charles II, which occurred on February 6, 1684—5.

* As a sign that Echard was accurate in this passage—he has been termed “a compiler and an uncritical compiler at that”—we may quote Evelyn's *Diary* for April 10, 1696, which at least confirms one of Echard's statements:

The quarters of Sir William Perkins and Sir John Friend, lately executed in the plot, with Perkins's head were set up at Temple Bar, a dismal sight which many pitied. I think there never was such at Temple Bar till now, except once in the time of King Charles II, namely of Sir Thomas Armstrong.

Narcissus Luttrell also writes (*op. cit.* Vol. i. p. 312) 20 June 1684, “Sir Thomas Armstrong's quarters are disposed off; a forequarter is sett on Temple bar, his head on Westminster, another quarter is sent down to the town of Stafford for which he was a parliament man.”

† Laurence Echard, *History of England*, folio, 1707—1718, Vol. III. p. 714, folio, 1720, p. 1043.

‡ T. M., *History of Independency*, 1660, p. 114, says that Harrison's was placed on the south end of Westminster Hall on Oct. 3, 1660, and on p. 118 we read “Mr Axtel was bowelled and quartered and so returned back [i.e. to Newgate] and disposed as the former.” The “former” means Scroop (or Jones) and this would mean that his head was placed on London Bridge. On p. 116 we read that Cooke's head was “set on a pole on the North east end of Westminster Hall (on the left of Mr Harrison's) looking towards London.” It would appear as if Harrison's must have been on the east side towards the south, but if so, it must have been later removed to the north end. If Harrison's head was put on the North end, then Cooke's must have been on its right, unless we suppose the “left” to mean on the spectator's left when looking at the east face of the Hall! There is some contradiction in T. M.'s statements.

Possibly further difficulty is introduced by Pepys; in the entry into his *Diary* for October 21, 1660, we read:

George Vines carried me up to the top of his turret, where there is Cooke's head set up for a traitor and Harrison's set up on the other side of Westminster Hall. Here I could see them plainly, as also a very fair prospect about London.

Presumably George Vines' “turret” was the west tower, at the north end of the Hall.



The Wilkinson Head. Occipital view showing the wedge of flesh cut out by the axe, with at least two axe cuts inside the wedge, also an axe cut above the vertebra, where severance has been finally made by two blows of the axe. The fracture of the occipital may be seen about half an inch to the left of the piece chiselled out.



The Wilkinson Head. Enlarged occipital view showing the hair on back of head, the chisel and saw marks on the cincture, and the abnormality of the sutures in this region.

After 1684 we have no trustworthy information as to the head of Oliver Cromwell. We may take it that it was 24 to 25 years at least on the Hall. We have no record of a reward being offered for its recovery, except the tradition attached to the Samuel Russell Head*. It is possible that it did not remain up after the flight of James II in 1688†, for there would, amid the feelings which two great-grandsons of Henry Darnley had created, be less stress laid upon Cromwell as a "horrid regicide," and less importance attached to his head as a warning to all who might premeditate an attack on a king by divine right. William and Mary, we may remind the reader, were to be joint sovereigns *by vote of the people*.

The head of Cromwell had disappeared; it was not, if Mr Wilkinson's Head be genuine, to reappear till 1710, or some may assert till about 1773. If it be not genuine, we must agree with Dr Welldon that it disappeared for ever; since, however much the disappearance during 63 or 89 years may weaken the evidence for this Head, no other claimant head‡ or skull has any pretensions in the least comparable either in externals or even in traditions with this. We shall now consider what Cromwell's real head must have looked like, had it survived a quarter of a century on the top of Westminster Hall, and whether there is any feature of the Wilkinson Head which obviously contradicts the appearance we should anticipate. It is needful, however, in the first place to give a somewhat more detailed description of the Wilkinson Head than has so far ever been provided.

SECTION III.

10. *Comparison of Cromwell's Head and the Wilkinson Head on the non-measured Characteristics.*

Cromwell's Head.

(1) Had the skull-cap removed and the brain weighed. James I's skull-cap was removed by saw and chisel (see our p. 297).

(2) Cromwell's skull-cap would be stitched on again, for those death-masks which show the cincture, or the wrap, indicate that it must have been refixed.

Wilkinson Head.

(1) Has had the skull-cap removed by saw and a small piece probably by chisel. See Plate XXV.

(2) Shows thread holes round the border of skull-cap. The shrinkage of the flesh round the cincture indicates that the skull-cap was separated before embalmment.

* See our p. 280.

† If it survived on its pole to James's accession it was extremely likely to be maintained there while he ruled.

‡ Beside the discredited Ashmolean skull now in the craniological section of the Oxford Museum, an amusing list of Cromwellian embalmed heads or skulls will be found in *Notes and Queries*, Vol. cl. Jan.—June, 1926, p. 392. We have also to note the head of Cromwell, which was rescued by a surgeon from the mob, who were dragging it about in Red Lion Square, or Kingsgate Street, at the time. This surgeon left an only daughter who married a puritan cutler at Sheffield named Fletcher, etc. This story savours of Samuel Russell's tale of the head having belonged to the daughter of a *soldier*, not a surgeon, who had the head for her marriage portion! See *Notes and Queries*, 3rd Series, Vol. v. pp. 265—266. This tale is attributed to Joseph Hunter, who is said to have found it in a diary of the time. Hunter's MSS. are now in the British Museum.

(3) Cromwell's body was embalmed. There is some doubt as to the thoroughness of this embalmmnt. This could only arise from Bate's statement in the *Elenchus* but we do not know whether this refers only to what took place immediately after the autopsy, or whether a more thorough embalmmnt did not follow later.

It is reasonably certain that a death-mask of Cromwell was taken, ten to fourteen days after his death (p. 362). It follows therefore that the account given by Bate is incorrect, grossly exaggerated, or that putrescence was checked by a later more perfect embalmmnt. There is no reason to doubt that Cromwell's body went on Sept. 20 to Somerset House, lay under the bed of state with the effigy, and was interred in Henry VII's Chapel on Wednesday Oct. 27. As to the effectiveness of the embalmmnt, we do not think much one way or the other can be deduced from Sainthill's description of the condition of the body at Tyburn.

(3) Has been embalmed. The present condition of the flesh is like tanned leather. It has been held that this Head has been very badly embalmed, largely to prove that the Head does not disagree with Bate's statement in the *Elenchus**.

On the other hand some have asserted† that it has been extremely well embalmed, and, to meet the evidence drawn from Bate (see our p. 275) and fearing that the genuine head would not be well embalmed, have tried to interpret Sainthill's words "very fresh embalmed" as indicating that Cromwell's head was actually well embalmed. On the whole it seems to us that this Head must have been very thoroughly preserved, or the flesh could not have lasted at least 161 and very probably 224 years without falling to pieces. If it has lasted that time, it may well have lasted 276 years, even if, during some 25 or 26 of those additional years, it was exposed to the weather. We do not understand by "well embalmed" a process which has succeeded in preserving excellently the features of the subject, but rather one which has prevented most of the facial skin and the membranes from perishing, by converting them into leather.

* See W. J. Andrews, *Notes and Queries*, Vol. cl. 1926, pp. 318—319.

† See Howarth, *loc. cit.* and our p. 318 fn. §. There is another type of critic, who takes the position (i) that nobody has justified the statement that the Wilkinson Head has been embalmed, and (ii) that English embalmers did their work very superficially, and consequently Cromwell's head had no lasting powers! See *Notes and Queries*, Vol. cl. p. 408. There is no doubt that a cheap form of embalmmnt was current in London at least in the last quarter of the 17th century, but it hardly claimed to preserve the body indefinitely, and was not likely to be used in the case of royal or illustrious persons. Two advertisements will suffice to show its character. Both are taken from the *London Gazette*.

No. 1608, Monday March 24 to Thursday March 31 [? 27], 1681.

Thomas Warren of London, Apothecary, living at the Heart and Anchor in St Lawrence Lane near Cheapside, having after great cost and trouble, found out a most curious way of preserving dead Bodies from Putrefaction, or change of Colour, without Disboweling, Searclothing, or cutting any part, and undertaking for 5 l. to secure any dead Body above ground for several years; he caused publick Notice to be given thereof; but some persons, perceiving the great satisfaction, this his Invention has given to all persons that made use of it, have pretended to make use of his Powder, as bought from him: Wherefore he desires it may be known, that there was never any yet sold of it, nor is it to be had but from himself.

No. 1959, Monday August 18 to Thursday August 21, 1684.

William Russel Coffin-maker, who hath the Art of preserving Dead Bodies without Embowelling, Searclothing, cutting, or mangling any part thereof, and that used it to the great satisfaction of all those Honorable Persons by whom he hath been employed, lives at the Sign of the Four Coffins in Fleet-street; Coffins ready made, and the Body preserved for Five Pounds.

These advertisements appeared some quarter of a century after Cromwell's death, and as his body had the brains and bowels etc. removed and was wrapped in cerecloth, it is clear that this superficial method of embalming was not applied in his case.

(4) Cromwell's head was hewn off while his whole body was still in its cerecloths, for Sainthill tells us that owing to these it took eight blows of the axe to remove the head. There is no doubt therefore that the neck must have been badly hewn about. We know that the head was placed upon a pole on Westminster Hall. There appears to be no extant account of how the heads of traitors were attached to their supporting poles. We know that such poles were 15 to 20 feet in length and therefore must have been of stout build. The conception that such heads were placed on *pikes*, which would not have the requisite length or substance needful seems improbable, nor are the diamond or laurel leaf sections of the pike, however suitable for piercing flesh, the best for ramming through bone. If Cromwell's head had been blown down, it might either have been thrown off its spike, or the pole might have snapped.

(5) The pictures of Cromwell show generally a mass of light brown (? reddish) hair. See Plate XLII. Lady Payne-Gallwey's miniatures (one certainly by Cooper) both show Cromwell with greying light brown hair. The portrait of Cromwell by Cooper in Sidney Sussex College shows grey hair, and a portrait in profile (a copy by Bernard Lees) in the possession of the Duke of Portland shows Cromwell aged, and his hair very thin. There is no doubt that he had changed considerably in appearance before he died.

All the portraits of Cromwell show the short cut "beardlet" below the under lip; this was however the fashion of the day. Bradshaw wore the beardlet long so that it came to a point below the chin, and Ireton's beard was of the same type*.

(6) Cromwell was aged 59 at the time of his death. "Before I came to him as he rode at the head of his life guard, I saw and felt a waft of death go forth against him; and when I came to him he looked like a dead man" (George Fox, *Diary*, Vol. 1. p. 440). Noble† remarks of Cromwell: "It is certain that in old age he was

(4) This Head has been attached to an oaken pole, surmounted by an iron spike of square section tapering to a point. The top of the oaken pole to a length of eight inches has been broken off. The iron prong—something like the straightened prong of a pitchfork—is carried by a rudely shaped flat inverted Ω collar, nailed onto the top of the post. The collar and nails are well shown in our Plates XXXII—XXXIV reproducing skiagrams of the Head. The pole has been long in contact with the Head, for some of the worm holes pass through the Head and the pole. The spike where it has penetrated the skull-cap has rusted away, but inside the brain-box it has been less attacked. The iron prong and collar are rough blacksmith's work. This prong has been so forcibly thrust through the skull-cap that it has split it from the place of penetration to the right border: see our Plate XXXVI (d).

(5) The hair on the Head is sparse, it is remarkably fine and of a light reddish brown tinge. This fineness might possibly be due to the perishing of the substance of the hair, and the redness might be indicative of the diffused pigment left, after the melanotic pigment had largely perished‡. Again it may be that the hair was grey and the redness arises from staining by preservatives. Charles I had grizzled black hair at the time of his execution. At his exhumation in 1813 his scalp hair was a beautiful dark brown and his beard a redder brown. This was probably the combined effect of the preservatives and the 165 years of entombment. There was no prolonged weathering as in the case of Cromwell's head. The hair is pressed close to the Head as if it had received a set from moisture, which might be from the preservatives and the binding of cere-cloth or from the drip of rain. All we can say is that the sparse hair has been pressed close to the scalp.

The chin shows the beardlet; this, without proving it to be Cromwell's, indicates the period from which it came, if we dismiss the idea of fraud.

* Besides the two regicides mentioned in the text only Scott and Pennington had the Charles I type of beard. Of the remainder whose portraits we have examined, fourteen had beardlets like Cromwell and five no beards at all.

† *Memoirs of the Protectoral House of Cromwell*, Vol. 1. p. 292, 3rd Edition, 1787.

‡ A prolonged study was made, as to whether it is possible to learn anything from hair, which has been exposed or bleached, as to its former degree of pigmentation, but no very definite light was thrown on the original pigmentation of the hair of the Wilkinson Head by this microscopical work. See p. 278.

but a very coarse looking man, and this for many reasons; the number and greatness of his cares; the inclemency of the weather, which, as a soldier, he was obliged to endure, and perhaps the loss of his teeth; the difference of his face is very discernible in comparing those portraits of him which were taken when he was lieutenant-general...with those of his coins or medals struck but a short time before his death." And again, if we turn to the cavalier accounts (besides the frequent references to the "red nos'd Noll") we read* "But Cromwell wants neither wardrobe nor armour, his face is naturally buff, and his skin may furnish him with a rusty coat of mail; you would think he had been christened in a lime pit, tann'd alive, and his countenance still continues mangy. We cry out against superstition, and yet worship a piece of wainscot, and idolise an unblanch'd almond." This is no doubt exaggeratedly satirical, but there must have been an element of truth therein.

On the Cooper miniature of Cromwell in the possession of Lady Payne-Gallwey, there are a series of spots on the face, and on the back of the case the following note:

Mem. 29th May, 1812. Mr West is of opinion that this miniature was painted from Walker's picture, for when Cromwell sat to Walker he insisted on every imperfection in his face being represented; so this accounts for the spots, which seem to have been occasioned by accident from gunpowder.

The tale about Cromwell and the painter is reported by Horace Walpole of Lely (see our p. 350), and probably West had this tale in mind. No one, however, could mistake a Cooper miniature of Cromwell for the copy of a Walker painting. Nor again could a Lely painting be mistaken for a Cooper miniature. If the Walpole tale be true, and it sounds true, then the painter to whom Cromwell gave the direction may have been Cooper, and this miniature the result. It would therefore have great historical value. We regret being unable to give a reproduction of it. From a very poor photograph we have seen, it would appear as if the spots extended to the collar.

(7) Cromwell's brain-weight was stated within 10 years of his death (see our pp. 276—277) to be $6\frac{1}{4}$ pounds. This has been recognised by anatomists as an exceedingly improbable statement. And this is true even if all the water in the brain-box were weighed with the brain. Assuming that we may take the density of the "water" and brain material to be about unity and that 1 cm.^3 of water = 1 gramme = $15\cdot4323$ grains, and in ignorance of what is meant by

(6) In the case of the Head its owner retained nearly all his teeth up to death. They have fallen out, with the exception of an upper *R* 3rd molar and a lower *R* 3rd molar, since death†. But we know of no historical record of Cromwell losing his teeth in life; it may be only a supposition of Noble. The nose of the Head inclines to the left cheek, but is too battered to allow us to assert anything about it. The *present* condition of the skin of the face would almost exactly correspond to the cavalier's description of Cromwell's face, but as the tanning of the skin is almost certainly due to the embalming, to preservatives and possibly to weather, we cannot lay any stress whatever on the correspondence. On the other hand such a skin as the Cavaliers attributed to Cromwell might much aid in the preservation of his head! The skin of the Wilkinson Head by no means suggests that the owner had a smooth and tender skin. It appears besprinkled with pimples. See Plates XXVII and XXX.

(7) The Head gives the following *skull* measurements:

Glabella-Occipital Length, $L=192 \text{ mm.}$

Maximum Parietal Breadth, $B=151 \text{ mm.}$

Auricular Height, $OH=111\cdot5 (?) \text{ mm.}$

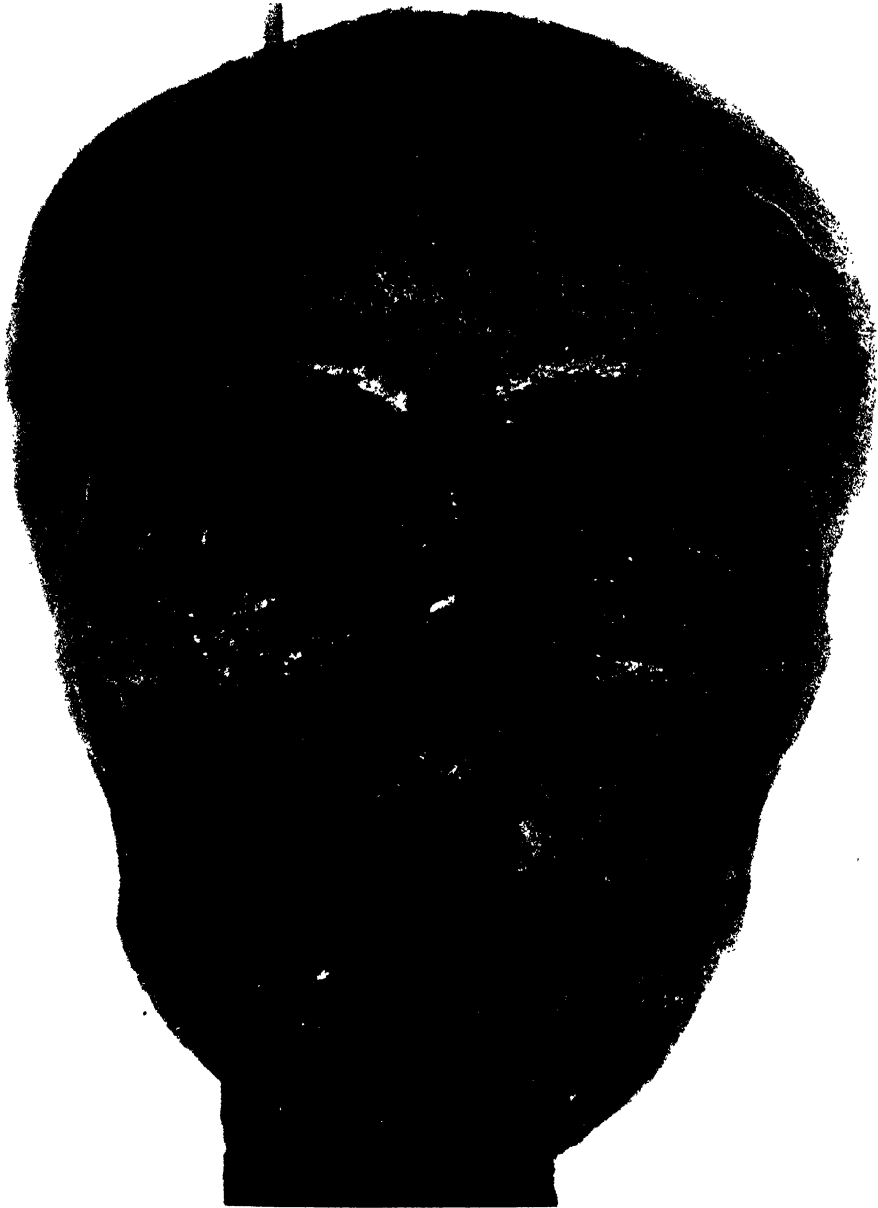
OH has been estimated by comparison of the male transverse type contour of the St Bride Burial Ground (Farringdon Street) crania with the transverse contour of the Head.

* Samuel Butler, *Posthumous Works*, 1754, pp. 281—282.

† Only two teeth were certainly lost before death: see fuller account below.

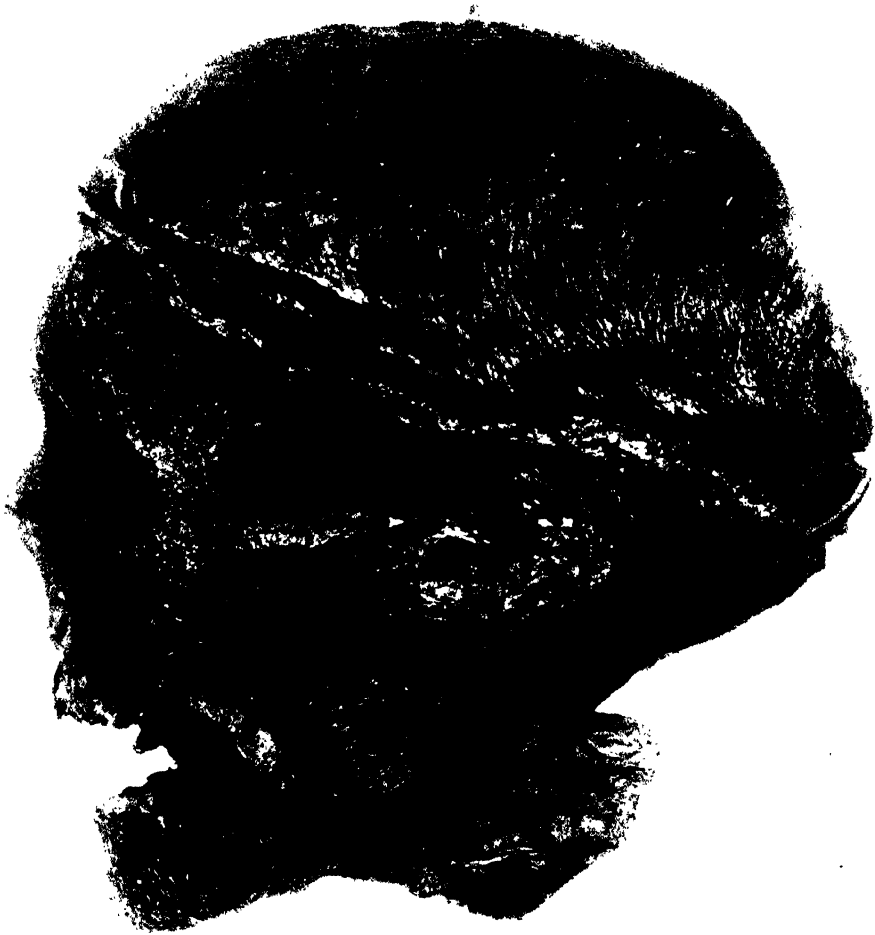


The Wilkinson Head. Full face, showing broken nose, shrinkage of eyelids, the cincture, hole for wart, flowing moustache on right side (cf. Plate LXV) and broken lips



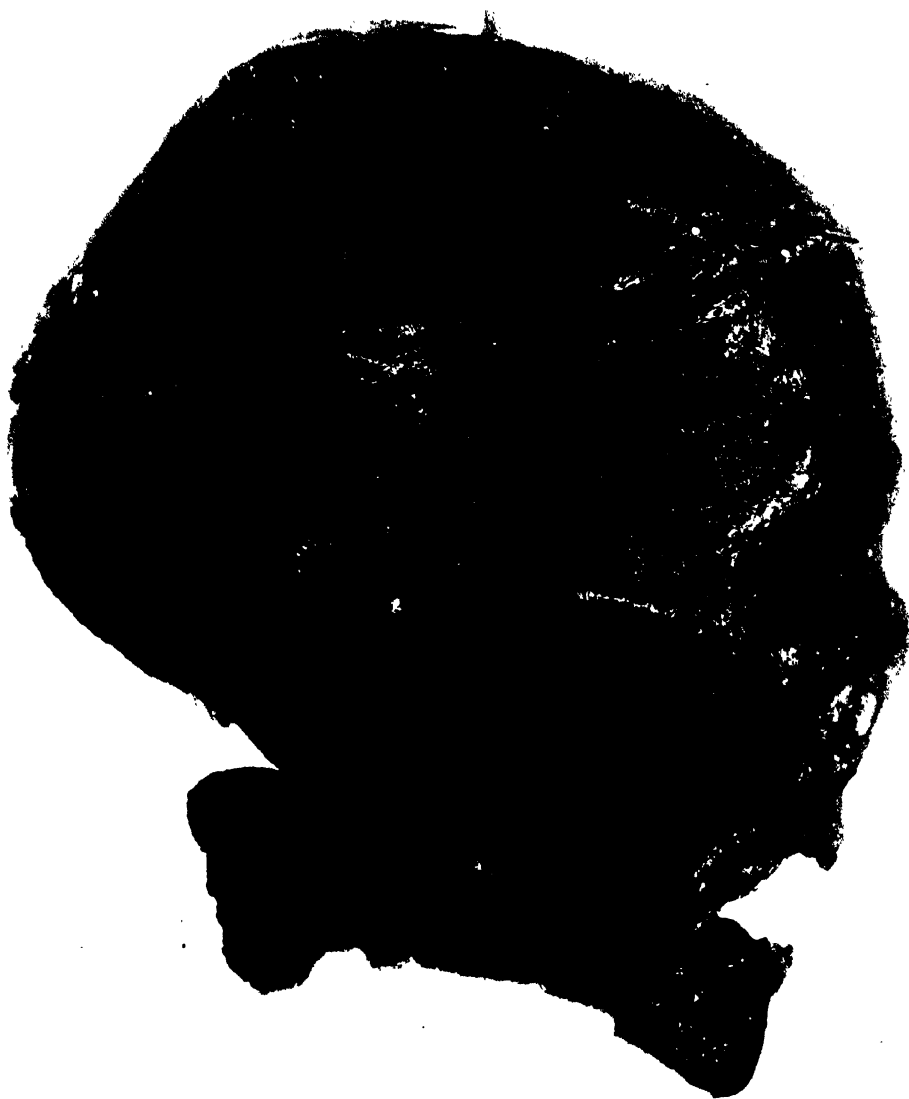
The Wilkinson Head. Full face, with the pole vertical, corroded point of iron spike, and the many pimples.

Pearson and Morant: *The Cromwell Head*



The Wilkinson Head. Left Profile, showing flesh shrunk back from cincture, distorted ear, and nature of hair on scalp and chin. The fracture of the occipital is shown by a white line on the extreme right.

Pearson and Morant: *The Cromwell Head*

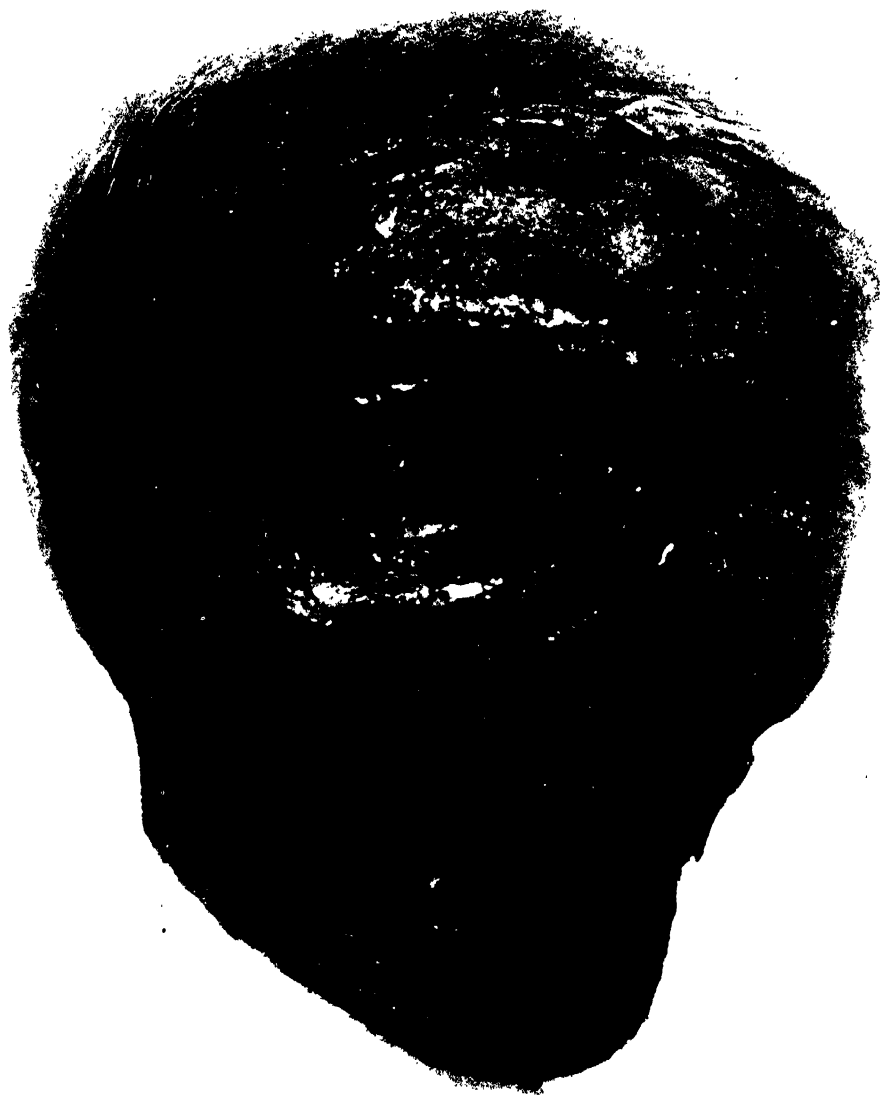


The Wilkinson Head. Right Profile, showing ear cut off, and wedge cut out of back of neck.



The Wilkinson Head. Three-quarter face, showing the wart-cavity, the pimples, and the flowing moustache on right (compare British Museum Wax Death-mask, Plate LXV).

Pearson and Morant: *The Cromwell Head*



The Wilkinson Head. Three-quarter face, looking slightly down.

the pound, and using either Apothecaries' or Avoirdupois measure we have:

For Cromwell's skull capacity,

Apothecaries'	Avoirdupois
$6\frac{1}{2} \times 12$ oz.	$6\frac{1}{2} \times 16$ oz.
36,000 grains	43,750·5 grains
2332·8 cm. ³	2835·0 cm. ³

The first of which, if not impossible, is exceedingly improbable, and the second practically out of the question. Of course the pound may have meant something very different to the medical men of the 17th century, or, what is more probable, the widespread superstition that great weight of brain signifies preponderating intellect led the physicians and surgeons appointed to embalm the body of His Highness the Protector, or one of their number, to sycophantic exaggeration. The portraits of Cromwell do not suggest an abnormally large head. While this brain-weight might have provided positive confirmation, we cannot take it to be of value as a disqualification.

Using the Reconstruction Formula for the capacity of male English crania (*Biometrika*, Vol. xviii. p. 34) or

$$C = \cdot 000416 \times L. B. OH + 247 \cdot 86 \pm \frac{44 \cdot 3}{\sqrt{n}}$$

we find the capacity C of the Wilkinson Head to be $1592 \cdot 6$ cm.³ $\pm 44 \cdot 3$. We can safely assert that the cranial capacity of this Head must have been between 1482 and 1703 cm.³ There is no approach whatever to the values of the capacity indicated by $6\frac{1}{2}$ lbs. Indeed a capacity of $1592 \cdot 6$ cm.³ corresponds to about $3\frac{1}{2}$ lbs. Avoirdupois, and $4\frac{1}{4}$ lbs. in Apothecaries' measure. The mean capacity of the Farringdon Street male crania being 1481·5 with a standard deviation of 130·1, we see that the owner of this Head would stand about the 20th grade in 100 Englishmen or 6th in 30 cases. This is about the position the eminent chemist, Sir William Ramsay, occupied. The owner of this Head had a brain well above the average, but not one of outstanding size.

On the whole we have not succeeded in finding any characteristic of the Wilkinson Head, which directly disqualifies it from being the head of Cromwell. This by no means demonstrates that it is the head of Cromwell, but had we discovered any such characteristic a more detailed relation of the Head to the portraits and busts, etc. of Cromwell would have been of small purpose.

11. *Age at Death of the Man who owned the Head.*

The point which tended at first to influence our judgment was the question of age at death, and for this reason we have gone more fully into the two characters by which it is customary to give a rough estimate of the age of a skull, namely the condition of the sutures and the state of the teeth. In the case of the former of these characters, however, while in a long series of skulls of a given age we can state something *on the average* as to the closure and obliteration of the sutures, it is not possible to fix at all definitely the age from the condition of the sutures. If Broca's or some similar system of scoring for the condition of the sutures be used, then the *average* score of a skull of known age is accompanied by a standard deviation measuring the variability of the score, which gives a very wide margin to the age prediction even when the condition of the external and internal sutures is fully accessible and visible. In the present case only very small portions of the sutures are visible, namely where the external flesh has before embalmment retreated from the cincture, and where the pia mater lining the skull-cap has done so likewise. We have therefore had to help ourselves out with the appearance of the sutures in skiagrams.

In the case of the teeth the sockets are accessible and by aid of the usual dental mirror and other instruments, together with a study of the skiagrams*, a fairly full account can be given. In our description we were most kindly guided by Mr C. Bowdler Henry to whom our very cordial thanks are here expressed.

Notes on the Sutures of the Wilkinson Head.

A direct examination of the Head shows that the states of the small parts of the sutures visible are as follows:

METOPIC. *Ecto-cranial.* A short length is exposed on the skull-cap immediately above the cut. This is apparently closing but it can be traced easily.

CORONAL. *Ecto-cranial.* A short length is exposed on the right side above the pterion and there the suture is closed and nearly obliterated.

Endo-cranial. A short length of the line of the coronal suture is exposed, but there is no sign of the suture, so it seems to have been obliterated.

SAGITTAL. No part of this suture is exposed.

LAMBDOID. *Ecto-cranial.* The parts exposed on the right and left sides are clearly open. On the left side there is a wormian bone with a maximum length of 10 mm. situated below the cut, and above it is a larger wormian bone which was sawn through. No wormian bones can be seen on the right side.

INTER-PARIETAL. *Ecto-cranial.* These sutures are somewhat confusing. There are traces of a closed suture running horizontally across the supra-occipital and to be seen on the skull-cap and lower part of the skull. This appears to be the horizontal suture of an inter-parietal bone. On the right side there is also a trace of what seems to be the closed suture between an *os triangulare* and an *os pentagonale*, but this is only seen on the skull-cap and it is not clear why the continuation of it should not be visible on the part of the skull below the cut. A photograph taken illustrates this point: see Plate XXV.

OTHER SUTURES. *Endo-cranial.* The region of part of the suture between the temporal squama and parietal bone is exposed on the right side, but there is no trace of a suture here, so it is apparently obliterated.

Skiagrams of the skull-cap of the Head were taken to examine the condition of those parts of the sutures which are not exposed. These were from above whole plate and 3" x 2" size, and from below whole plate. Skiagrams were also taken of three skull-caps from Spitalfields, the views from above being 3" x 2" showing a length of the sagittal suture near the vertex only, and from below whole plate. See our Plates XXXV and XXXVI. These skull-caps are:

S.F. 672. Young or young adult. All sutures clearly open externally and internally.

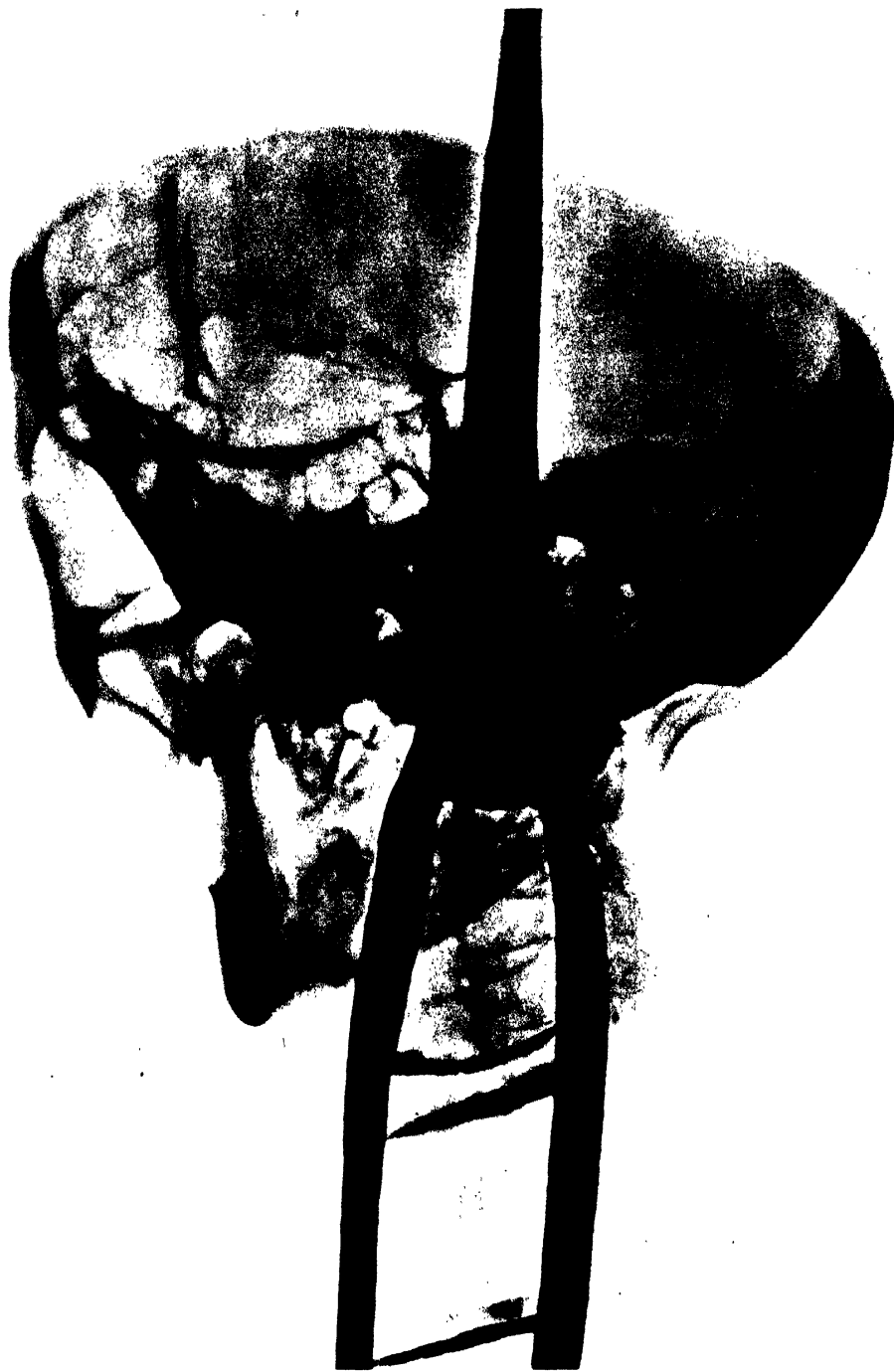
* For the numerous skiagrams taken (only a few of which are reproduced) we have warmly to thank Dr G. O. Dalton of Ipswich, W. H. Trethowen, Esq., F.R.C.S., and Drs Yeo and Hilton of the Radiographic Department, University College Hospital.

Pearson and Morant: *The Cromwell Head*



Skiagram of the Wilkinson Head in profile, showing iron spike and its side-pieces, the mandible, the cincture, and the place where the vertebral column has been severed.

Pearson and Morant: *The Cromwell Head*



Skiagram of the Wilkinson Head with the skull-cap removed, showing the metal spike and the three nails, which still attach it to the oak pole. On the border of the cincture to the right may be seen, as a nearly vertical pale bar, the start of the occipital fracture.



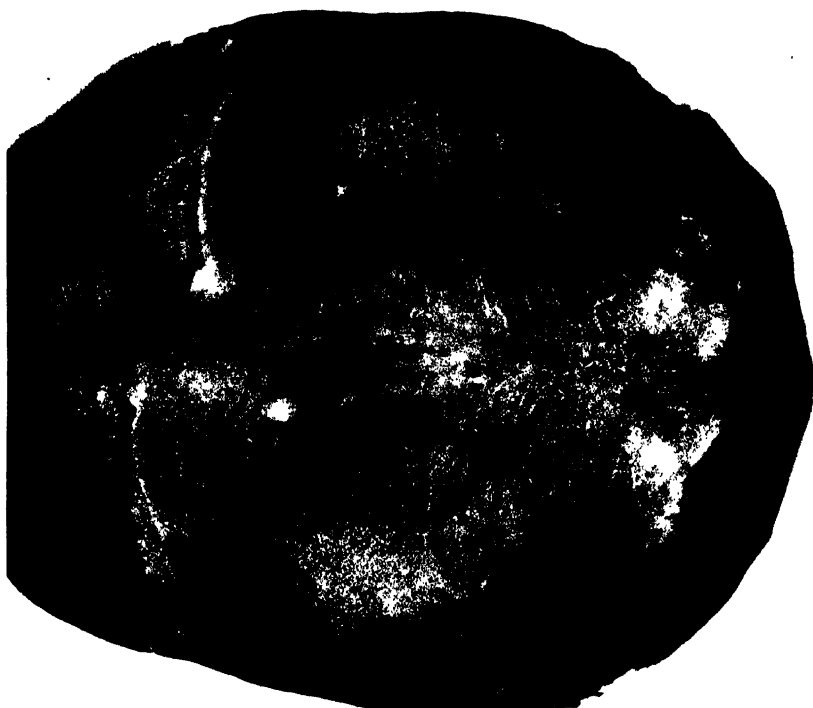
Skiagram of the Wilkinson Head, showing a further view of the iron spike, the cincture and mandible.

Skiagrams of Skull-Gaps.



(a)

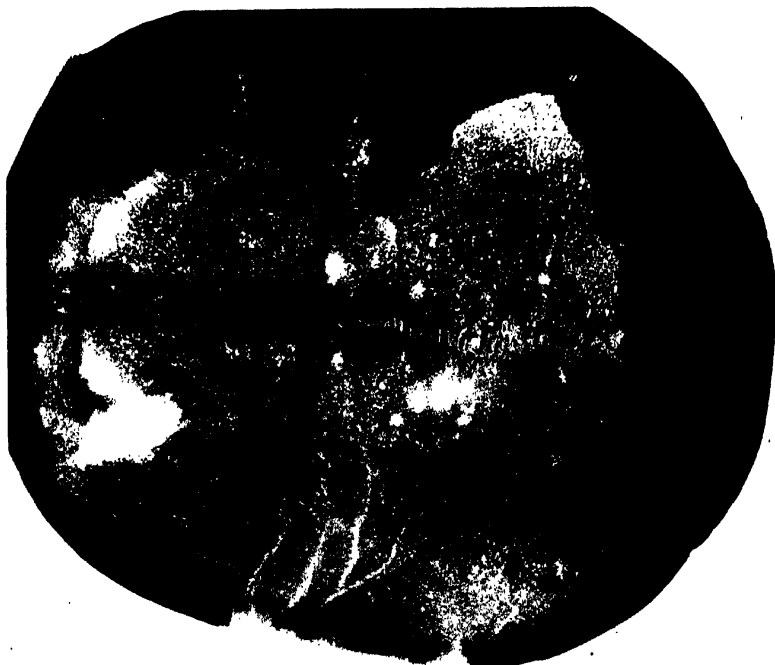
Young Adult; sutures open. No Pacchionian pits.



(b)

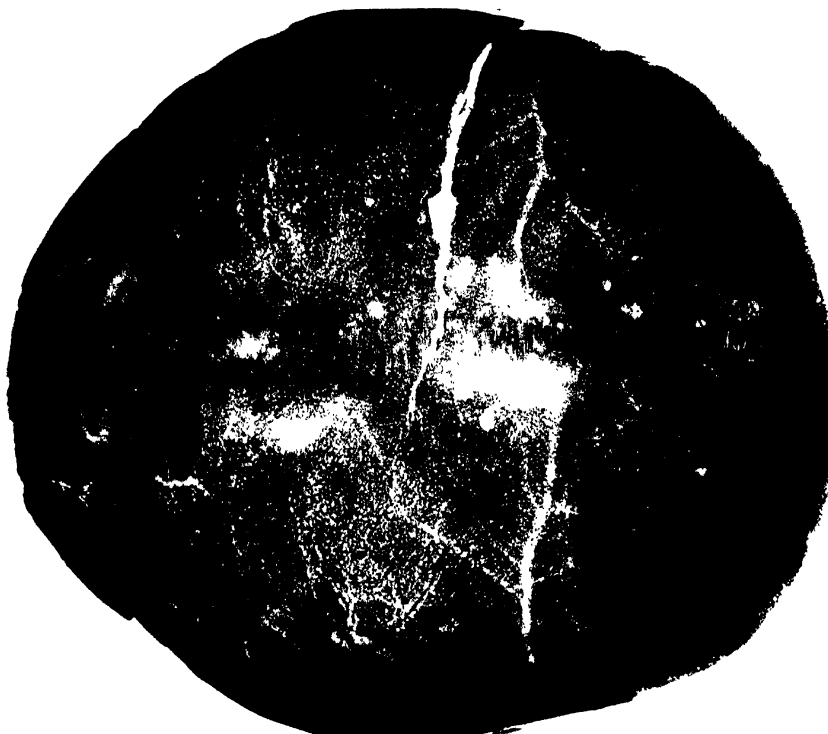
Adult; internally sutures largely obliterated, externally beginning to close. A few Pacchionian pits.

Skiagrams of Skull-Caps.



(c)

Oldish Adult, externally sutures largely closed or obliterated, internally nearly fully obliterated, many Pacchionian pits.



(d)

Wilkinson Head, showing hole where spike was rammed through fracturing the right parietal. Sutures closing or closed. Numerous Pacchionian pits (Oldish Adult).

S.F. 375. *Ecto-cranial sutures*—metopic beginning to close, coronal beginning to close, sagittal closed at obelion but otherwise open, lambdoid clearly open.

Endo-cranial sutures—metopic, coronal and sagittal obliterated, lambdoid open.

S.F. 657. *Ecto-cranial sutures*—coronal closed near bregma and open at pteria, sagittal obliterated except for 2 cms. immediately behind bregma where it is closing, lambdoid obliterated near lambda and open near asteria.

Endo-cranial sutures—coronal obliterated, sagittal obliterated except for 1 cm. above lambda where it is open, lambdoid obliterated except close to asteria where it is open.

A comparison of the internal-external whole plate views shows that for the Head the sutures are in a far more advanced state of closure than for the youngest of the Spitalfields skull-caps (S.F. 672). The film of the latter shows the lines of both ecto-cranial and endo-cranial sutures for the whole length of the sagittal suture and for part of the lambdoid. No trace of the internal sutures can be seen on the film of the Head and its sagittal suture is probably obliterated internally. The lines of the external metopic, coronal and sagittal sutures can be traced rather more easily on the film of the Head than on that of the Spitalfields skull-cap of intermediate age (S.F. 375) and far more easily than on the oldest skull-cap (S.F. 657). The sagittal suture cannot be traced at the obelion and it is probably closed externally, or beginning to close there. The external-internal skiagrams suggest the same conclusions. The sutures of the Head are probably in the same, or in a rather less advanced, stage of closure than those of the skull-cap S.F. 375 and this accords with what can be seen directly.

From the two kinds of evidence it seems reasonable to conclude that for the Head the endo-cranial sutures are all obliterated or nearly obliterated, while of the external sutures the metopic is beginning to close, the coronal is beginning to close and is nearly obliterated near the pteria, the sagittal is open or beginning to close except at the obelion where it is obliterated, and the lambdoid is open. For the Hythe and Spitalfields series it was found (see *Biometrika*, Vol. xxiv. p. 172 and Vol. xxiii. p. 218) that externally the sagittal suture was almost invariably the first, or one of the first, to begin closing while the coronal and lambdoid closed at about the same time as one another. Also, when the metopic suture persisted to an adult stage it appeared to close at about the same time as the sagittal. The suggested condition of the sutures of the Head is in accordance with these observations.

What do these results as to the sutures enable us to predict about the probable age of the owner of the Wilkinson Head? The chief investigators of this topic—on which much remains to be found out—are Frédéric* and Wingate Todd†. The former author gives detailed information about the large number of both European and non-European crania in the Strasburg collection. He draws general conclusions from his data, but provides no means of simply answering the question: 'This being the state of the sutures as to closing, what is the probable age?'

* *Zeitschrift für Morphologie und Anthropologie*, Bd. ix. S. 373—456 (1906).

† *American Journal of Physical Anthropology*, Vol. vii. pp. 326—384, 1924, Vol. viii. pp. 23—45, 1925.

He uses Broca's divisions for the sutures and practically Broca's method of reckoning*: 0 = open, 1 = $\frac{1}{4}$ closed, 2 = $\frac{1}{2}$ closed, 3 = $\frac{3}{4}$ closed, and 4 = wholly closed. This scheme is also followed by Wingate Todd. Most unfortunately Todd does not give the data for each of his individual skulls, all he gives us are charts indicating a sort of mean value for the closure or partial closure of each suture or section of a suture. The important point, the variability round such mean value, is not provided. It is thus impossible to ascertain from his diagrams any measure of the probability that a skull of given age will have a suture-closure of the given value. Further, what we really want is a table the other way round, i.e. giving for each suture-closure value the mean age and variation in age. Lastly one of the most important points is that we are not sure whether "closure" and "obliteration" mean precisely the same thing with Wingate Todd and Frédéric, or indeed with other writers. It might be argued that closure must precede obliteration, and there is a possibility for the play of much personal equation until there is a finer definition of what is to be understood by "closure." This is well illustrated by what Wingate Todd writes as to "lapsed unions" (i. p. 337); in such cases the suture is visible, but according to him such cases must be classed and he did class them as "closed," although according to Broca's diagrams and Frédéric's practice they could not be considered "obliterated." It will be seen that until there be a concordat as to what is meant by "closure," "Verwachsung," "obliteration," and until scientific statistical methods have been applied to really adequate material, say 50 crania in each five-year group, it will not be possible to reach an age appreciation with any satisfactory degree of probability†.

Let us see if, on the basis of Frédéric's and Wingate Todd's data, we can produce any definite unlikelihood that the Wilkinson Head belonged to a man of age 59.

First, on the Wilkinson Head all the endo-cranial sutures are closed. According to Todd the sagittal suture in all four sections is closed at 35; the Head therefore probably belonged to a man aged 35 or over (Diagram I, p. 344). According to Diagram I, p. 350, the whole of the coronal suture is not closed till after 40; the age of the man to whom the Head belonged was probably over 40. Turning to the endo-cranial lambdoid suture (Diagram I, p. 354) this suture is not fully closed till between 45 and 50; the Head suggests a man probably at least 47.

Thus, as far as the endo-cranial sutures are concerned there is nothing to oppose the view that the Head was that of a man of any age over 47.

We come next to the ecto-cranial sutures. If we suppose that the closure of the sutures starts as a rule at the inner table and creeps up through the bone to the outer table terminating with final obliteration after closure, then the need for exact definition of, and distinctions, if needful, between "closure" and "obliteration" becomes very obvious. This is of peculiar importance having regard to Wingate Todd's classification of "lapsed unions‡." Are such "lapsed unions" to be found in

* Wingate Todd follows Frédéric in reversing the numbers as used by Broca who termed 4 completely open and 0 completely closed.

† We hold that a series of skiagrams corresponding for each suture to a definite grade of closure might be the best way to reach not only a standard, but to standardise skulls.

‡ What do such "lapsed unions" look like on skiagrams?

senile crania? If many of us have been including them in the non-obiterated instead of the obliterated or closed class, we shall find that Wingate Todd's ecto-cranial suture diagrams give rather earlier ages for their closure than we should otherwise expect.

Ecto-cranial sagittal suture. This suture is obliterated at the obelion; elsewhere it is open or beginning to close. Wingate Todd's Diagram (II, p. 25) indicates that for a man of 60 we should expect the pars obelica of this suture fully closed, while the pars bregmatica, pars verticalis and pars lambdica are between $\frac{3}{4}$ and wholly closed. Unfortunately the same diagram shows that between 70 and 75 years of age these same three parts of the suture are only closed between $\frac{1}{2}$ and $\frac{3}{4}$. No doubt this is due to the smallness of the samples and the great variability in age of suture closure; but it indicates how little stress can be placed on a statement that apart from the pars obelica the sagittal suture will be externally 3 to 4 parts obliterated at 60 years of age.

Turning to the coronal suture; for the Head it is beginning to close and is nearly obliterated at the pars pterica. According to Wingate Todd's Diagram (II, p. 25) this closure is to be expected in a man of 60 years of age, although his curve is erratic. The pars complicata should be not quite half closed and the pars bregmatica three parts closed. Here again the curves are strangely erratic, for at age 75 the pars complicata is given as only about $\frac{1}{3}$ closed, although $\frac{4}{5}$ closed at 60!

Lastly, turning to the lambdoid suture which is described as open in the Head, not one of its three sections is given as open after 25 years of age in Wingate Todd's Diagram (II, p. 25). The pars lambdica is given as $\frac{3}{4}$ closed, the pars media as about $\frac{1}{2}$ closed, and the pars asterica as about $\frac{1}{4}$ closed at 60 years of age; but between 70—75 years, and so 12 years later, the pars lambdica and the pars media are both less than $\frac{1}{2}$ closed and the pars asterica is open! It will be clear to the reader that the samples in Wingate Todd's series are inadequate to settle any really definite age from the diagrams of the ecto-cranial sutures.

If we turn to Frédéric we find no attempt at statistical reduction and all we can do is to examine his *Tabelle VI*² (following S. 456). He has one skull only of a man—from Elsass—aged exactly Cromwell's 59 years.

Ecto-cranial Sutures.

<i>Frédéric's Elsassier - aged 59</i>		<i>Wilkinson Head</i>
Metopic Suture	Pars supra nasalis	$\frac{1}{2}$ closed
	Glabella Portion	closed
	Bregmatic Portion	closed
Coronal Suture	Open throughout	
Sagittal Suture	Pars bregmatica	$\frac{1}{4}$ closed
	Pars verticalis	$\frac{1}{4}$ closed
	Pars obelica	closed
	Pars lambdica	$\frac{1}{2}$ closed
Lambdoid Suture	R. wholly open	R. wholly open
	L. wholly open except in pars lambdoidea, where it is $\frac{1}{4}$ closed	L. wholly open

The comparison of the two skulls shows practically as large an amount of open sutures in the Elsässer of 59 as in the Wilkinson Head. Further in the case of the endo-cranial sutures, while these are all closed in the latter, the lambdoid suture is only $\frac{3}{4}$ closed in the former, the coronal and sagittal being fully closed.

It would thus seem that no strong evidence against the Wilkinson Head being that of a man of 59 can be based on this examination of the sutures*. On the whole we think the sutures are more compatible with those of a man aged 59 than of a man aged 40 (Ireton).

Notes on the Teeth of the Wilkinson Head.

Mandible. Left Side. No teeth had been lost before death anterior to the third molar. The third molar had either been lost before death, or it had never erupted, or had never existed. The skiagrams show no sign of an unerupted tooth below the alveolar border, and the smooth surface of the bone—as far as it is visible—suggests that it is more probable there never was a third molar than that it had erupted and been lost before death. All the other teeth had certainly been lost in their entirety post-mortem, except in the cases of the first molar of which there is a broken anterior root and the second pre-molar, of which the broken root remains. In these cases it is also clear that the fractures were post-mortem.

Right Side. No teeth were lost before death, but all have been lost since death except the third molar which is complete but considerably worn, and the first molar of which the broken anterior root still exists but the fracture was post-mortem.

Upper Jaw. Left Side. The second pre-molar was lost before death, but all the other teeth including the third molar were lost post-mortem. There are no broken roots *in situ*.

Right Side. The first molar was the only tooth lost before death. All other teeth have been *completely* lost post-mortem except (a) the broken root of the first pre-molar, (b) the broken antero-buccal root of the second molar, both fractured post-mortem, and (c) the third molar which is complete, but considerably worn.

The third molars on the right side are the only complete teeth *in situ*. The upper of these is loose in its socket, and can be moved upwards some 2 mm. Further the lower molar has a slight cant to the left so that it appears nearer to the median plane than the upper by some 4 mm. This displacement of the mandible† is probably due to the thrusting in of the prong and pole, and if the Head be really Cromwell's we might anticipate that the chin would be too much to the left of the face when fitted to portraits, or if chin be fitted to chin, the left orbit of the Head would be brought too low.

* If we may trust Wingate Todd's Diagram (I, p. 364) the suture between temporal squama and parietal is entirely open at 60, and not till 65 is it even $\frac{1}{2}$ closed; its obliteration, even if exceptional, would be an argument for the Wilkinson Head being far from young.

† One of the questions we asked ourselves at an early stage of our investigations was whether the jaw had possibly been dislocated when the spike was forced through the Head between the branches of the mandible, and the pole came into contact with them; after some difficulty we obtained skiagrams showing the head of the condyle in its socket, but these are only clearly intelligible by aid of stereoscopy.

The upper second pre-molar on the left side and the upper first molar on the right side are the only teeth which were certainly lost before death, their alveolar margins being completely absorbed. The lower left third molar may possibly also have been lost before death.

The alveolar margins of the teeth lost post-mortem, and particularly those of the front teeth, are well preserved. The condition of the inter-dental bone of the front teeth suggests an absence of pyorrhœa in that region at least. If the Head be that of a person who died at the age of fifty-nine, his dentition by modern standards* was remarkably well preserved.

If Noble's statement was more than a suggestion to account for change of expression, if it had been based on actual records, then the Wilkinson Head could certainly not be Cromwell's, but the dentition of that Head is not so excellent that many men of fifty-nine of those days and even of today could not show as good. Cromwell as a child must have led a healthy out-door life, and as a soldier rough and coarse fare might exercise his jaws, and rather wear away than start decay in his teeth. Cromwell was not a man who aged in youth, it was in the last few years of his life that illness and political worries aged him. There is nothing in either the sutures as far as visible, or in the dentition of the Wilkinson Head which compels us to say that it cannot be Cromwell's; while it might well be that of a man of fifty, it would not be impossible as the head of a man who died at 70 years of age.

As far as the teeth of the actual Cromwell are concerned they would hardly be loose in their sockets at his disinterment, and if this Head be his, it must have suffered much hard treatment since. Not only have nearly all the teeth been shaken out (or extracted purposely), not only has the bulk of the hair gone, but the skull has at some time been *fractured*. The fracture might well have arisen from a fall. It ends abruptly on the cincture, and extends from its border downwards some 20 mm. of the exposed surface of the occipital bone. It is not possible to say how far it extends down the occipital, because of the covering internally and externally by the skin. None of the skiagrams indicates the complete fracture, but, judging by the separation of the bone parts, it must descend far into the bone. The fracture may be seen on our Plate XXIV, just halfway from the left end of the cavity made by the chisel to the point where the hair on the left overhangs the cincture. It appears solely as a white line on the photograph (Plate XXVIII) of the left profile of the Head. It is not an unreasonable site for a fracture if the Head had fallen either on the occipital or on the left parietal.

* [The "standard" is largely based on the opinion of modern dentists working among a town population, and they rarely see the individuals with thoroughly sound dentition. My father died at 86 with the loss of only 2 to 3 molars, all his other teeth being *in situ*; my father-in-law died at 70 with all his teeth complete; and a sister-in-law, aged 60, has not yet lost a tooth. A second dentist of wide experience remarked on seeing the Cooper miniatures that the muscular development of Cromwell's jaw was not consistent with his having lost many teeth at the time of these portraits being painted (1653—1658). A third dentist stated that he considered above 30% of the dentist-seeking population had practically perfect dentition at 60 years of age. K.P.]

While we are concerning ourselves with the Head we may draw attention to the manner in which the Head has been severed from the trunk. As we have said the actual severance occurred by division of the fourth vertebra, but the bungler who performed the task first attempted severance much higher up, and here he appears to have cut out a solid wedge of the embalmed flesh (see Plate XXIX). Even where the severance was finally made he does not seem to have confined himself to a single blow. Thus far we can say that at least four blows were used. But if we examine the attempted severance, assisted if needful by a lens, there appear to be further marks of the edge of the axe within the bottom of the wedge-shaped cut (see Plate XXIV). We think that at least six, and possibly more blows of the axe have been given. If we may, and we think we must, trust Sainthill's account (see our p. 314) this suggests Cromwell's rather than Ireton's head, which latter, granted the Head is a head decapitated after embalmmment, appears to be our only alternative*. To settle this point a little more definitely a measured drawing was enlarged from a reproduction of the head of the portrait of Henry Ireton in the possession of Mrs R. B. Polhill-Drabble, and fitted with the sketch of a photograph of the Wilkinson Head with the same aspect (see our Plate XL). The Head fitted *vertically* rather better even than it does to some of the Cromwell portraits, but considered in the horizontal plane of the pupils was quite clearly impossible. The contour of the Head on the left side of Ireton's face projects right over the hair, and would project half as much again, if we replaced the embalmed skin by its appropriate amount of living flesh (see our Plates LXXXV—LXXXVII). We believe that the alternative hypothesis that the Head is that of Ireton (which if we discard tradition has *a priori* as much to be said for it as the hypothesis that it is Cromwell's Head) must be discarded†.

Another characteristic feature of the skin of the Head is the profusion of pimples: see our Plates XXVI and XXX. Are these evidences of the very "roughnesses, pimples and warts" (see our p. 350) that Cromwell himself is said to have pointed out to Lely? We attempted to compare the single ear on the Head with Cromwell's ear. Not only, however, is the embalmed ear so dried up that it is impossible to ascertain its original form, but further we could find no representation of Cromwell's ear, except that on the Florence bust (see our Plate LXXXIII). Not knowing the circumstances under which this bust was produced we cannot be certain that it is a truthful model of Cromwell's ear. Accordingly any prolonged study was scarcely likely to be profitable. We may note, however, one point we observed: both ears have an unusually straight upper border; this applies to both edges of the rim. Cf. Plates XXVIII and LXXXIII. A last point with regard to this Head may be

* A letter of T. C. to the *Monthly Magazine* (March 14, 1821, p. 110) makes the statement that of the bodies of the three regicides only Cromwell's had been embalmed. This is directly opposed to Sainthill's account of the aspect of the bodies at Tyburn and to the fact that Ireton died in Ireland on Nov. 26, 1651 and was not buried in Henry VII's Chapel, Westminster, till Feb. 6, 1652. He had been more than eight years buried when he was exhumed, but his face was said to be recognisable: see our pp. 307 and 313—314.

† The conditions of the sutures are compatible with those of a man of 40 or of 50, but on the whole we think that they and the condition of the skin point to the greater age.

pointed out to our readers, if they will examine Plate XXXVI (*d*). On the skiagram may be noted a number of whitish spots chiefly on the frontal and parietals. No suspicion of the existence of these spots could have arisen without the aid of X-rays, for the bone is not exposed. The spots mark an abnormality of the cranial vault. We sought Sir Arthur Keith's opinion and he most kindly replied as follows:

As to that skull, there can be no doubt, I think, that the white spots you ask about are Pacchionian impressions or pits. They are in the right position for such depressions.

It should be the skull of an oldish person* and the bone formation is not normal. The condition is one of those classed as osteoporosis. I have seen the condition in the skulls of the insane†.

Sir Arthur had no knowledge of the possible owner of the skull; and Cromwell was certainly not insane, but he did suffer from a religious monomania; he believed that he really received inspiration from the Deity, which unfailingly guided him in his actions. Perhaps he could not have achieved what he did without such a belief, and the power to impress this belief on his subordinates.

With regard to the Pacchionian depressions, the majority of anatomists consider them to increase with age‡. With regard to their frequency, Dr Dudley Buxton most kindly examined the number of Pacchionian depressions in the case of 116 skull-caps in the Anatomical Collections at Oxford. Of these 66 had belonged to sane and 50 to insane persons. The mean age of the sane was 32·42 years with a standard deviation of 10·07 years, and of the insane 36·80 years and 11·12 years respectively. The mean number of depressions for the sane was 2·80, with a standard deviation of 2·101, and the corresponding quantities for the insane were 3·36 and 2·744. Thus the insane were sensibly older and more variable in number of pits than the sane. In fact only two of the 66 sane had an age over 50 while the 50 insane had four over 60. But this, while hindering any conclusion as to the frequency of these depressions in aged persons, gave some information as to the insane. No sane person had macroscopically more than eight depressions and there was only one with eight, but there were four of the insane with 10 pits, and only one of the four was aged (67), there being two (21 and 24) among the youngest ages. This gives some support to Sir Arthur Keith's view that he had noticed the condition in the skulls of the insane. It is clear that the Wilkinson Head has a large number of these Pacchionian depressions (Plate XXXVI (*d*)), a far larger number than was exhibited by any of Dudley Buxton's normal skull-caps, and it seems justifiable on this ground to class it as "oldish." A fuller investigation into the question of the incidence of these depressions with age will, it is hoped, be shortly undertaken. The more so as one of the present writers believes that such a study may throw more light on age

* This is undoubtedly against the possibility of its being Ireton's head.

† Le Gros Clark (*Journal of Anatomy*, 1920-21, p. 40) suggests that in conditions which are associated with increased pressure of the cerebro-spinal fluid, the Pacchionian bodies increase in size and number, e.g. in cases of paralytic dementia (G.P.I.), chronic nephritis and arteriosclerosis, and advancing age.

‡ See for example: Henry Gray's *Anatomy*, 15th Edition, 1901, p. 29, and again, pp. 620-621: "They [these pits or depressions] are usually found after the seventh year; and from this period onwards increase in number as age advances. Occasionally they are wanting."

than the closure of sutures can do, and the other is of opinion that sutures can give a better appreciation of age than Pacchionian depressions.

Stopping of the Nostrils. Had this been cotton wool as suggested in Maria Edgeworth's letter (see our p. 293) it would have told against the originality of the Head or at least of the stopping.

Canon Wilkinson kindly presented Dr Morant with a small sample of the stopping from the nose of the Head. Dr A. Davin of the Linen Research Institute, Belfast, most kindly made a microscopic examination of the fibres of this stopping. On Plate XXXVII will be found photomicroscopic reproductions of flax and cotton fibres $\times 120$.

(a) Represents *flax* fibres from medium quality cloths.

(b) Reproduces *cotton* fibres from modern medical lint.

The reader will see at once that the flax fibres differ from the cotton by being straight and untwisted.

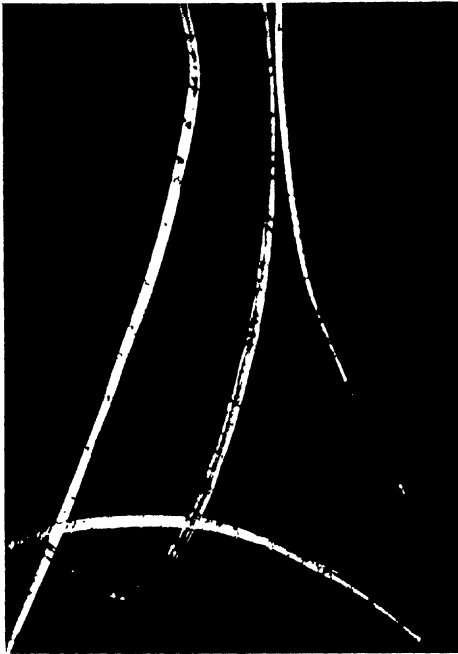
(c) Provides a sample of the fibres from an Egyptian mummy wrapping taken from a tomb of the 1st Dynasty. The fibres are clearly flax, being untwisted and correspond with (a).

(d) Gives the fibres from the stopping of the nose of the Wilkinson Head. They are again flax, and the reproduction suggests that they are scrapings from linen, i.e. true lint*. Hence the stopping of the nose of the embalmed Head carries us back to a time when medical lint was made of flax, not cotton. But when did cotton wool come into general use for medical and surgical purposes?

We have made some attempt to trace the introduction of cotton in place of flax in the case of medical lint. It must, however, be remembered that while cotton was known much earlier the first shipments of cotton from America did not reach this country till 1770 and from the East Indies not till 1783. Also there were Acts of Parliament against the wearing of cotton in the first quarter of the 18th century†. A letter to the authorities of the Royal College of Surgeons to ask whether any of Hunter's specimens contained a sample of medical stopping led to the discovery of some stuffing which had been used to fill a large ovarian cyst, stitched up after being stuffed, the stitches appearing to be the original ones. The cyst was removed after death from a patient who died in 1783. This sample of stuffing proved under the microscope to be *animal* wool, not cotton wool. The specimen is No. 8317.1 in the Pathological Collection. A statement has been made that cotton wool was first

* Cf. the old slang "lint-scraper" for a young surgeon, as one employed in scraping linen to produce medical lint.

† The following dates, which we owe to Dr A. Davin, are of interest, especially in relation to the calico-printer Claudius Du Puy and his intestacy. Calico imported into England, 1601; Muslins imported, 1670; Calico-printing introduced, 1696; Wearing of cotton goods prohibited in England, 1700; Act of Parliament imposing fine of £5 on wearer and £20 on vendor of cotton goods, 1721; Act of Parliament allowing goods with linen warp and cotton weft to be printed on paying an excise duty of 6d. per sq. yd., 1780; Cotton stockings first woven, 1780; Cotton first imported from America, 1770; from East Indies, 1783; Muslin first woven in England, 1778.



(a) Scrapings from Linen. Straight Flax Fibres.



(b) Modern Medical Lint. Twisted Cotton Fibres.



(c) Egyptian Mummy.
Straight Flax Fibres.



(d) Sample from pad in nostril of Wilkinson Head.
Straight Flax Fibres.

used for surgical purposes with the army in Flanders in the first half of the 18th century. If verified, it would not aid us, as it is clear that it was not in general use even in 1783, and consequently no limiting date can be fixed by the flax stoppings of the nostrils of the Wilkinson Head.

12. *Cromwell's Helmet and Hat.*

We may make here some remarks on the girth measurement of the Wilkinson Head, which, if somewhat inconclusive, are possibly not without interest. The maximum length from glabella to occipital is some 194 mm. This is from the dried flesh at the glabella to the exposed bone at the occipital. The thickness of the skin at the glabella amounts to 1—2, say, 1.5 mm. Thus we will take the cranial length at 192.5 mm. = L . The maximum parietal breadth is 155 mm., or allowing for the dried flesh may be taken as 150.7 mm. = B . Now if we suppose the contour to be approximately elliptic, the eccentricity of this ellipse, e , will be given by:

$$e^2 = 1 - (B/L)^2 = 1 - (.782,837)^2 = .387,135,$$

or the ellipse is of eccentricity $e = .6222$.

This gives $e = \sin(38^\circ 28' .6)$ approximately, whence the entire circumference* = $2 \times 192.5 \times 1.40543 = 541$ mm. This is for the cranial "horizontal" circumference usually denoted by U . The indication is that of a somewhat but not exceptional excess over the mean U of 17th century male English crania; thus for three cranial series, U for Whitechapel = 524 mm., for Liverpool Street = 527 mm., for Farringdon Street = 530 mm. The excess is in accordance with what we have noted of the cranial capacity. We may remark at once that Cromwell would not require a very outsized hat or helmet. We may allow him a length in life of 200 mm., and a breadth in life of 160 mm. and this would probably be adequate to include hair. Proceeding in the same manner we find $e = .6 = \sin(36^\circ 52' .19)$, and for the living circumference $U = 2 \times 200 \times 1.41807 = 567$ mm., of course only approximately.

Now it must be remembered that the hat circumference is somewhat less than the living U circumference or the hat would slip over the Head. Allowing 15 mm. for this difference, a closed band of 552 mm. was forwarded to the Reverend Paul Cromwell Bush, who is the lucky possessor of Cromwell's broad-brimmed Long Parliament hat† and his helmet, with a request for the long diameter of the hat, and the degree of fitting of the band to hat and helmet. It was with some curiosity we waited for his answer, for if the band was too large for hat or helmet, the Wilkinson Head could not be that of Cromwell.

Mr Cromwell Bush most courteously fulfilled our requests and replied as follows:

As to the Protector's Helmet the band you sent goes into that with a good margin, but much of the original lining of the helmet has perished.

* Using Legendre's Tables for $E(38^\circ 28' .6)$ and $\phi = 90^\circ$ on p. 322 of Vol. II. of his *Traité des Fonctions Elliptiques*, 1826.

† Possibly worn also in 1648 at the trial of Charles I: see our Plate XII.

This seemed satisfactory, as the helmet should sink well over the back of the Head. Mr Cromwell Bush continued:

I have not had time till today to take the Long Parliament Hat out of its case and make the necessary measurements. The inside of the crown measures $7\frac{1}{2}$ inches across as near as I can tell, and the band you sent me easily fits in with a small amount to spare. Whether the hat originally had any sort of lining I do not know—if so and allowing for the hair as well it would have been a tight fit.

Here again we must remark that the hat would slip down onto the ears if it had not a *less* length than the glabellar occipital, and the fact that the band left ample margin for the helmet and only a narrow margin for the hat indicates that they are not of the same size, which is what we should have anticipated; the Wilkinson Head may reasonably have coincided with Cromwell's head which must have been between them*.

$7\frac{1}{2}$ inches = 190.5 mm. and a girth of 552 mm. would give a breadth of 160.3 mm.; the hat would have been supported by its lesser length than the Head, not by its equal breadth. We do not think anybody's head could be individualised by a hat, but we think a negative case could be made out from a hat, certainly *not* fitting, but this is not so here.

SECTION IV.

Comparison of the Wilkinson Head with the Painted Portraits, Coins, Medals, Life and Death Masks, and Busts of Cromwell.

Thus far our inquiry has been threefold. We have in the first place investigated the history of the Wilkinson Head. We have seen that this history carries us at least back to 1773, when we find it in the possession of a somewhat disreputable actor, Samuel Russell, who was said to be and, if we can believe the tale, claimed to be a descendant of the Protector through one or other of the Russell families. We have not succeeded in linking him up with either the Chippenham or Fordham Russells. How it came into his possession we do not know, but von Uffenbach's description of the head, reputed to be that of Cromwell, in Claudius Du Puy's museum accords so well in the little that is said with the Wilkinson Head that we take it that this head was in existence at least as early as 1710. How it came to Samuel Russell we do not know. Two men, who might together have adequate knowledge to commit a fraud, come too late into the story to explain in that way the existence of the Head. Samuel Russell would have neither the knowledge nor

* The Head itself was tried in the hat when the latter was in the possession of Mr Oliver Cromwell of Cheshunt, the great-grandson of the Protector and the last descendant of Cromwell to bear his name. Mr Oliver Cromwell of Cheshunt was the great-great-great-grandson of the Rev. Cromwell Bush of The Vicarage, Chewton Mendip, through the female line; and the latter has inherited in this way his most interesting Cromwelliana.

In *The Weekly Dispatch*, June 17, 1821, p. 191, we read:

Mr Cromwell, of Cheshunt, has now in his possession the hat of his ancestor, Oliver Cromwell, by which the skull [?] supposed to be the Protector's, which with two others were after the Restoration affixed over the entrance [?] to Westminster Hall, until the reign of Queen Anne [?] has been tried and no doubt is now entertained of its identity.

ability to commit such a forgery, and it would be entirely out of the question, if the Head be the same as that formerly in Du Puy's possession. In fact, if it be a forgery, so skilful a one seems impossible before the publication of Noble's work in 1784. What fraudulent artificer would realise that the skull-cap must be removed before, and the head chopped off and by several blows after embalmment? What forger would have broken the nose, made the lips appear to be decayed away, left only two teeth in the head, and removed the hair which is a marked feature of Cromwell's portraits? The Head is not a fraud, it contains much of its history in itself. But that does not definitely prove it Cromwell's; the traditions concerning it are not improbable in themselves, but they are not proven, and are not unique.

Secondly we have followed up the story of Cromwell's remains; we find no vestige of verisimilitude in the legends as to the disposal of Cromwell's body. It was undoubtedly buried in Henry VII's Chapel at Westminster, dug up by order of Parliament, carried in its coffin on a sledge to Tyburn, pulled out of its gorgeous coffin there, and hung in its sixfold cerecloths on the triple gallows. At sunset it was taken down, beheaded, and the trunk with those of Ireton and Bradshaw buried under the gallows. Whether the coffin was afterwards removed anywhere, or the trunk dug up and taken to Newburgh Priory, we cannot say. The head of Cromwell was placed four or five days after decapitation on the south end of Westminster Hall probably towards the east side, and remained there at least till 1684 and possibly till the end of James II's reign. We have no evidence of what became of it after this period. It is conceivable that it would not be a very welcome sight to rulers who like Cromwell did not claim that they succeeded by divine right, but that they had been chosen by "vote of the people." The Wilkinson Head, whether due to the embalmment or to being dipped in preservatives after decapitation, is still in such a condition that it could stand a good many years of weathering, and if it were demonstrated to be Cromwell's head might have withstood 20 to 30 years' exposure* once before.

Thirdly we have discussed the chief characters of the Wilkinson Head, not turning to metric characters, and have been unable to find anything which renders it highly improbable that it could be the genuine head of Cromwell, but this does not demonstrate that it is. It might, for example, have been equally likely to be the head of Cromwell's son-in-law, Ireton, who was also embalmed, if badly, decapitated after embalmment, and spiked and poled on the south end of Westminster Hall. This possibility has been dismissed owing to Ireton's age and frontal breadth.

The only way to test the truth or falsehood of the Wilkinson tradition is to appeal as we do in this section to the portraits and masks.

* From this aspect it is interesting to remind the reader that Jeremy Bentham in his pamphlet on *The Auto-Icon*, suggested that the well embalmed bodies of a man's ancestors, their auto-icons, should be placed along the avenue to the family mansion. Clearly he did not fear weathering would affect them, and he usually knew what he was talking about. His auto-icon, although not exposed to the weather, has stood a century of the stone-destroying London atmosphere without disintegration, and there is little doubt would easily have survived 20 or 30 years of weathering.

13. *Painted Portraits of Oliver Cromwell.*

At least three contemporary masters of the brush have provided us with portraits of Oliver Cromwell, namely Walker, Cooper and Lely. There is no doubt that pictures of Cromwell were painted by two of these artists and with the highest probability by the third; we have records of their doing so; but if we are to count as genuine all the portraits said to be by Walker and Cooper then a great deal of the time of these painters must have been occupied solely with Cromwell. Our first object must be to discover portraits the history of which is reasonably certain, and such portraits must in the first place be sought in the families which are directly descended from Cromwell. We do not propose here to provide an exhaustive list of the portraits of Cromwell, but only to select out of the very copious material those most likely to decide the problem we are studying, the genuineness* or spuriousness of the Wilkinson Head.

(i) We will consider first the Walker paintings. These may be said to be of four types: (a) Cromwell stands with a baton in his right hand, his left hand rests on a helmet. (b) Cromwell stands with a baton in his right hand, a page standing behind him on his right ties his sash. (c) Cromwell stands with a baton in his right hand, a page stands in front on his left, tying a sash. Behind is a table and helmet. (d) A half-length in armour without baton.

(a) A portrait of this type is in the possession of Mrs R. B. Polhill-Drabble at Woodside, Sundridge, Sevenoaks. Tradition states that it was a wedding gift† of Cromwell to his daughter Bridget when she married Henry Ireton, 1646. Cromwell was then Lieutenant-General of the Horse under Fairfax, and the marriage took place at Holton St Bartholomew, six miles from Oxford, which Fairfax was besieging. The son of this marriage, Henry Ireton, died without issue, and in his will‡ left this portrait of his grandfather to his nephew David Polhill, his sister Elizabeth Ireton's son, whose son Charles united the blood of Cromwell, Ireton and, through his mother, of Hampden. Charles Polhill's granddaughter, who inherited the Cromwelliana, married a Beadnell, and again her daughter married Mr R. B. Drabble. This brings us to the present owner, Mrs R. B. Polhill-Drabble. Thus the history of the picture appears to be well authenticated.

A second portrait of this type, with baton but without page, is at Warwick Castle.

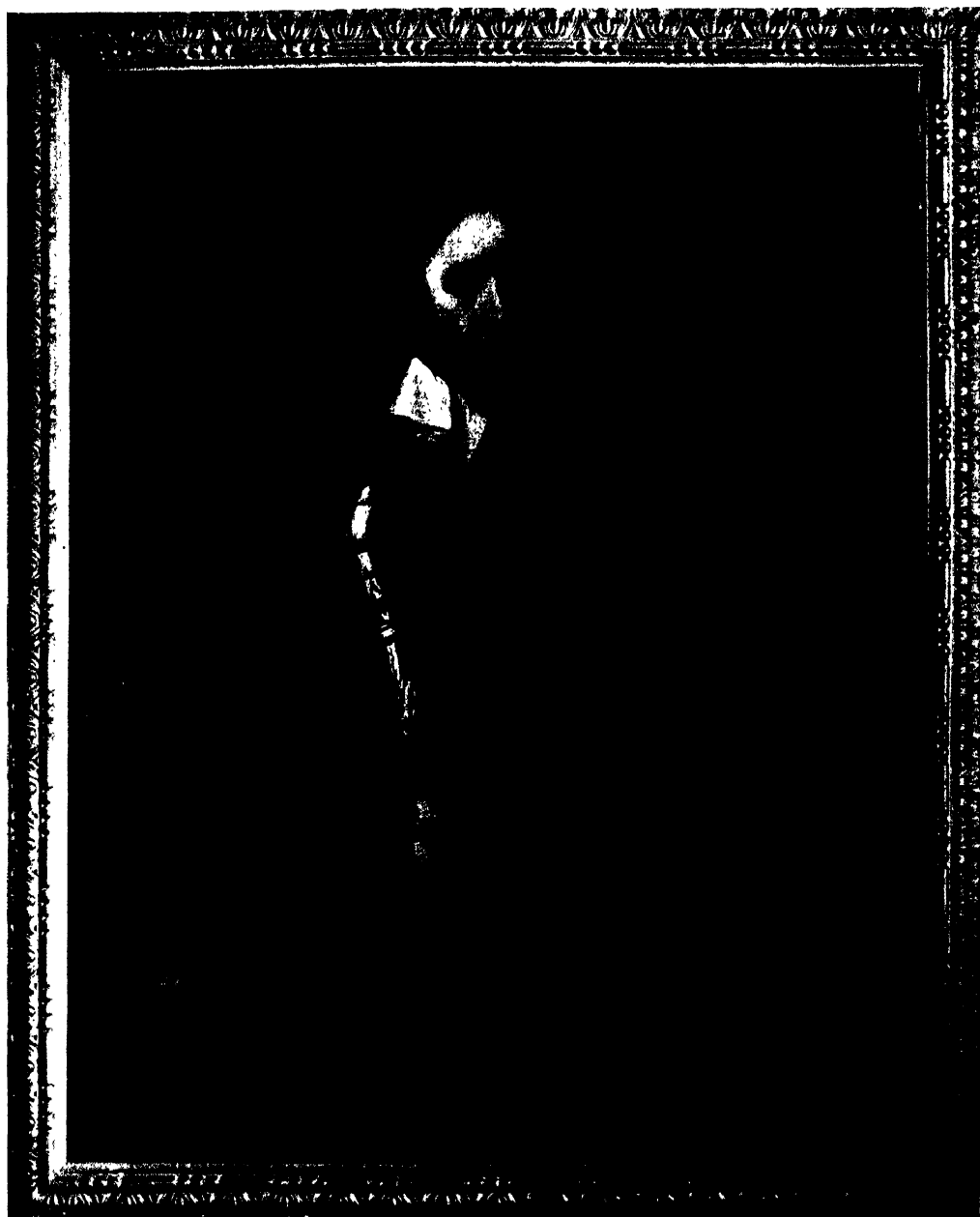
(b) A portrait of this type is in the National Portrait Gallery, No. 536; see our Plate XXXIX. The history of this picture seems reasonably authentic. It was bequeathed to the British Museum in 1784 by Sir Robert Rich, Bt., and is said to have been given by Cromwell to the baronet's great-grandfather, Colonel Robert Rich, who married in 1657 Cromwell's daughter Frances. It was transferred to the National Portrait Gallery in 1879. Frances Cromwell being married in 1657, her

* We use the word "genuineness" to signify the genuine head of Cromwell. The Head is certainly genuine in the sense that it is not a forgery.

† If the baton denotes that Cromwell was Captain-General and Commander-in-Chief the gift must have been later than 1650, and thus hardly a wedding present.

‡ 1712, at Somerset House.

Pearson and Morant: *The Cromwell Head*



The Walker Portrait of Cromwell in the possession of the Rev. Paul Cromwell Bush.
By kind permission.

Pearson and Morant: *The Cromwell Head*



The Walker Portrait of Cromwell in the National Portrait Gallery, No. 536.

Pearson and Morant: *The Cromwell Head*

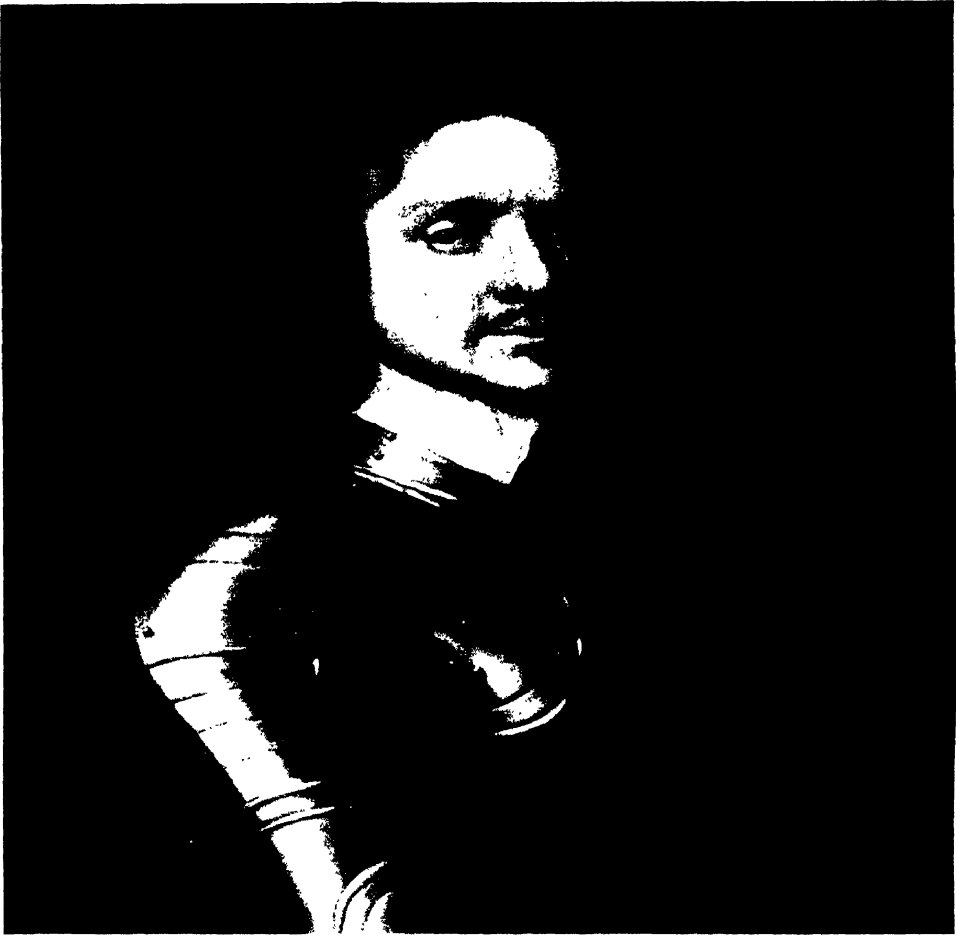


The Walker Portrait of Cromwell in Justitsministeriet in Copenhagen. The details are far more complete than in the National Portrait Gallery, No. 536. See Plate XXXIX.

Pearson and Morant: *The Cromwell Head*



The Walker Portrait of Cromwell in the Tangye Collection, London Museum.



(a)



(b) Original Height 2.2 cm.

The Stockholm Portraits of Cromwell (Walker). I.

- (a) Traditionally the portrait Cromwell sent to Queen Christina with Milton's Latin verses. Gripsholm, No. 1213.
- (b) Gouache on ivory. Formerly preserved in the castle of Drottningholm. In gold capsule and back enamelled in turquoise blue. An original by Walker, or a copy of a Walker. It has been associated with Whitelocke's mission to Sweden.



(c)

The Stockholm Portraits of Cromwell (Walker). II.

(c) Gripsholm, No. 93, after Walker.

Chiefly notable for its emblematic frame.

father was then Lord Protector, and it seems unlikely that Walker would have painted a "baton" picture at that date, unless as a replica of an existing picture. The picture must have descended to the relatives of the Hon. Robert Rich*, and not to the offspring of Frances' second marriage with Sir John Russell of Chippenham.

A Walker picture of this class is also in the possession of Sir John Payne Gallwey; it came originally from Thirkleby Park, Thirsk. Sir John is in the direct descent from Cromwell through the Franklands and the Russells of Chippenham, and so from Frances, fourth daughter of Cromwell.

The picture is a three-quarter-length with a page tying on a sash and Cromwell holds the baton in his right hand. The following inscription is painted on the base of the picture on the left hand: "Oliver Cromwell, Lord Protector. This picture formerly owned by Henry Cromwell, Ld. Lieut. of Ireland, 4th son of Oliver Cromwell (Copy in the British Museum)." On the bottom right-hand corner is painted in block letters: WALKER. The picture's history is no doubt authentic, but the inscription is of late date (*post* 1784?). The baton denotes the Lord-General rather than the Lord-Protector period. Henry Cromwell was "Deputy for Ireland." Thus the words "Copy in the British Museum" can hardly be taken in themselves as giving priority to the Payne-Gallwey picture over the Rich picture.

A famous portrait of this type is in the possession of Earl Spencer at Althorp Park. It is reproduced as the frontispiece to the Goupil-Gardiner *Oliver Cromwell*, 1899. An equally fine portrait (see our Plate XXXIX (*bis*)) is in the Justitsministeri at Copenhagen.

(c) A portrait of the third type belongs to the Rev. Paul Cromwell Bush, Vicar of Chewton Mendip, Bath. He is descended through the female line from Mr Oliver Cromwell of Cheshunt, the last descendant of the Protector to bear his name. Oliver Cromwell of Cheshunt was a grandson of Henry Cromwell, the son of the Protector Oliver. Mr Cromwell Bush possesses the receipt† of "Mr Walker the limner" for this painting, and there can be no doubt of its being genuine and probably painted from the life. By kind permission of Mr Cromwell Bush the

* Considering the great admiration the Earl of Warwick had for Cromwell the picture may have been given at an earlier date to him and not to his grandson.

† The receipt runs as follows:

MR WATERHOUSE

My Ladye [the Protectress] desires you to pay to Mr Walker the Limner the somme of twenty four pound for the draught of his highnesse picture so I rest

Whitehall, 15th of Jan.
1655.

y^r Lovinge friend

SIMON CANNON.

January 25. 1655.

Rec ^d then of Nath: Waterhouse Esq ^r	} 24£
the sume of twenty four pounds	
as above.....(?)	

ROBERT WALKER.

It is not clear from this receipt whether "the draught of his highnesse picture" was a replica of an earlier picture painted by Walker, or the result of a fresh sitting in the year 1654 (or 1655). We should hardly have anticipated in the latter case that a Walker of 1654 could differ so much from a Cooper of 1657!

picture is reproduced on our Plate XXXVIII. The head seems to have greater character than the other Walker portraits of Cromwell we have come across.

(d) A fourth class of Walker portraits represents Cromwell, half-length in armour, three-quarter-face, without baton. We are not aware that any such portrait is still owned by a family descended from Cromwell. There is a picture of this type in the Tangye Collection at the London Museum: see our Plate XL. Two further portraits are at Stockholm: see our Plates XL (*bis*) and (*ter*). The picture in the Pitti Palace at Florence, usually attributed to Lely, has been described by Lionel Cust* as the best example of this type of Walker. We shall, however, return to this point. Walker is supposed to have died in the same year (1658) as Cromwell, and his portraits give us Cromwell at an earlier stage of his career than those of Cooper. It seems probable that they all belonged to the period before Cromwell became Protector.

A study of the cut of the collar in the Walker pictures might assist in dating them. The type (d) without baton have a collar which meets at the top of the neck in front, but is cut out like an inverted Ω ; see our Plate XL. Those with the baton have a still wider opening in front: see our Plates XXXVIII and XXXIX. Walker does not make a collar with a sharp inverted Λ in front like Cooper does. The Florence portrait has a collar with the Λ and a crumpled left side, quite unlike Walker's collars. The crumple and the patch of reflected light on the gorget immediately below it in the Lely portraits (see our Plates XLI and XLI (*bis*)) are strangely akin to the large Chequers Court miniature (see our Plate XLVII) and the crude oil painting of Cromwell in the National Portrait Gallery (see our Plate XLIX)! This collar-work differs wholly from that of Walker or Cooper.

The certainly genuine portraits of Walker differ very essentially from those of Cooper. In the case of Walker we have ample hair round a rather long face; the face is young compared to that in Cooper's miniatures, less coarse and yet with less strength. There is little, if any, trace of the wide square jaw with which Cromwell has been credited, probably on the strength of Cooper's paintings. Accordingly if Walker be admitted to have painted with some degree of truth, then the wide square jaw of Cromwell was largely a fleshy or muscular development of later years and not a skeletal feature. The square chin of the masks taken late in life or after death confirms Cooper, but Walker must be borne in mind when we consider the Wilkinson Head.

(ii) When we turn from Walker to Lely we have less definite evidence of still extant genuine portraits by the latter master. The Florence portrait of Cromwell has always been attributed to Lely until recent times, when Cust changed his first opinion, and attributed it to Walker (see above). We have grave doubts about the accuracy of this re-attribution. As we have already pointed out Walker in his unquestionable portraits always gives Cromwell a definite form of collar; so also does Cooper. The Florence portrait has a third form of collar; it is crumpled on

* *D.N.B.* Vol. LIX. p. 82, "Walker, Robert," 1899. In 1893 Cust apparently accepted the Pitti portrait as by Lely, see *D.N.B.* Vol. XXIII. p. 19.

Pearson and Morant: *The Cromwell Head*



The Lely Portrait of Cromwell, 1650 (2), in the Pitti Gallery at Florence, from a photograph by Anderson.

Pearson and Morant: *The Cromwell Head*



Portrait by Sir Peter Paul Rubens

Portrait by Sir Peter Paul Rubens

OLIVAR *Rex Ang. Soc. & Hel. PROTECTOR*
Collectore Gulielmo Powlett Esq.

The Lely Portrait of Cromwell, 1653, from the larger mezzotint prepared by Faber in 1735
of the picture owned at that date by William Powlett, Esq.

Pearson and Morant: *The Cromwell Head*



The Lely portrait 1653

Re-engraved 1740

OLIVER CROMWELL. *Lord Protector*

The Lely Portrait of Cromwell, 1653, from the smaller mezzotint prepared by Faber in 1740 of the Powlett-Cavendish painting.

both sides, and below the crumple on the left side is a bright spot of reflection on the armour. These things do not appear on either the Walker or Cooper portraits. Further the head is much stronger and represents a conception different from those provided by the other two artists. It stands so to speak midway between the two, although this may point to an intervening age period. That there have been other portraits by Lely and, perhaps, still are, we shall now proceed to indicate.

In 1735 J. Faber issued a mezzotint engraving of a portrait of Cromwell, with his own name and date, and on the left below *Petrus Lely pinx.* 1653 (see our Plate XLI (*bis*)). In this engraving we have the crumpled collar and the strong stern face of the Florence portrait, but it certainly was not an engraving from the Florence picture. It is clearly from a picture by the same artist, but not from his Florence picture. It shows more of the arm-pieces than the latter does and the shoulder-pieces are of a slightly different pattern. The original picture belonged in 1735 to William Powlett. In 1740 Faber made a smaller mezzotint engraving of the same painting which appears as frontispiece to Peck's *Memoirs of Oliver Cromwell*, 1740. This fine print does not state where the original picture is (see our Plate XLI (*ter*)). In 1750 Faber reissued the large mezzotint with the date 1735 changed to 1750, and the words "e collectione Gulielmi Powlett, Gen." altered to "e collectione Dom. J. Cavendesh" by erasure of the former name. Thus in 1750, we have a mezzotint portrait of Cromwell from an original attributed to Lely, which is clearly by the same artist as the Florence portrait, but is not engraved from the latter. The owner was presumably Lord John Cavendish, fourth son of the third Duke of Devonshire, and in 1782-3 Chancellor of the Exchequer. Powlett probably sold the picture to Lord John Cavendish, but if the latter owned it in 1750 he must have started his picture collection at or before 18 years of age. He died unmarried in 1796. His pictures do not appear to have returned to the Duke of Devonshire's collections at Hardwick or Chatsworth*. Two of Lord John Cavendish's sisters married Ponsonbys, a third married the first Earl of Orford. Inquiry about the picture from the Earl of Bessborough failed to receive an answer. Had the picture passed to the Earl of Orford, it might be anticipated that Horace Walpole would have known of it; we shall see later that he did not.

Certain Waynwright letters are mentioned in the *Sixth Report of the Royal Commission on Historical Manuscripts*, 1877. On p. 426^b we read that Waynwright mentions a portrait seemingly of Cromwell painted by Lely (? 1651). On Oct. 6, 1654, Waynwright writing to Bradshaw, then in Copenhagen, says: "The picture you sent for [I] have bespoken of one Mr Lilley [Lely], the best artist in England who hath undertaken to do it rarely" and again a week later (Oct. 13, 1654): "I have bought you a curious picture, exactly done by Mr Lilley, who drew it for his Highness, and hath since drawn it for the Portuguese and Dutch ambassadors it cost me 12*l.* present money; I could [have] had it cheaper, but not so good." (pp. 437^b—438^a.)

* Information from the Duke's Librarian, who most kindly visited Hardwick to see if an unassigned portrait could possibly be the missing Cromwell.

Now here we have evidence of four portraits of Cromwell by Lely:

(a) One for the Protector himself. (b) One for Bradshaw presumably for presentation in Denmark. (c) One for the Dutch Ambassador. (d) One for the Portuguese Ambassador.

Besides these we have (e) the Florence portrait probably procured by the Tuscan Ambassador, and (f) Lord John Cavendish's portrait. Of these it seems possible that (a) and (e) may be the same: see Appendix to this paper. There seems to have been a plethora of Lely portraits! As (b) might have been the portrait Cromwell presented to Queen Christina of Sweden (accompanied by Milton's verses beginning:

Bellipotens virgo, septem Regina Trionum)

it appeared well to include Stockholm among the places where inquiry should be made. Our investigations were not very fruitful in the discovery of Lely portraits, but certainly produced interesting results.

We inquired from the Director of the Galleries and Museums of Florence, if it were known when and how the Lely picture came to Florence. He replied that it was unknown, but that:

Dans un ancien catalogue de la collection est cette note: "Cromwell fut peint par Lely quand il commandait l'armée à la bataille de Dunbar et de Worcester, dans la 51^{ème} année." Mais on ne sait pas d'où cette information a été puisée.

The battle of Dunbar was fought on September 3, 1650, and that of Worcester September 3, 1651. Cromwell was born in 1599, hence he was aged 51 and 52 at the time of these battles. Whether the note above refers to the dates of the battles or to the age of Cromwell ('sa' for 'la') there is really not much difference, it suggests that Lely painted Cromwell in his fifty-first or fifty-second year. That is before he became Lord-Protector. The Director also informs us that:

Le portrait est entouré d'une bordure qui en fait une sorte de médaillon, et qui est certainement originale; mais aucune lettre n'est inscrite sur cette bordure. Egalement aucune signature apparaît ni sur le devant ni sur le revers du tableau.

Now the Goupil-Gardiner monograph on Cromwell has an excellent reproduction of the Florence Cromwell (Plate to face p. 180), but it is framed in a border with the letters P. R. O. C. in the four corners, which should probably be interpreted as Protector Regni, Oliver Cromwell, and would date the picture as 1653 or later, and is thus misleading. It would appear from this that the oval border or mount with the letters P. R. O. C. on it is not part of the original, but introduced by the publishers of the work*.

The question as to whether there was a portrait of Cromwell at the Hague, met with a negative, and a suggestion that the National Portrait Gallery in London was the proper place to make inquiries about Cromwell portraits. Enquiry at the Rijks Museum, Amsterdam, met with the polite reply that they had no portrait of Cromwell, but 14 engravings. (According to J. J. Foster, there is in the Rijks

* Mr C. K. Adams of the National Portrait Gallery points out to us that this oval border is a copy of that round the engraving of a Walker Cromwell by Van der Velde. He also has given us a reference which proves that a Cromwell portrait was at Florence in 1706, and thus could not be that engraved by Faber: see Appendix to this paper.

Museum, a miniature on enamel of Cromwell attributed to Cooper*.) The Director of the National Museum of Art, Lisbon, was unable to find any trace of a Cromwell portrait in Portugal.

Director Carl Johan Lamm of the National Museum, Stockholm, sent a very full and courteous reply to our inquiry. There are three portraits of Cromwell known to be in Swedish collections.

(a) Gripsholm, No. 93, oil on canvas, 39 × 31 cm. after Walker. This picture has a somewhat remarkable frame with the words *Olivier Cromvel* below, and above, a torch, a mask and a snake, the body of which passes through a broken crown: see our Plate XL (*ter*).

(b) Gripsholm, No. 1213, oil on canvas, 75 × 60 cm. By or after Walker. This is Walker Class (*d*). It was restored by G. Jaensson, 1903. According to a label on the back, it was given to the Castle of Gripsholm by the Countess Dowager Gyllenborg, born Axelsson, in 1830 and according to tradition it was the gift of Cromwell to Queen Christina of Sweden†. If this tradition be correct the portrait sent to Christina was by Walker and not by Lely: see our Plate XL (*bis*) (*a*).

(c) A miniature deposited in the National Museum from the Statens Historiska Museum. Formerly in the Castle of Drottningholm. Gouache on ivory, height 2.2 cm., in a gold capsule with the back enamelled in turquoise blue: see our Plate XL (*bis*) (*b*). The Director suggests that this was painted a short time before Whitelocke's embassy to Sweden in 1653—4. It seems to correspond closely with the bracelet miniatures of Cromwell, one of which is at Chequers Court and a second one belonged to the Polhill family, see our p. 351 fn. ‡. Probably by Walker, or a copy in small from him, as is also that at Chequers Court.

From Copenhagen through the kindness of Professor Harald Westergaard we obtained a most valuable letter from Director Otto Andrup of the National-historiske Museum paa Frederiksborg. He informed us that portraits of both Cromwell and Fairfax were included in an inventory of the pictures in the Kunstkammer taken in 1673. In 1690 they are mentioned in the Helttekammer list as half-length figures, and in that of 1730 it is said that the pictures are life-size. Later Director Andrup started a hunt for these pictures; they were hung for a time in the Old Christiansborg, but the Cromwell was later transferred to the Ministry of Justice, where it now hangs, and where the Director has recently seen it: see our Plate XXXIX (*bis*). The Cromwell picture is not a pair to the Fairfax, which is smaller. The former painting is about 130 × 110 cm. Cromwell is in armour, and holds a baton. A page in red velvet to the left of the spectator aids with his dressing. It is clear that we have here a fine picture by Walker of Class (*b*), superior to the National Gallery and comparable with the Althorp portrait. The Director further states that he has noted in the Inventory of 1690 "en Tegning af Cromwell." We

* *Op. cit.* Cooper's Portraits of Cromwell, pp. 27—31.

† One of the miniatures at Chequers Court has a gold back upon which is inscribed that it was presented to Queen Christina. If so, how and when did it come back to England and find its way to Chequers Court?

were hoping he might be able to discover where this now is, it might chance to be a "limning" by Lely, but he has failed to locate it. Thus our search, if not very helpful with regard to Lely, has brought to light three more Cromwell portraits, copies or originals by Walker, and one bracelet miniature by Walker(?). The search has been admittedly superficial as little could be really ascertained without a personal visit to some of the countries referred to, but this has so far not been feasible.

Neither in the Florence nor the Cavendish portrait is Cromwell shown with any imperfections of the face beyond the wart over the right eye*.

Till recently the statement that Lely did paint Cromwell has been largely based on what Horace Walpole reports in his *Anecdotes of Painting*†, namely that a certain Captain Winde told John Sheffield, Duke of Buckingham, that Oliver undoubtedly was painted by Lely, and that while sitting to him, Oliver said: "Mr Lely, I desire that you would use all your skill to paint my picture truly like me, and not flatter me at all; but remark all these roughnesses, pimples, warts and everything; otherwise I will never pay a farthing for it." The last remarks hardly confirm the statement that the Florence portrait is the one to which Captain Winde was referring. Further the Catalogue note to the Florence portrait gives Cromwell's age as 51 to 52 which would date the painting as 1650 to 1651. The Faber mezzotint states that Lely painted in 1653 the picture engraved, and the title underneath is "Oliver Cromwell, Lord Protector," which fits in with the date 1653. This confirms our view that Faber did not engrave from a portrait that later went to Florence but from another picture by Lely. While the anecdote cited by Walpole can hardly apply to the Florence portrait, yet, as we have already seen, the Payne-Gallwey miniature of Cromwell, stated to be by Cooper, is said to show the pimples, etc. See our p. 330. The confusion is intensified by the attribution of the Walpole-Lely anecdote by Benjamin West to Walker, and the suggestion that Cooper copied a painting of Walker's that had the pimples!

While there is little chance of even a layman confusing a Walker portrait of Cromwell with one by Cooper, we are not prepared to accept the verdict that the Florence portrait is by Walker, or a copy from a Walker. Without being dogmatic, for we are only laymen, there is something individual about that picture. It appears to stand somewhat midway between the Walkers and the Coopers. This may be either because we are dealing with another painter's conception of the man, or because the portrait deals with an intermediate stage in Cromwell's life. We see the puffiness developing, but it has not reached the characteristic expression given to it by Cooper.

If the Florence portrait be not by Lely, then there is now no portrait known of Cromwell which can claim to be by him. We are not in a position to prove that the Florence portrait is by Lely, but we seriously doubt whether the modern ascription to Walker is valid. Meanwhile we think it entirely reasonable to leave the traditional ascription to Lely standing, if only as a means of distinguishing

* Compare for example our Plates XLI and XLI (*bis*).

† Edition Dallaway and Wornum, p. 444. Walpole is here simply following Vertue's note books: see *Volume XVIII of the Walpole Society* 1929-1930; *Vertue's Note Books*, Vol. 1. p. 91.



(a)



(b)



(c)

"Cooper" Miniatures of Cromwell.

(a) and (b) The Duke of Devonshire's Miniatures. (c) Marquess of Crewe's Miniature.



The Houbraken engraving of Cromwell (1747 or before), professing to be a copy of the Duke of Devonshire's Miniature (Plate XLII (a)), but looking more like a reversal of the Marquess of Crewe's Miniature ((c) on same Plate).

Pearson and Morant: *The Cromwell Head*



The unfinished Miniature of Cromwell by Cooper in the possession of the Duke of Buccleuch. Reproduced from their engraving by kind permission of Messrs P. and D. Colnaghi & Co.

Pearson and Morant: *The Cromwell Head*



Viscount Harcourt's Miniature of Cromwell by Cooper.

the type of Cromwell shown in the Florence portrait and the Faber mezzotint from those peculiar to Walker and Cooper.

(iii) Lastly we come to Cooper. Here as we have already indicated (see our pp. 269 and 344) it is almost impossible to enumerate the material or to classify and distinguish originals, replicas and copies*. Nor is it requisite for our purpose to do so. The passion for collecting miniatures has been so intense, and the prices offered so high, that Cooper miniatures or even miniatures claiming to be by him have passed rapidly from hand to hand and may often be found in the possession of those having no claim to Cromwellian descent†. In many such cases the history of the individual miniature has never been recorded or has been long lost‡. For our purposes the miniatures under discussion may be divided into two classes. The first class or type shows Cromwell in right profile. A good example of this type is that in the Duke of Devonshire's possession; it is a drawing in pen, and brown tinted: see our Plate XLII (a). The armour was never completed on this drawing. In or before 1747 Houbraken produced his well-known engraving of Cromwell: see our Plate XLIII. This is the left profile, or has been reversed, and on the print itself it is said to be from a portrait in the possession of the Duke of Devonshire. Another miniature of this type is that in the possession of the Marquess of Crewe, which is reproduced as (c) on our Plate XLII. The Crewe miniature seems hardly a direct copy of the Devonshire miniature, although it does seem to be a copy rather than an original Cooper§.

The inter-relationship of the "Cooper" miniatures of Cromwell can never be worked out until they all are collected in one room for comparison and study, and we can suggest no better field for a winter exhibition of the Royal Academy, than that of all the paintings, miniatures and busts of Cromwell, with the possible extension to those of other leaders of the Commonwealth period.

The second type of Cooper miniature is the three-quarter face looking to the subject's left. This contains all the characteristics of Cromwell we have already noted, including the wart. The most familiar portrait of this type is the Duke of Buccleuch's unfinished miniature (see our Plate XLIV), but there are other finished miniatures of almost equal excellence. Thus we may note the Duke of Devonshire's (see our Plate XLII (b)) and Lord Harcourt's (see our Plate XLV).

* See for a list of the Cooper miniatures of Cromwell the works of J. J. Foster: *Samuel Foster and the English Miniature Painters of the Seventeenth Century*, and *A List alphabetically arranged of the Works of English Miniature Painters of the Seventeenth Century*. Supplementary to the above, 1914—1916, pp. 27—31. We have ourselves come across more than thirty cases.

† Even the Cooper miniature of Ireton, preserved till recently in the ownership of the Polhill family, his descendants, has now passed from their hands.

‡ That Cromwell's children had paintings or portraits of him is proved by several being still in the hands of their descendants. The Protectress gave to her daughters on their marriage bracelets with a miniature of their father. One of these a few years back was still with the Polhill family, but was sold at Christie's, and a second coming from the Russells is to be found in the miniature case at Chequers Court. It is, however, too minute to be of any service for comparison with the Wilkinson Head.

§ It has a known history, which does not, however, carry us far enough back. It was sold for a hundred guineas by Lady Cornwallis to Sir Joshua Reynolds. He bequeathed it to Richard Burke, who left it to Frances, Lady Crewe, from whom it descended to her granddaughter Lady Houghton and so to the present Marquess of Crewe.

It scarcely appears to be the source of a somewhat crude oil painting No. 588 in the National Portrait Gallery (see our Plate XLIX). A far better picture of this type is No. 514 in the same gallery (see our Plate XLVIII), and there are no less than three miniatures at Chequers Court of the same type and closely allied to this No. 514 and ultimately to the Buccleuch miniature. There is a curious if slight difference which might help in establishing the relationship of the portraits of this type. The Marquess of Ripon has a miniature like No. 514 with the initials S. C. and dated 1657 with the armour completely finished. The armour is not completely finished in No. 514. Now in No. 514 the gorget comes over the left shoulder piece. In the Harcourt miniature it comes beneath (cf. our Plates XLV and XLVIII). In the large Chequers Court miniature (see Plate XLVII) it also comes beneath and this is true for the smallest Chequers miniature of this type (see Plate XLVI (a)). In the medium-sized miniature (see Plate XLVI (c)) the border of the gorget has become little more than a meaningless chain of beads*!

Lord Harcourt's miniature, which is in many ways strikingly effective, is dated 1657 and initialled S. C.†. The Duke of Buccleuch's miniature is said to represent Cromwell at the age of 58, or to be also from 1657, the year before Cromwell's death‡. It is difficult to believe that one can be a copy of the other. Both face to the left shoulder, but the Harcourt miniature has a tilt of the head to the right

* It would almost seem as if the borders of the gorget had become ropes in the Cambridge bust, and the shoulder pieces equally disguised. In the case of the Ashmolean bust the gorget has disappeared, and the shoulder pieces have become highly ornamented, but are hardly recognisable as armour.

† The monogram and date are clear even in our photograph close to the left of the frame: see Plate XLV. Rapin's date, 1653, for his reproduction of this miniature must be in error; see his *History of England*, Vol. II. p. 591.

‡ The history of the Buccleuch miniature is somewhat vague. According to Vertue¹ the story is as follows:

The Picture of Oliver Cromwell, a limning by Samuel Cooper, the head only finish'd, suppos'd to be the best of him, being the very picture Oliver Sate for,—the other as Mr Graham's was, and others copy'd from this and touch't up by the Life. This picture Cooper kept for himself till Oliver died, when his son Richard Protector heard of it. Sent a Gentleman for it with twenty pounds to pay for it (that being Mr Coopers price), but Cooper knowing its super-excellence to the rest, and what he could make by copies, would not part with it under a hundred pounds which the Protector was Obliged to give him; he gave it to his Sister the Lady Falconbridge, and she before she died made a Present of it to S^r Thomas Franklin in whose possession it now is [?circa 1713]. It was Richard Cromwell that paid Cooper for it after the death of Oliver.

Presumably "S^r Thomas Franklin" is the Sir Thomas Frankland who married Elizabeth, daughter of Frances Cromwell by her second husband, Sir John Russell. In this case we should have expected the miniature to have reappeared at Chequers Court. Walpole in his *Anecdotes of Painting* (Edw. Dallaway, Vol. III. p. 117) speaks of "this fine head" as in the possession of Lady Frankland, widow of Sir Thomas, a descendant of Cromwell (she was his granddaughter). Vertue engraved the head. G. G. Foster in his *List of English Miniature Painters* adds to this (pp. 27-31): "This miniature, which is justly regarded as one of the best authenticated and finest portraits of Cromwell in existence, is, according to Dallas, the one which Cromwell surprised Cooper in the act of copying, and took away with him in anger. It was brought, according to the Montagu House Catalogue, to Messrs Colnaghi about 1860, by a gentleman, the legal adviser of a lady, to whom it had been presented by a member of the Frankland family, into whose possession it came through Miss Claypole." Here we have two wholly different legends. One asserts that Richard Cromwell bought the miniature of Cooper and handed it to his sister Mary, Lady Fauconberg, and the other that Oliver Cromwell seized the miniature and gave it to his daughter Elizabeth, Mrs Claypole. In both cases it appears to have passed to the Franklands, but did not reach Chequers Court.

¹ *Walpole Society*, Vol. 18; *Vertue's Note Books*, Vol. I. p. 31.

The smaller Miniatures at Chequers Court.



(a)

The "cased" Miniature by Cooper.

(b)

Distant copy from a Cooper.
By Petitot ?

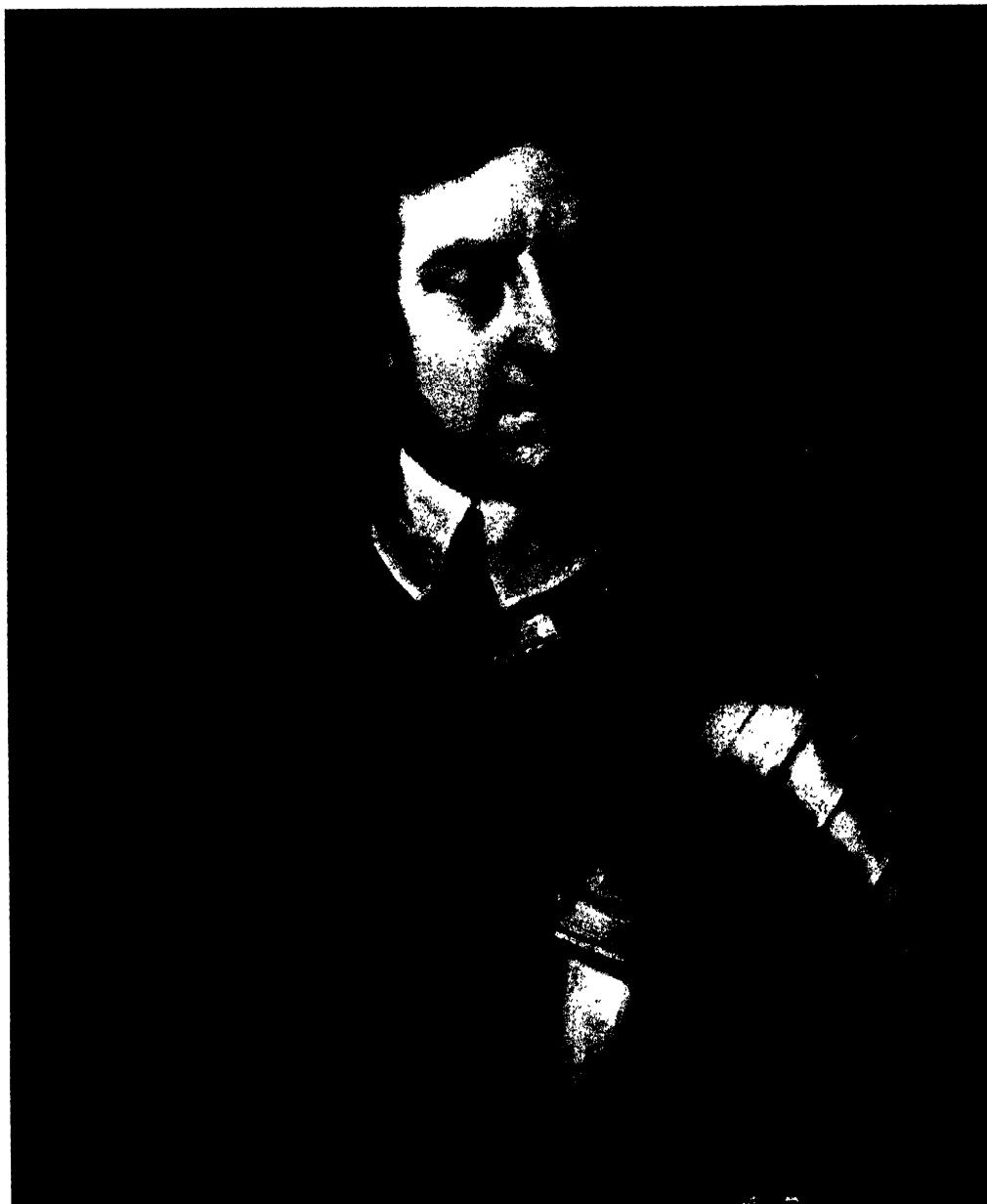
(c)

Miniature from Cromwell's later life to illustrate
the developed "puffiness." Artist ?

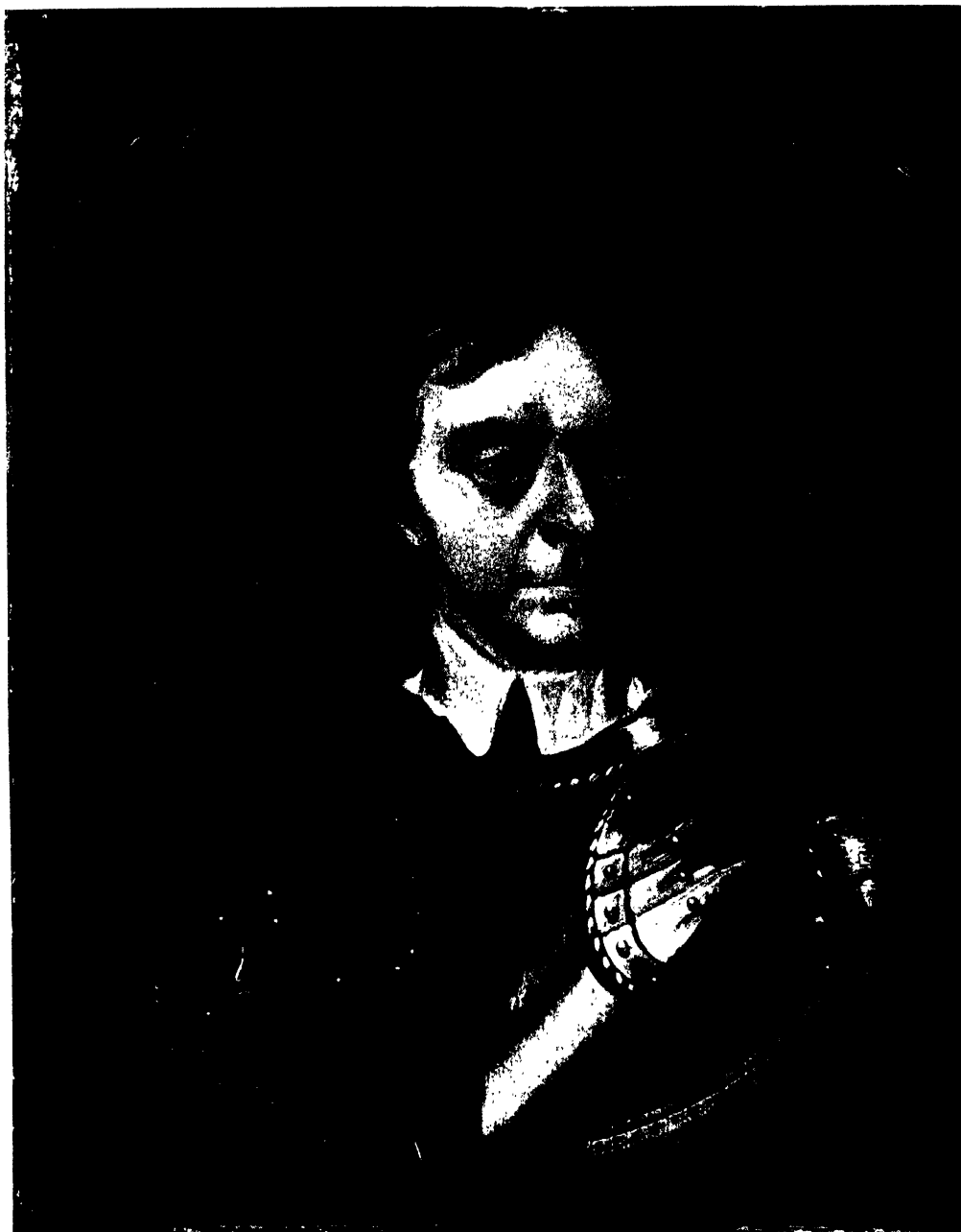
Pearson and Morant: *The Cromwell Head*



The large Miniature at Chequers Court. Cooper? The crumple in the collar and spot of reflection on armour below link it curiously with the Lely portrait (Plate XLI), and the National Portrait Gallery, No. 588 (Plate XLIX).



National Portrait Gallery, No. 514. Said to be a copy of the Buccleuch Miniature (Plate XLIV), the face has been drawn out vertically, and the head given a tilt to the right, which associates it more with the Harcourt Miniature than with the Buccleuch.



National Portrait Gallery, No. 588. Painter uncertain. Crude painting on a small scale in oils. The crumple in the collar and the reflection on armour immediately below indicate that it has some relation to the Lely and the larger Chequers Court Miniature. (See Plates XLI and XLVII.)

shoulder, so that the right border of the face is on the slope; there is no trace of this in the Buccleuch miniature, but this tilt of the head to the right shoulder reappears in the Cambridge bust, giving to the bust an expression of command which it fails to provide in the miniature. Indeed, if the eye dwells too long on the sloping straight line bordering the face and neck in the latter, one seems to find something of an artist's failure in the drawing there. The slighter tilt in the National Portrait Gallery picture (No. 514) seems more effective. The armour is far more highly finished in the Harcourt miniature than in No. 514.

Again, the Harcourt miniature is closely related to the Duke of Devonshire's three-quarter face of Cromwell, which shows the same straight line as border on the right to the face, and even the same two bright spots on right and left shoulder pieces close to the gorget. There is no doubt of the one being a replica of the other, and that the National Gallery No. 514 is further removed from both of them. Close to the Buccleuch and Harcourt miniatures comes the fine "cased" miniature at Chequers Court: see our Plate XLVI (*a*). If these miniatures all date from 1657 we see that the fleshiness of the cheeks had established itself at least in the year before Cromwell's death. The large miniature at Chequers Court (Plate XLVII) and the curious miniature (Plate XLVI (*c*)) are either later productions showing still increasing puffiness, or the work of an artist whom this feature of Cromwell's later physiognomy had markedly impressed, and who had then used it, as weaker brushmen will, as his leading index of identity. The somewhat crude oil-colour painting of Cromwell in the National Portrait Gallery (see our Plate XLIX) has we think the large Chequers Court miniature (Plate XLVII) as its prototype or near relative. Note the dent in the left side of the collar, and the reflection on the gorget immediately below it*; the left shoulder piece in both passes above the gorget, and curiously enough there is what appears to be a narrow defective strip running from the lowest corner of the left shoulder piece over the apparent frame in both!

There is another point of interest about these puffy-cheeked Coopers or pseudo-Coopers, they show more hair on the forehead than do the profiles. In the Devonshire and Crewe miniatures, the bald patch is on the right frontal with a wisp of hair straggling over it; it is difficult to believe that this wisp of hair could produce, however well combed out, the hairy mass over the right frontal in the Harcourt miniature, the large Chequers Court miniature, or the National Portrait Gallery No. 514! Hence the artists of the latter portraits either flattered Cromwell, or the profile miniatures belong to a later period than the three-quarter face pictures, i.e. to the last year of Cromwell's life. As far as we are aware no date has so far been connected with the profile portraits. Further, had Cromwell's left profile been as in the Chequers Court miniature (Plate XLVI (*b*)) or the Houbraken engraving (Plate XLIII) it would have been impossible to twist the wisp of hair

* As we have remarked (see p. 346), these occur also in the Florence portrait. A further study of the details of the Lely type of Cromwell portrait leads us to believe that the large Chequers Court miniature and the N. P. G. No. 588 (Plates XLVII and XLIX) are linked, however distantly, with Lely rather than Cooper.

over the right frontal without showing a bald left frontal on the three-quarter face portraits. Thus the miniature (if of Cromwell) as well as the engraving are *reversed*. While the Devonshire and Crewe profile miniatures show a different type of collar to the three-quarter face portraits, this longer collar is still further extended in the two reversed portraits, and removes them from being exact copies of any originals of Cromwell. The Chequers Court copy (Plate XLVI (b)) of a miniature by Petitot, or it may be a Petitot original*, has thus no authority, and Houbraken's engraving, while very characteristic, does not claim to be more than a copy of the Duke of Devonshire's profile portrait.

In the latter, the right ear of Cromwell is clearly visible and there is no armour. In the Marquess of Crewe's profile portrait the ear has almost disappeared under the hair, and the armour is introduced; in Houbraken's engraving the ear has practically been entirely removed and the armour appears, but is rather fancifully arranged. We are inclined to think that Houbraken really used the Crewe miniature, which in his day might have been generally recognised as a copy of the Devonshire Cooper†.

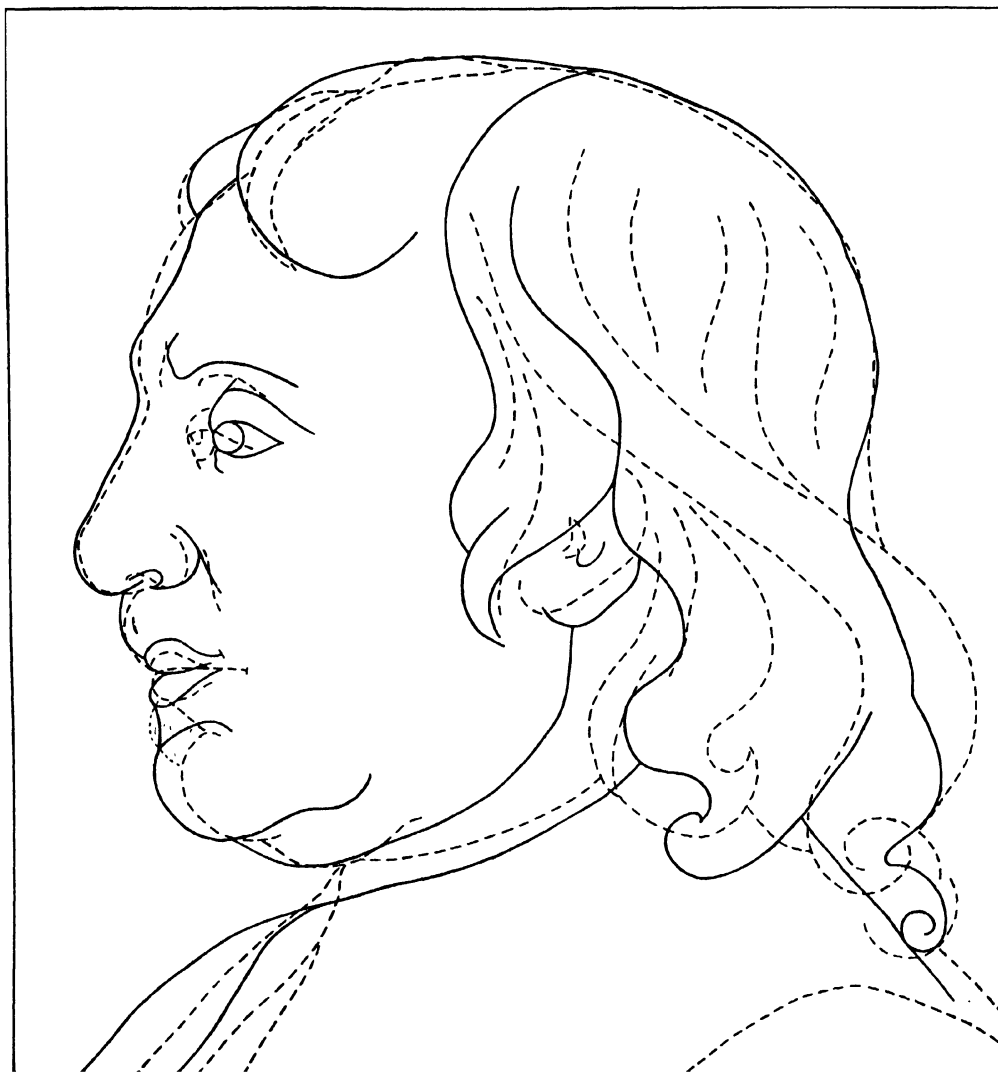
To sum up with regard to the three-quarter faced "Cooper" miniatures. The Duke of Devonshire's miniature (Plate XLII (b)), the Chequers Court "cased" miniature (Plate XLVI (a)), and the Harcourt miniature (Plate XLV) are closely linked together, and we are inclined to believe the Chequers Court the more characteristic and possibly the original of the three. We hold that the Buccleuch miniature, although dated the same year as the Harcourt miniature, is an independent production. The National Portrait Gallery No. 514 (Plate XLVIII) may have been based on the Buccleuch miniature, the simplicity of the armour is in favour of this; it is certainly not directly copied from the Harcourt group, as the gorget border is above the left shoulder piece and the two bright spots on the shoulder pieces fail.

But what we want, especially for our present purposes, to draw the reader's attention to is that portraits from the last year or year and a half of Cromwell's life indicate a puffiness of the cheeks and an increasing puffiness, which does not exist in the Walker portraits of an earlier period and which has much strengthened the idea that Cromwell had a massive square jaw, such as we apparently find in these miniatures and the death-masks.

Before we discuss the plates showing the Head fitted to the various types of Cromwellian portraits we have discussed above, we must again remind the reader

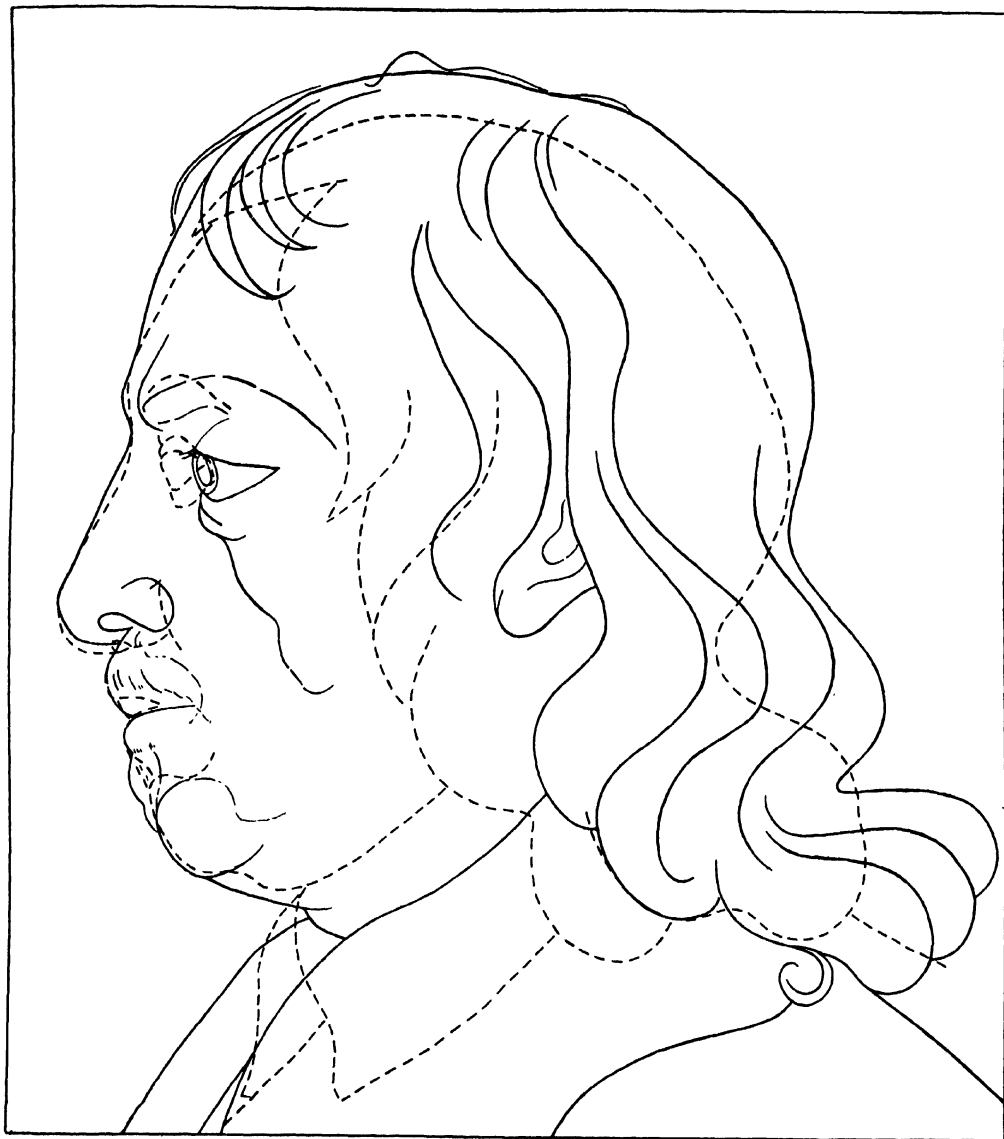
* Compare the Chequers Court miniature on our Plate XLVI (b) with that formerly in the Tangye Collection, and reproduced in both Sir Richard Tangye's books, i.e. *The Two Protectors: Oliver and Richard Cromwell*, London, 1899, p. 99, and *The Cromwellian Collection of MSS., Miniatures, Medals, etc.* Privately printed, 1905, on the title-page: see also pp. 40 and 121. Sir Richard spells the artist's name as Pettitot three times and Pettitott twice.

† Mr O. K. Adams of the National Portrait Gallery kindly provided us with references to eight miniatures attributed generally to Cooper, of Cromwell's right profile. Mr B. S. Long believes that none of these is from the life. But the Duke of Devonshire's appears to have the greatest claim to originality.



Two Miniatures, both attributed to Cooper, fitted to each other.
Note eyes, chins and foreheads.

Pearson and Morant: *The Cromwell Head*



Dunbar Medal (see Plate LXI) fitted to Cooper Miniature. Note receding forehead and position of eye (!). Medal - - - , Miniature — .



Dunbar Medal (see Plate LXI) fitted with the Chequers Court Life Mask.
Note chins and noses. Medal ———, Life Mask - - - - .

of the difficulties arising from these attempts at fitting. If the head in the portrait be not a profile we have had to photograph the Wilkinson Head in as nearly as possible the same, usually three-quarter face pose. But when this has been done, the photograph of the Head and the picture will not be of the same size, especially in the case of miniatures; or, again, in the case of a larger picture it may still not be life size. Accordingly, in each case a drawing has been made of the picture, taking care that the proportions are the same, and then the appropriate photograph of the Head has been fitted to this by making some one measure identical in the two, and using, when it seemed advantageous, a Corade Precision Pantagraph to change proportionately the photograph of the Head. The glabella defined for our *present* purposes as the point in which the tangent to the upper borders of the eyebrows meets the median plane, and the subnasale defined as the point in the same plane where upper lip meets the nose, and again the point where the lip line meets the median plane, are generally recognisable points on the picture and may serve to determine the terminals of a measurable line. But the matter in the case of the Head is far more difficult. The glabella of this can be determined, but the lower terminus of the required measurements is vague. The subnasale of the Head, *as skull*, may not be difficult of determination, because the dried skin is drawn tight to the bone, but it is a different matter to fix where lip and nose met when they were clothed with their full amount of flesh. The living subnasale was clearly considerably below the skeletal subnasale: see the mid-sagittal contour on our Plate LXXXV. A certain amount of personal equation is inevitable in this measurement*. Again, the lips of the Wilkinson Head being much broken away, there is no lip line; if we take it as midway between the alveolar point on the upper and intradental on lower jaw, then because the mandible is to some extent displaced some allowance should be made for this. We can make some rough appreciation of what this allowance ought to be, and it will be dealt with later.

To indicate the real difficulty of the problem arising from artistic individuality we have compared together certain portraits of Cromwell:

(a) Plate L gives a comparison of two profile miniatures, the Devonshire and the Crewe, both attributed to Cooper. Continuous line Devonshire, broken line Crewe. Both were prepared for other purposes. Compare the position of the eyes, the shape of the foreheads, and the chins! It is clear that if the Head fitted well one of these, it could not fit the other.

(b) Plate LI compares the Dunbar medal (Plate LXI) with the Cooper miniature (Plate XLII), the medal being the broken line. Compare the forehead and position of the eyes!

(c) Plate LII compares the Dunbar medal with the profile of the Chequers Court life mask. The medal is given by the continuous line, the mask by the broken line. Note chins and noses!

* The external ocular distance can only be taken on full-face portraits, busts or masks, but again it is difficult of ascertainment on the Head, because the canthi, external and internal, owing to the shrinkage of the eyelids have drawn together, and the allowance for this is again a matter of personal judgment.

Numerous other comparisons were made, especially with the coins among themselves and with the miniatures, but the divergencies between the portraits themselves were as great as between the portraits and the Head. In particular, the coins failed to agree either with each other* or with the masks. After some attempts, we concluded that actual measurements on the painted portraits were not likely to be very profitable. Not only did different painters have different conceptions of the man, but different portraits by the same artist were for the purposes of measurement incompatible one with another. We think that something of this is due to the relative position of painter and subject. If the painter stands and the subject sits, then, especially in the case of baldness, the painting shows a high brow. If the painter is below his subject, then the reverse may be the case. For each of the portraits we have dealt with, the Wilkinson Head was placed with as much accuracy as we could muster in the pose of the subject's head and photographed. These photographs, a separate one for each picture, were then made the basis for outline drawings of the Head, which were compared against the drawing from the painting. Often more than one reproduction, altered in size from the photograph of the Head, had to be made before we were at all satisfied that the best fit had been obtained.

14. *Fit of the Portraits of the Wilkinson Head.*

We will now consider the fit of some of the portraits to the Wilkinson Head, taking the Walker, the Lely and the Cooper types in what we think to be their chronological order.

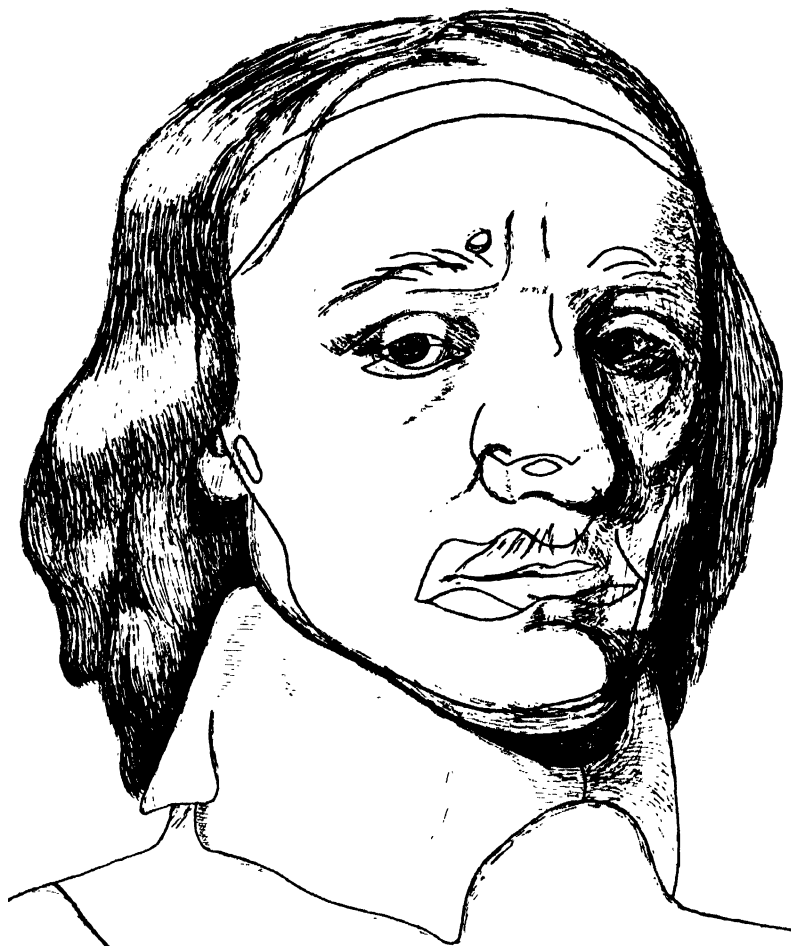
(i) *Walker Type Cromwell.* Plate LIV gives a sketch of the Hinchbrooke portrait in the possession of the Earl of Sandwich, fitted with the outline of the Wilkinson Head. The chin, nose, eyebrows, and the wart are in excellent position, the eyelids of the picture, if closed, might fit fairly well, but the Walker forehead is too high for the Wilkinson Head. Possibly a better fit would have been obtained had the camera been slightly raised or the Head inclined, that is to say more in the position of the Head in Plate XXVII than of Plate XXVI, which would have shown more of the forehead above the cincture. On the other hand, Walker may have been prone to emphasise the brows of his subjects. Plate LIII gives a sketch of the Walker portrait from the Tangye collection in the London Museum (see our Plate XL). The fit is about equally good with that of the Hinchbrooke portrait; but in the same manner it fails on the left frontal. Having gained our experience from the fit of one portrait to a second, we lay less stress on this defect of the Walker fits than the reader may feel inclined to do. A slight further tilt of the Wilkinson Head, or a half-inch higher camera, would have largely met the defect.

(ii) *Lely Type Cromwell.* Plate LV shows a sketch of the Florence portrait set against the outline of a photograph of the Head placed, as nearly as we were able to judge, in the attitude of the subject of the painting. The fit is, perhaps, somewhat

* We have not reproduced our fittings of coins and medals with each other, as the thoughtful reader has only to examine Plate LXI to be convinced of their divergence.

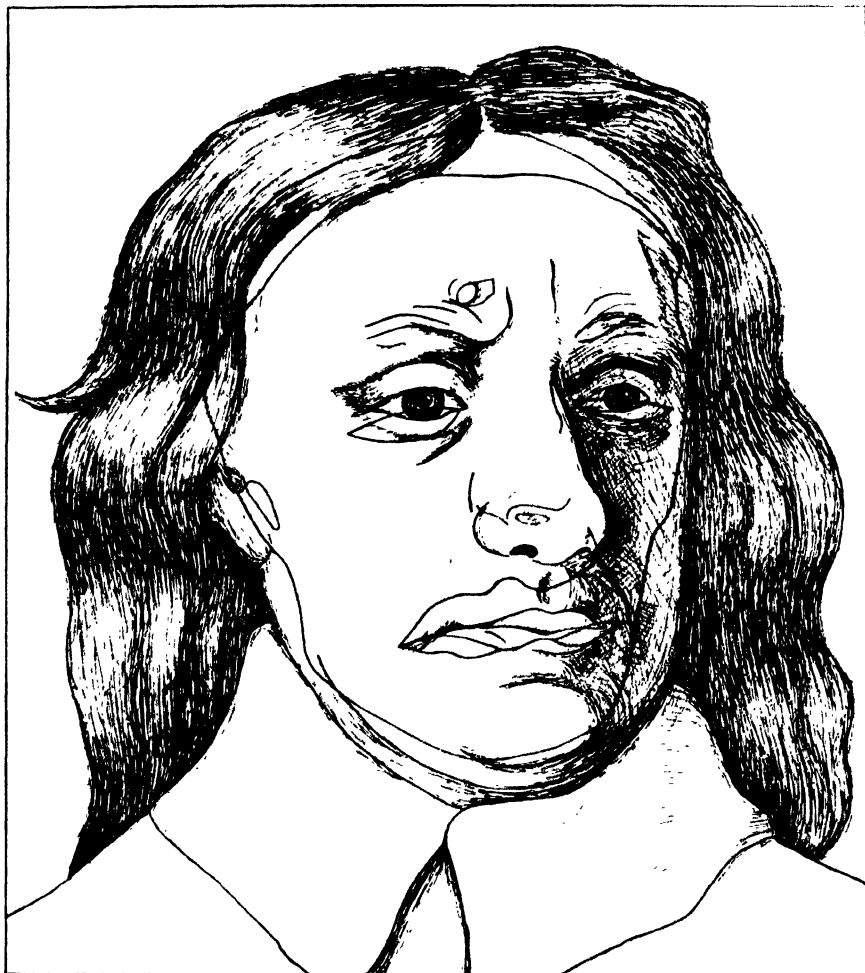


The Walker Portrait in the Tangye Collection, London Museum,
fitted with the Wilkinson Head.

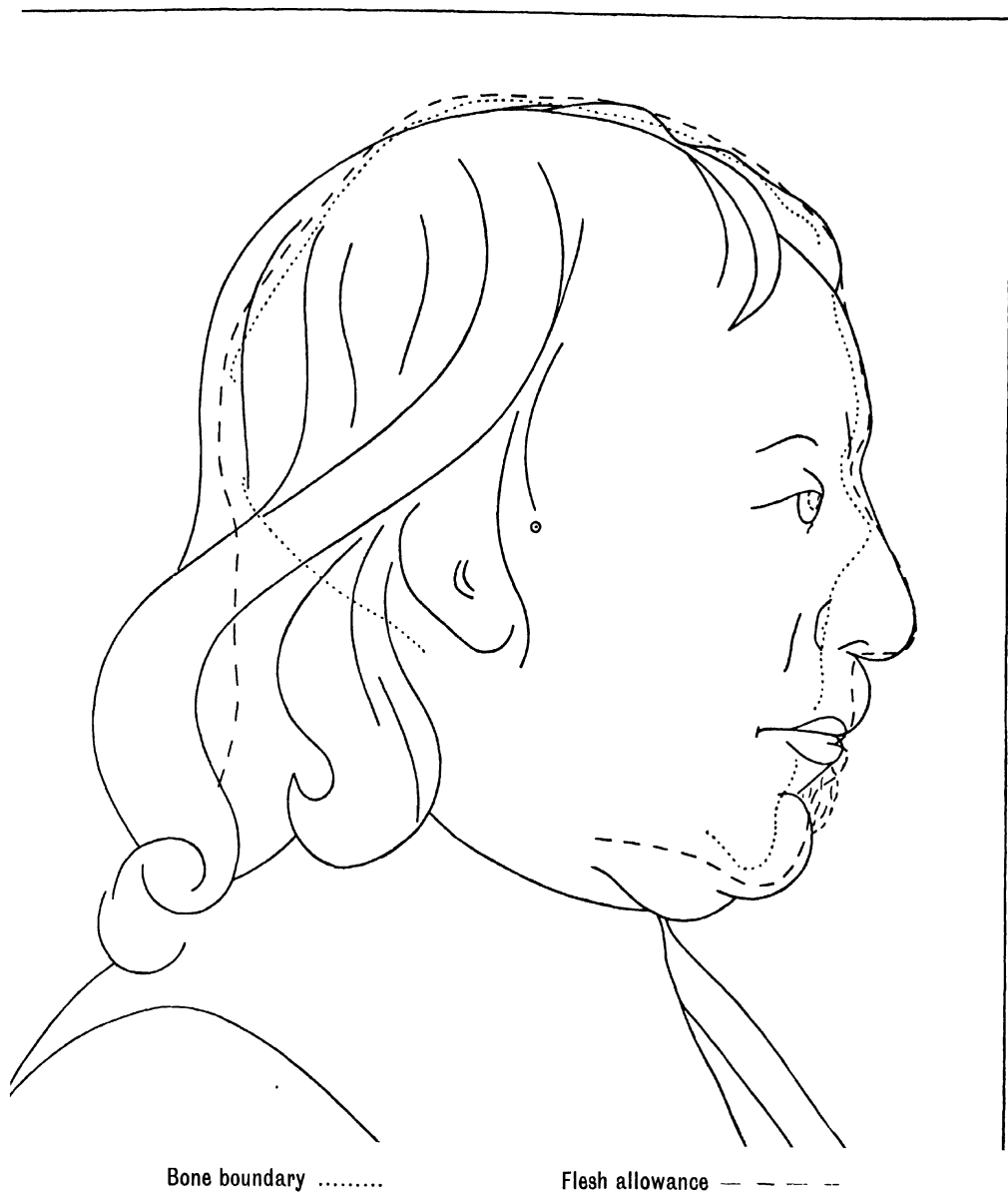


Sketch of the Walker Portrait at Hinchingsbrooke fitted with the Wilkinson Head. Chief failure, left frontal.

Pearson and Morant: *The Cromwell Head*

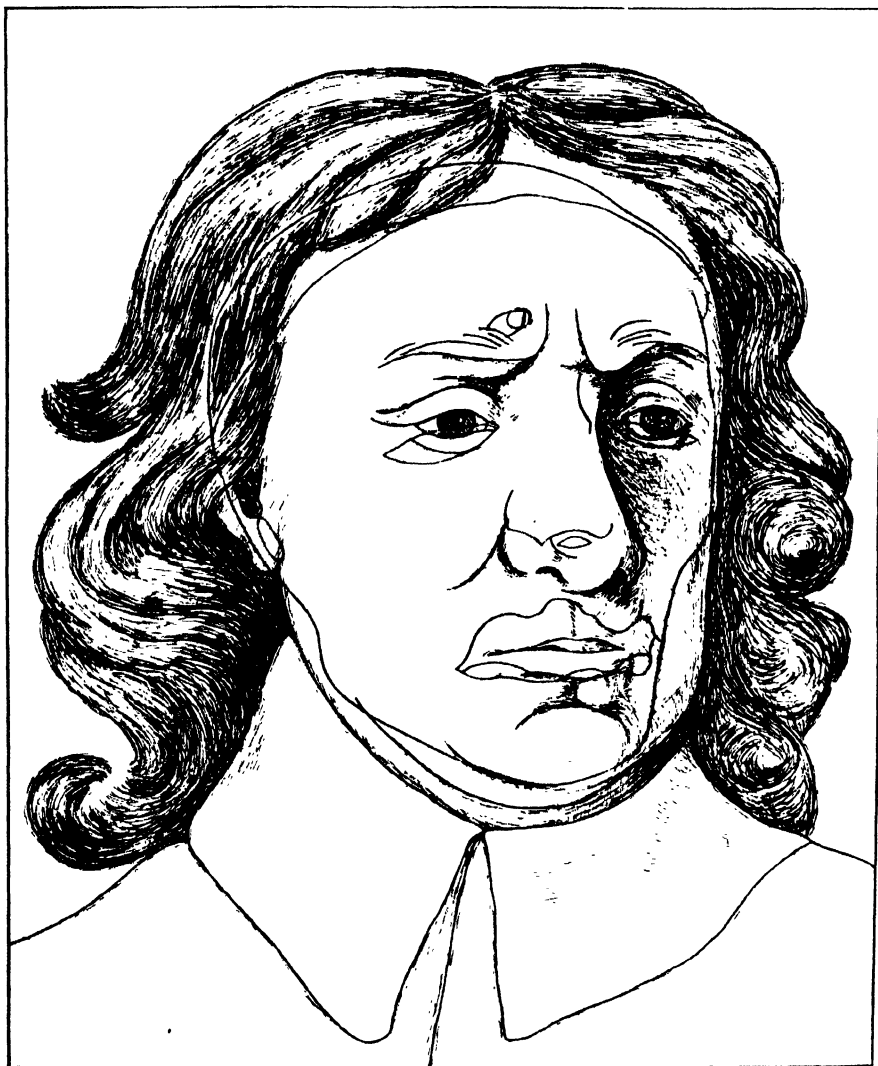


Sketch of the Lely Florence Portrait of Cromwell
fitted with the Wilkinson Head.



The Cooper Profile Miniature (Plate XLII (a)) fitted with the Profile Contour of the Wilkinson Head.

Pearson and Morant: *The Cromwell Head*



Sketch of the "Cooper" Portrait of Cromwell at Sidney Sussex College, Cambridge, fitted with the Wilkinson Head.



Sketch of the Buccleuch Miniature fitted with the Wilkinson Head.
Note position of wart-cavity and the left frontal.



Sketch (by a different Artist) of the Buccleuch Miniature, fitted with the facial outlines of a different photograph of the Wilkinson Head.



Sketch of the Walker Portrait of Ireton in the possession of Mrs R. B. Polhill-Drabble, fitted with the Wilkinson Head. The fit is good vertically, but the Head is far too broad horizontally to be Ireton's.

better than that of the Walkers. The left frontal fit is improved, the wart, chin, nose and mouth are fairly good, the right auricular passage is seen to be somewhat out of place. A slight clockwise twist of the head would have bettered this. It might have worsened the chin fit, but we have to remember that the chin is somewhat displaced. If the Head may be considered to pass the Walker test, it certainly will the Lely.

(iii) *Cooper Type Cromwell*. Plate LVII shows a drawing of the Sidney Sussex College portrait of Cromwell. The nose, mouth, chin and wart are in reasonable positions. The right auricular orifice is too low, and there is, as in the Walkers, too much left frontal above the Head. A raising of the camera and a slight rotation of the drawing of the Head would have bettered matters.

Plate LVIII shows a sketch of the Head superposed on the Buccleuch miniature. The mouth, nose, chin and eyes* fit fairly well, the auricular orifice is better, but the hole in the dried skin, where the wart is supposed to have been, is somewhat too high; it seems, however, to us that with regard to the eyebrow the draughtsman might have placed the wart on the sketch of the Buccleuch miniature somewhat higher. The auricular orifice is about the correct level, but as in several other cases the photographer has got the camera too low, i.e. has looked at the Wilkinson Head from above the level of the objective.

Plate LVI shows the Cooper profile of Cromwell with the Wilkinson Head with and without natural amount of flesh added. It will be seen that the flesh allowance causes the face to fit extremely well. Cromwell's upper lip in the miniature is too protuberant, and his lower lip too recedent for the amount of flesh provided. His 'double-chin'-ness exceeds the flesh allowance. The cranial contour is in fair accordance. But just as the Walker portraits give too high a head for the Wilkinson Head, so the Cooper miniatures give too low a head. Both cannot be true to Cromwell, and so the intermediate Wilkinson Head may well be.

The difficulty of adjusting the Head to the position in which the painter had viewed the subject, the cost of repeated photographs, the need for protracted labour and caution in posing the Head for photography as well as the great differences in the paintings themselves, led us finally to discard this method of testing the genuineness of the Wilkinson Head in favour of a more detailed study of the masks and busts. We had convinced ourselves before doing so that with time and labour it would be possible to improve our fits to the portraits, and that we could not assert under the circumstances that the fits discredited the possibility of the Head being Cromwell's†.

* The reader must bear in mind that the lower and upper eyelids of the Wilkinson Head are very tightly contracted and distorted: see Plates XXVI and XXVII.

† Another difficulty arises in the case of a solid head arranged in the same attitude as a portrait, which is not full face or profile. If a solid head and a mask or bust are seen from one position to be in corresponding aspects, they will appear to be so from other positions; but this is not so in the case of a solid head and a flat drawing or portrait; the camera must be placed in the same position as the eye of the observer who had made the correspondence, and we were in certain cases not careful enough in placing the camera, which the photographs indicated had been situated somewhat below the eye of the observer, so that the forehead of the Head was shortened. See Plates XL and LVIII.

15. *Cromwellian Medals and Coins.*

Originally we thought that the medals and coins of Cromwell might be of service in our task of investigating the genuineness of the Wilkinson Head. This possibility may, perhaps, have seemed to us the more reasonable owing to the judgments of Richard Southgate, the numismatist, and John Kirk, the medallist, in favour of the Head being Cromwell's. Southgate's assurance that "this is the real head of Oliver Cromwell," and Kirk's statement that he had "carefully examined it with the coin" and thought "the outline of the face exactly corresponds with it, so far as remains" (see our p. 281), seemed at least worth investigation. Through the kindness of Mr G. F. Hill, then Head of the Department of Coins in the British Museum, we were provided with casts of two Cromwell medals and of two of his coins. Photographs of these casts are given on our Plate LXI. Those of the Dunbar medal and of the two coins were in two stages enlarged up to life size by aid of a precision pantograph. But it soon became obvious that the drawings from these casts neither fitted each other, nor fitted other portraits of Cromwell. It is not possible to reproduce all our attempts here; it will be sufficient to deal solely with the relation of the Dunbar medal to the Cooper profile miniature and to the profile of the Chequers life mask. The fitting is shown on our Plates LI and LII. It has already been partially discussed (see p. 355). On Plate LI the fitting has been achieved by superposition of the two lip lines and the two "nasions"*. It will be seen that the medal profile has a forehead sloping well backward from Cooper's idea of the forehead, and the medal chin recedes in the like manner; the shape of the nasal bridge is wholly different, and the eye of the medal is too close to the nasal bridge. Thus if one of these is a good portrait of Cromwell, the other cannot be. On Plate LII the Chequers life-mask has been fitted to the medal in a somewhat different manner, the distances from lip-line to glabella* having been equalised. We now get a better fit of nasal bridge, eye and forehead, but the noses differ totally and again the medal chin is strikingly recedent. The first plate suffices as one example to indicate what different conceptions two independent artists, Symon and Cooper, both excellent in their special lines, could form of Oliver Cromwell! The second plate indicates how widely the medal differed from the actual man. The fairly complete study of the medals demonstrated that no one even by redrawing the medals and coins to the size of the Wilkinson Head could prove its genuineness by this procedure they differed as much from each other and from the portraits as they did from the Head. For a numismatist to assert, even without this procedure, that "this is the real head of Oliver Cromwell," or that "the outline of the face" as far as it remains exactly corresponds with that of a coin is idle opinion, in reality backed by no specialist knowledge. We do not believe that Sir Joshua Reynolds or Flaxman had any better basis for their dogmatic opinions of its genuineness than Carlyle had for his statement (see our pp. 271 and 281) about "fraudulent moonshine." At any rate, as far as our experience goes, coins are worse than paintings,

* For our use of this term see p. 369.

(a)



(c)



Medals and Coins of Cromwell.

(a) Lord Protector Medal, 1653.

(b) Dunbar Medal, 1650.

(c) Lord General Medal, 1650.

(d) Crown Piece, 1658.

and from neither type of artists can we obtain a standard portrait, i.e. one which would measurably fit the living subject; for had two artists done so their productions would fit each other, which they certainly do not.

16. *Masks and Busts of Cromwell.*

We will now consider the masks of Cromwell. Knowing what we do about the embalment and removal of the skull-cap of the Protector, we should anticipate two very different types of death-mask. (i) One taken before embalment; this would show the wart and no cincture, the eyes would not have fallen in, and if the face of the corpse had not been shaved, it should show the beardlet and a good deal of hair on the cheeks owing to Cromwell's illness. *No such death-mask* up to the present has to our knowledge been found. (ii) A death-mask taken after the embalment; this should show wart, cincture, and hair on the face, and if taken some time after death, the pads supporting the eyelids would be visible. *No such death-mask*, up to the present, has to our knowledge been found.

All the death-masks differ from (i) or (ii) in some particular, which disqualifies their passing for untouched casts from an original mould. Clearly some of them have been cast from anything but the original mould*. Thus the difficulty of fitting the death-masks to the Wilkinson Head (or to the portraits) is much complicated by the question of absolute size, as well as by the retouching of the mould immediately after it was taken.

We will give a list of the masks and busts with which we have been able to deal, noting how they appear to us to have been modified.

(A) *Life-Mask at Chequers Court.* We owe to the kindness of Mr Macdonald the permission obtained from the Trustees to photograph this, as well as some of the Cromwell miniatures, in the Cromwelliana at Chequers Court. The mask is said to have been taken in 1655 by request of Cromwell's family. We may assume it to have passed with most (not quite all) the Cromwell relics from Frances Cromwell, the Protector's fourth daughter, to her daughter Elizabeth Russell, who married Sir Thomas Frankland. Through Franklands and Greenhills the relics of Frances Cromwell in part passed to Mrs Frankland Russell Astley of Chequers Court, where they now form a portion of the treasures of the country house of the Prime Ministers of Great Britain.

This mask is probably the most satisfactory of all we possess†. See our Plates LXII, LXIII and LXIV. As the eyes are open the lids and the pupils must be due to the modeller. They clearly lack something of reality. Measurements were taken on the mask. They do not indicate any exaggeration on life size, or indeed on the Wilkinson Head, and we have taken this as our standard for the reduction of other masks. The mask exhibits the thick under lip with the beardlet, the upper lip

* There is a death-mask of Cromwell in the Museum of the former Biometric Laboratory purchased many years ago, which is some 18% above life size, and which must have been through 9 to 10 moulds

† It is plaster bronzed, which made it somewhat difficult to photograph on a very dull December day in the Long Gallery at Chequers Court.

with its central convexity, and the somewhat coarse nose tending towards the left cheek. The position of the wart is determinable.

(B) *The Florence Bust.* We introduce this bust here, because it is stated to be based on a death-mask taken surreptitiously immediately after Cromwell's death. The earliest report we have of it was made by John Breval*, and goes back to 1738. On a visit to the Gallery at Florence, he speaks of "la stanza degli Idoli," and continues:

Between this room and the Tribune are four other Chambers—in the first...over the Door on the Inside is a cast of Oliver Cromwell taken upon his own Face, through the dextrous management of the Tuscan Resident in London, a few moments after his Decease. There is something more remarkably strong and expressive in it than in any Picture or Bust of that Usurper I had ever seen.

The bust at Florence is indeed "remarkably strong and expressive": see our Plates LXXXII and LXXXIII, which are taken from photographs of the casts from the Florence bust in the National Portrait Gallery for the purposes of the present paper. But difficulties at once arise about this bust; it is closely connected with a bust in the possession of the Duke of Grafton (see our Plate LXXXIV), which is plaster bronzed. It would appear that the Florence and Grafton busts had been based on the same original mould. The protruding under lip, the convexity at the centre of the upper lip, the coarse nose bent to the left and the wart are all here. There is no sign of hair on the scalp, which is compatible with a sculptor having used a mask, and having no model for the hair. The eyelids are impossible; there are no wrinkles in the lids, but it looks as if, the lids being closed in a mask, button-holes had been cut in them by the aid of a penknife! These eyelids are clearly the work of some very rude remodeller, by preference a worker in wood rather than a sculptor using clay.

But we now come to a point of maximum difficulty. The Florence bust has no hair on the upper lip, no beardlet, no hair on the cheeks. If it was taken from a mask either the corpse of Cromwell was shaved before the death-mask was taken, or the remodeller removed the whole of the hair from the face. Could the face of Cromwell have been shaved immediately after death? We cannot conceive it possible. The earlier death-mask taken after the embalment shows plenty of hair on the upper lip, and a luxuriant beardlet. The later death-mask, taken after the eyes had fallen in, shows still more hair on the face, especially round the border of the chin. The hair might have grown after death in the interval between the taking of the two death-masks—possibly a week to ten days—hair does grow after death—but it is more probably due to the remodeller or toucher-up of the first mask. What seems quite certain is that if Cromwell's face was shaved immediately after death, it could not in two or three days have developed the hair which we see on the earlier death-mask. We hold that the statement that the Florence bust was taken from a death-mask immediately after death must be discarded. If it was, the hair must have been removed from the

* *Remarks on General Parts of Europe, Relating chiefly to their Antiquities and History*.... London, 1738, pp. 154—155.

Pearson and Morant: *The Cromwell Head*



Life Mask of Cromwell at Chequers Court.

Full face (somewhat inclined).



Life Mask of Cromwell at Chequers Court.
Left Profile.



Life Mask of Cromwell at Chequers Court.
Right Profile.

face *after* the mask was taken. There is nothing to suggest such a crude proceeding in the mask itself, and for what purpose should the hair have been removed?

The alternatives are :

First, that Cromwell in life submitted to being clean shaven in order to have the mask taken. But why? He was not shaved for the Chequers life-mask in 1655.

Secondly, that for some purpose it was needful to prepare a hairless model of Cromwell's head, and that the Florence and Grafton busts have been taken from moulds made from this. The only purpose that we can think of is the making of a mould for the wax masks of the effigies. These effigies would doubtless have real hair attached to the upper lip and chin as well as to the scalp, very probably to the eyebrows and eyelids. But such effigies were works of art, if waxworks, and it seems possible that Symon* would prepare either in wood or plaster a hairless head of Cromwell to make the mould for the wax masks of his effigies. From this same mould the Grafton and Florence busts might be cast. There is certainly, notwithstanding the crudity of the eyes, the touch of no mean sculptor in the resulting head. Reduced to the Chequers mask as standard, there is a general agreement between measurements made on the Florence bust, and on those of the masks and other busts.

(C) *The British Museum Wax Mask.* This is a death-mask taken after the embalment, for the head is bound with a cloth to cover the cincture, and before the eyes have fallen in, or before the eyelids required pads for their support. It contains all the usual features—the heavy under lip, the upper lip with its central convexity, the beardlet, the flowing moustache and the coarse nose inclined to the left. The wart has been pared off, but it seems possible to identify the spot where it had been. See our Plates LXV and LXVI. There is a plaster reproduction of it in the Museum of the former Biometric Laboratory. See our Plates LXVII and LXVIII. The reader should note the small amount of hair on the lower part of the cheeks, or on the lower border of the chin. This mask seems by far the most satisfactory of the death-masks.

(D) *The Ashmolean Death-mask.* (See Plates LXIX and LXX). That this is a death-mask of Cromwell, hardly admits of discussion. But difficulties arise at once when we begin to consider it in detail. In the first place, we note that the usual characteristics occur; the flowing moustache, the beardlet, the convexity at the centre of the upper lip, the rather coarse nose inclined to the left cheek. But, as in the wax mask, the wart has disappeared, and there is no obvious sign of the cincture. Yet if the reader will examine our Plate LXIX carefully, he will see a wavy line across the left brow, and will realise that a coating of plaster has either been put into the mould or across the forehead of the cast itself, and this wavy line is produced by the thin border of this layer of plaster. This border of the plaster layer is less conspicuous over the right eye, but we think can be traced, and lies closer to the

* See our pp. 305 and 304 footnote *Officers of the Mint.*

eyebrow than on the left. Further, there are suspicious indentations across the forehead which may be signs of the cincture below. Roughly, on Plate LXIX, the internal interocular distance is 33·0 mm., and this is almost identical with the distance from the lower border of the left eyebrow at its centre to the portion of the indentation vertically above it*. The like measurements for the actual Wilkinson Head are 44·0 mm. and 44·0 mm. respectively, indicating the three-quarter reduction of Plate LXIX, and *pro tanto* an argument for the genuineness of the Head.

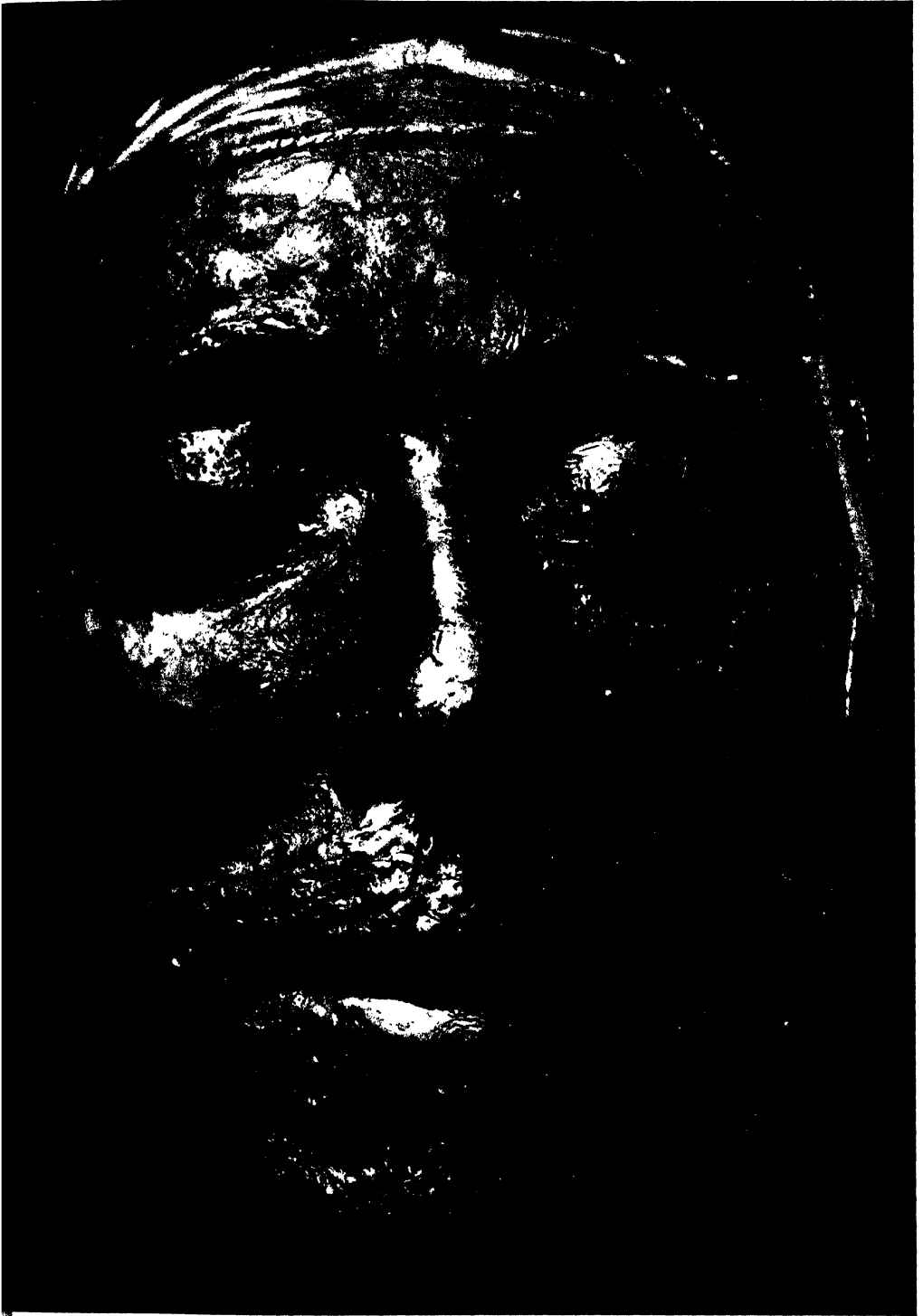
The next noteworthy feature is the appearance of the eye pads propping up the eyelids, since the eyes have fallen in. We are assured that this marks that the mask was taken 10 to 14 days after death. There is no sign of a bandage to support the jaw, but there is a very large increase in the hair on the lower border of the chin and on the lower part of the cheeks. Here we are met by a serious dilemma from which extrication is by no means easy. The wax mask (C) shows a bandage presumably to support the jaw, and a bandage presumably to cover the cincture, but there is no hair on the cheeks or lower border of the chin. The hair of Cromwell on the face is as we see it in the portraits. But in the Ashmolean mask we have a death-mask taken probably 7 to 11 days later with a good deal of hair where there was none on the wax mask. Now it is not improbable that Cromwell would have more hair on his face during his illness than when in ordinary health†. Hence this additional hair was either removed before the earlier death-mask was taken, or else the toucher-up removed it from the mask itself. In the former case it must have re-grown in the course of the 7 to 11 days interval. It is known that hair does continue to grow after death, but could it grow to this extent? Numerous inquiries made by us failed to elicit any reply to the question as to what rate hair can grow after death.

In the latter case doubts may be raised as to the chin in the British Museum mask. The re-toucher would not have been content with removing the wart, but must have removed the hair from the chin border and cheeks. Thus we should be as little able to trust the chin of the earlier mask as the forehead of the later. It is, however, quite certain that the toucher-up of the second mask would be extremely unlikely to introduce hair where it did not exist, especially where it was not worn by Cromwell in his usual health.

* This particular measurement was taken because it was nearly parallel to the focal plane of the camera.

† The hair on the chin border and cheeks of (D) is more in accordance with what may be seen on the Wilkinson Head itself. But this is not very helpful. For if the British Museum wax mask (C) really represents the face of Cromwell immediately after death, it may be argued that the Wilkinson Head does not show more hair than could have grown later. On the other hand, if the Ashmolean mask represents the state of affairs 10—14 days after death, the fact that the Wilkinson Head has less hair on the chin border and cheeks than that mask may be accounted for by rough handling, and possibly by the depredations of souvenir hunters. [The Rev. Paul Cromwell Bush informs us that one of his ancestors had allowed portions of the brim of Cromwell's Long Parliament hat to be cut off as souvenirs!]

We have seen a statement, but do not know on what authority it is based, it may be only a verbal tradition, that Cromwell's relatives did not like the death-mask, because they were not accustomed to seeing Cromwell with hair on his chin and cheeks.



Wax Death Mask of Cromwell in the British Museum, showing the bandage over the cincture. Full face.



Wax Death Mask of Cromwell in the British Museum. Left Profile.



Plaster Cast taken at some time from the British Museum Death Mask, but indicating more clearly the position of the Wart over the right eye. Biometric Laboratory, University College, London.



Left Profile of Plaster Cast in Biometric Laboratory,
University College, London.

One very important inference may be based on the Ashmolean type of death-mask. If it could be taken 10 to 14 days after death, it is clear that Cromwell's body was not at that time sealed up in lead. It was accessible, and accordingly the statement made by Bate becomes even more suspicious, if it be supposed to apply to an occurrence happening immediately after death.

The difficulties attending the Ashmolean mask are not lessened by the fact that other masks of this type, although corresponding in some respects closely with the Ashmolean, yet differ essentially from it in others. In particular one may refer to the widespread type of mask which we shall discuss under (E). In order to be somewhat clearer as to which might be the earlier form, (D) or (E), we sought for a death-mask still in the possession of a descendant of Cromwell, and found such in the ownership of the Rev. Paul Cromwell Bush. He has most kindly provided photographs of it, which are reproduced in Plates LXXI and LXXII. It will be seen at once that while the type is essentially that of the Ashmolean, it differs in certain particulars. This appears especially in the treatment of the forehead; we have here no wavy line marking the limit of the plaster-layer by which the saw cut has been covered up.

There is a mask, *plaster*, not wax as reported, at Warwick Castle; it has the wart, the hair on cheeks and chin, and the propping pads for the eyelids. It is said to be like the Ashmolean, which, however, has no wart. Accordingly, it belongs to the ten days after death type, and, without having seen it, we suspect it to be more like (E) than (D).

(D *bis*) *The Frankland Death Mask*. We may here note the death-mask possessed by Major A. P. Frankland, D.S.O.: see our Plate LXIII. This has the wart, which looks as if it had been stuck on, not cast from the original mould. The cincture has been hidden, but without exaggerating the forehead. The eye pads are there, and the hair on chin and cheeks. It seems to us the missing link between the earlier Ashmolean form and the "Distorted Ashmolean form" as represented by the masks in the possession of Canon Wilkinson or of Shrewsbury School. Major Frankland is a direct descendant of Elizabeth Russell, the granddaughter of Cromwell and daughter of Frances Cromwell. That the Frankland mask is the progenitor of the numerous "Distorted Ashmolean type" masks is clear, but we have been unable to convince ourselves that it is as near to the original mould as the Ashmolean and Cromwell Bush death-masks.

(E) *Distorted Ashmolean Mask*. This is a very curious distortion of the Ashmolean type of death-mask: see our Plates LXXIV, LXXV and LXXVI. As a rule in the distorted type of mask the two portions of the cast have been united along the median sagittal plane. Whether there were originally two separate moulds we cannot say, but in the process of re-moulding the horizontal measurements have become relatively smaller, and the vertical measurements are relatively larger than those of the Ashmolean death-mask, or indeed of the Chequers life-mask or the portraits. Hence, it is impossible to bring specimens of this type

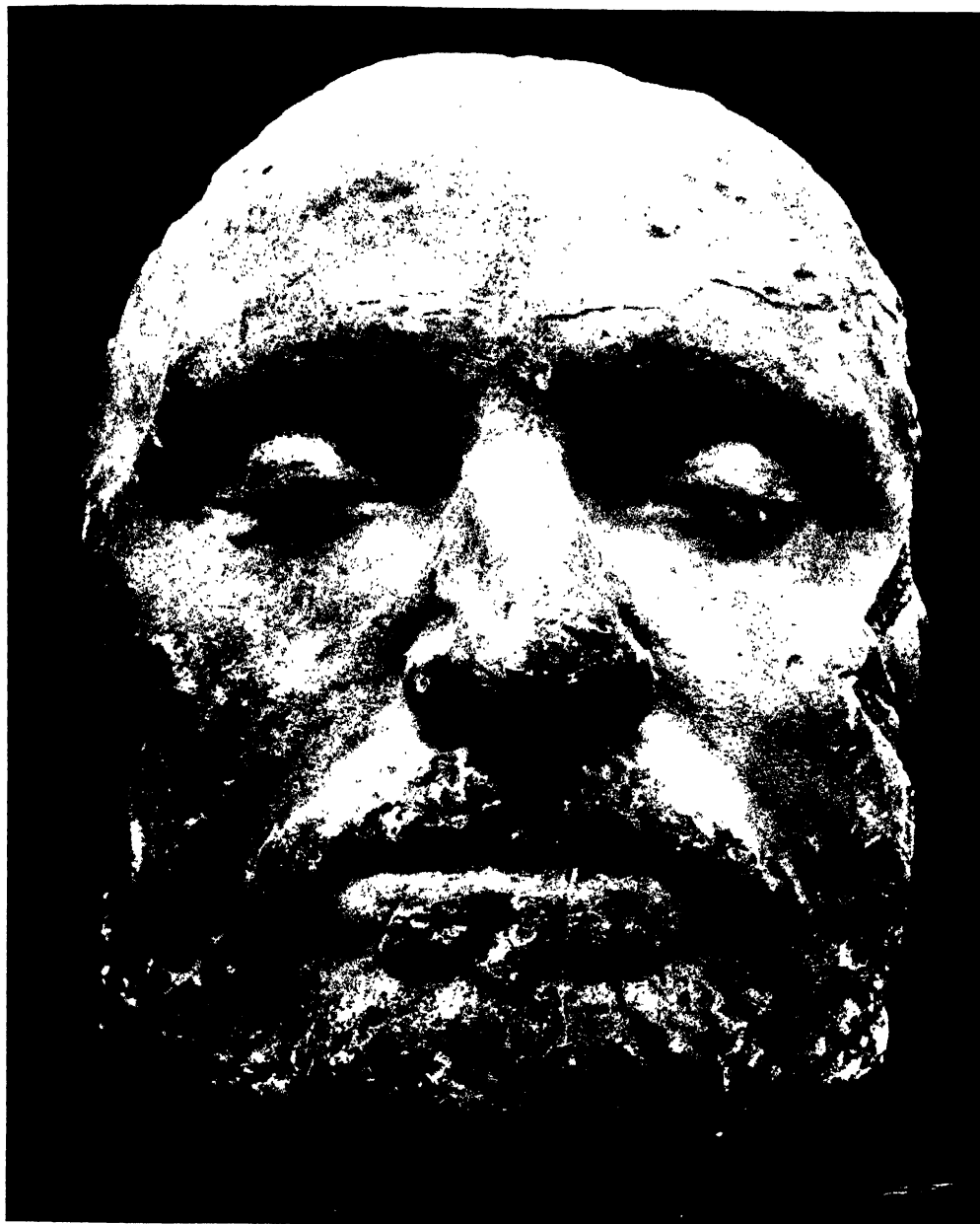
of mask into accord with other portrait material by using a single reducing factor; two must be used, one for the vertical and the other for the horizontal lengths. On this ground we have termed it the "Distorted Ashmolean" type. It is clear that it has been—after no doubt several intermediate stages—obtained from the same original or else from an original taken about the same time. There are the same eye pads, marking a period 10—14 days after death, there is the same increase of hair on the cheeks and on the lower part of the chin, beyond the earlier death-mask. But there are differences; the wart—it is true in a rather crude form—is there, and the re-modelling of the forehead has been differently achieved; one can find no trace of the cincture. Copies of this mask are in the possession of Canon Wilkinson and in the Library of Shrewsbury School*. Owing to the re-moulding, the touching-up and re-modelling, it cannot be considered a satisfactory death-mask, although when two factors are used the results are in not unreasonable accordance with the measurements on other masks and busts.

We will now turn to the two busts we have selected for comparison with the masks and the Wilkinson Head; they are the bust by Edward Pierce in the Ashmolean Museum and that by an unknown sculptor in the Library of King's College, Cambridge.

(F) *The Ashmolean Bust by Edward Pierce.* It is signed E. Pierce, but without date. If it was taken from life, then it must represent Cromwell in his full health and strength (say, about 1654—1655), and a very beautified Cromwell at that. As a work of art it is striking, and of course recognisably Cromwell. It is somewhat larger than life. It is stated to have been purchased by Lord Taunton from a dealer in London at the beginning of the nineteenth century, and to have passed from the Taunton Collection to Oxford in 1920. A terra-cotta model of this bust is in the National Portrait Gallery†. The thick under lip, the central convexity of the upper lip, the beardlet, the retreating and hairless chin, the nose inclined to the left, and the wart are all there, but they have lost their touch of vulgarity, and we recognise a man with magnetic strength to influence, and with the essential power to command. But it is hardly the Cromwell of any other bust, mask or painting.

* We have to express our warm thanks to Canon Wilkinson for allowing Dr Morant to photograph his mask, and to Mr J. B. Oldham, the Librarian, and Mr Pilcher for measurements and photographs of the Shrewsbury School Library mask.

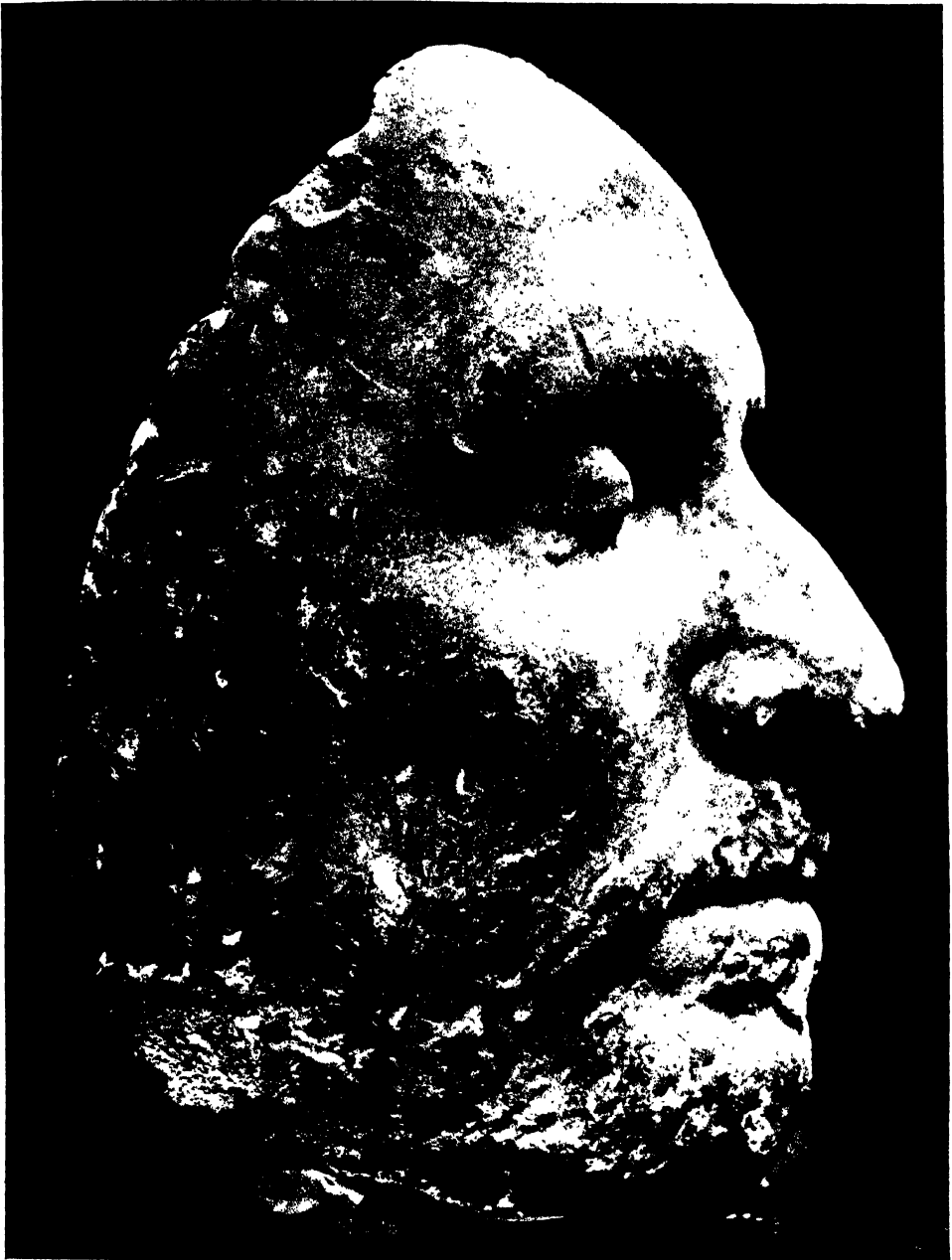
† The statement that the terra-cotta bust in the National Portrait Gallery is a counterpart of the Ashmolean Bust is due to Rachel Poole (*Edward Pierce, the Sculptor*, Vol. XI of the Walpole Society, 1922—1923, Oxford, 1923, pp. 33—45). On the other hand, Lionel Cust, writing in 1896 or earlier on Edward Pierce in the *Dictionary of National Biography*, stated that the terra-cotta in the National Portrait Gallery was the model of a marble bust of Cromwell "now in the possession of E. J. Stanley, Esq., at Quantock Lodge, Somerset." Possibly the discrepancy may be accounted for by the fact that Edward James Stanley of Cross Hall, Lancashire, married Mary Dorothy Labouchere, the eldest daughter of Lord Taunton. Thus there would not be two marble busts of Cromwell of the Ashmolean type to be reckoned with. But Cust seems to give priority to the terra-cotta when he speaks of it as the model of the marble.



Death Mask of Cromwell in the Ashmolean Museum. It shows the thin coating of plaster over the forehead put on to hide the cincture.

Full face.

Pearson and Morant: *The Cromwell Head*



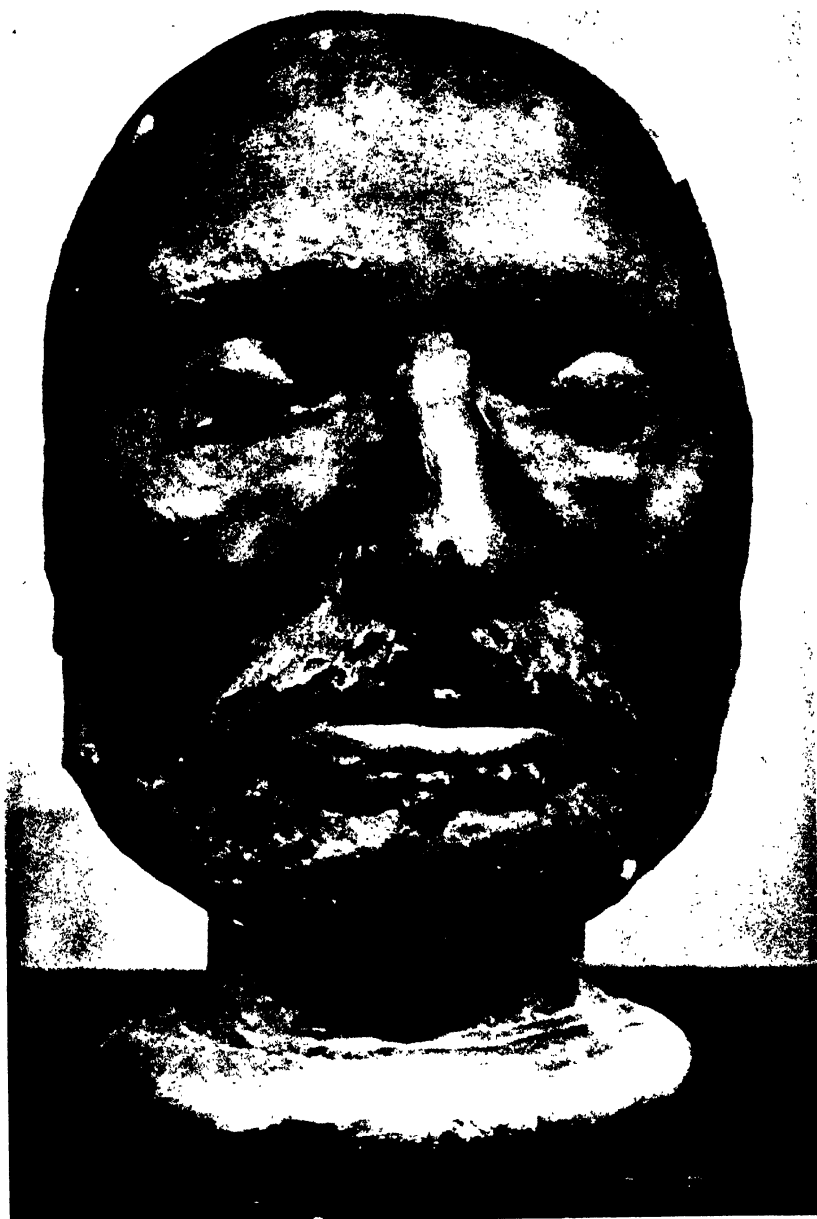
Death Mask of Cromwell in the Ashmolean Museum.
Right Profile.



Death Mask of Cromwell in the possession of the Rev. P. Cromwell Bush.
Full face.



Death Mask of Cromwell in the possession of the Rev. P. Cromwell Bush.
Profile.



Death Mask of Cromwell in the possession of Major A. P. Frankland, D.S.O. Original from which the Distorted Ashmolean Type of Death Mask has sprung. Nearly full face. By kind permission of Major Frankland and the Proprietors of the *Illustrated London News*.



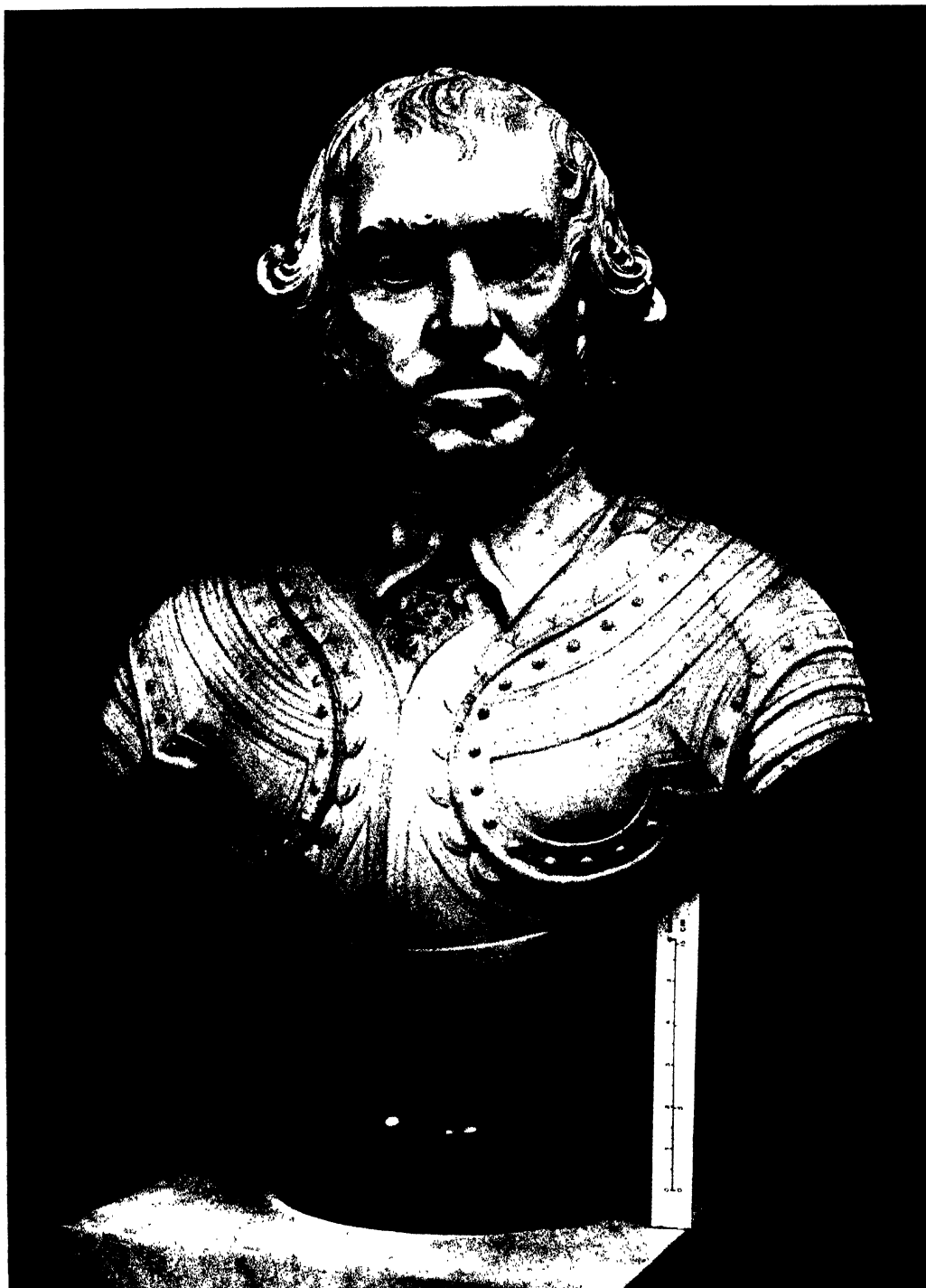
The Distorted Ashmolean Type of Cromwell Death Mask.
Full face.



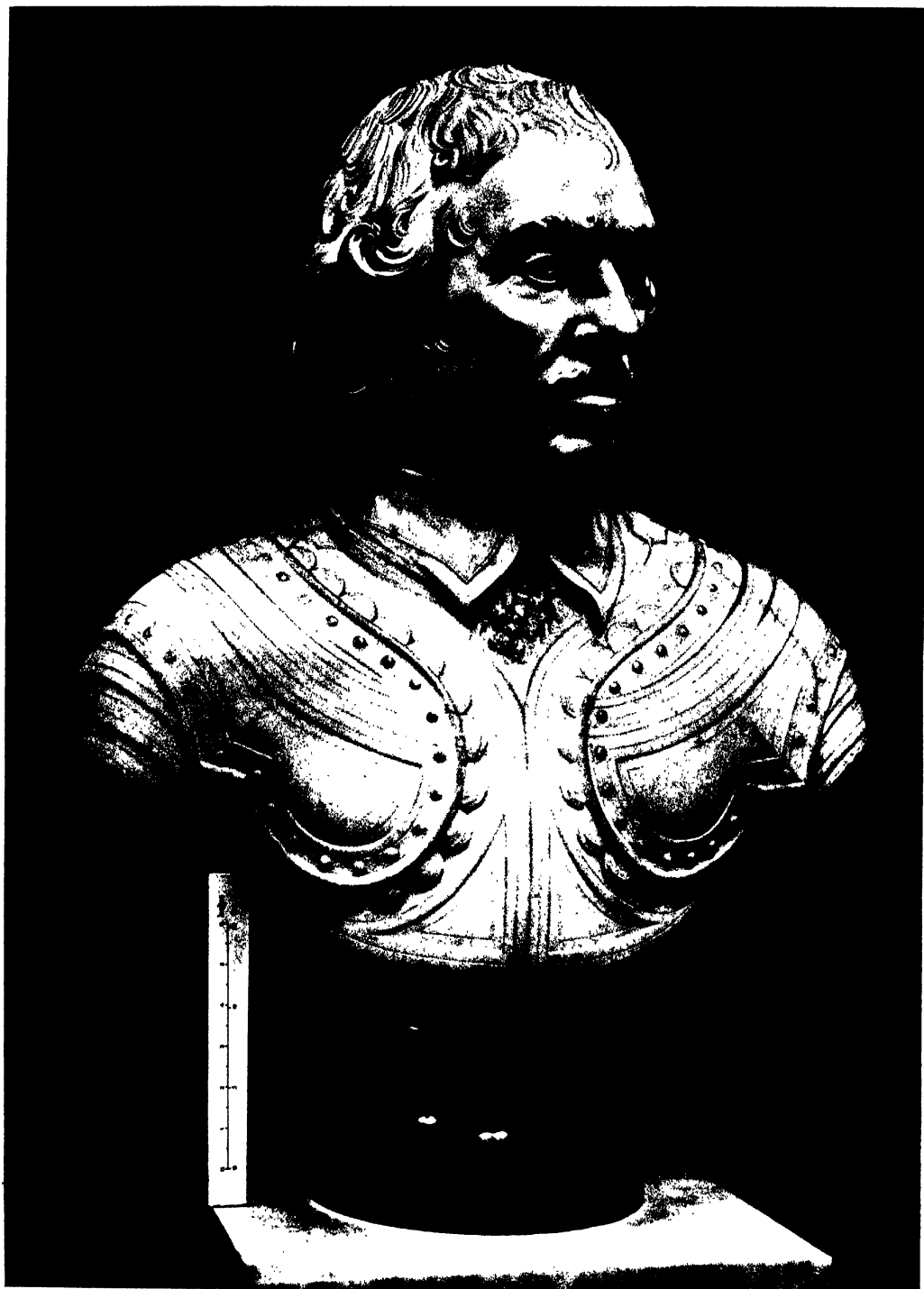
The Distorted Ashmolean Type of Cromwell Death Mask.
Right Profile.



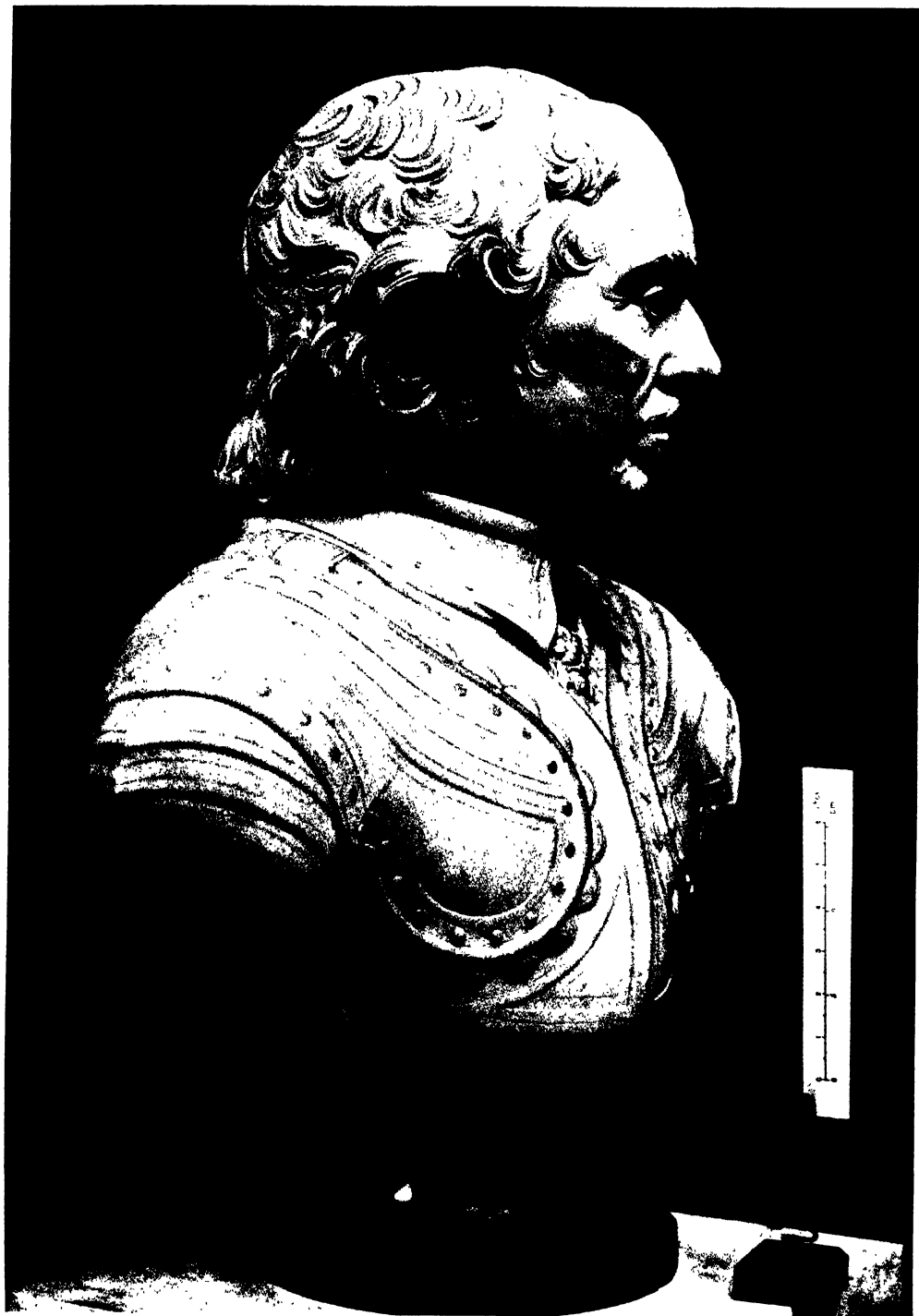
The Distorted Ashmolean Type of Cromwell Death Mask.
Left Profile.



Edward Pierce's Bust of Cromwell, Ashmolean Museum.
Full face.



Edward Pierce's Bust of Cromwell, Ashmolean Museum.
Three-quarter face.



Edward Pierce's Bust of Cromwell, Ashmolean Museum.
Right Profile.

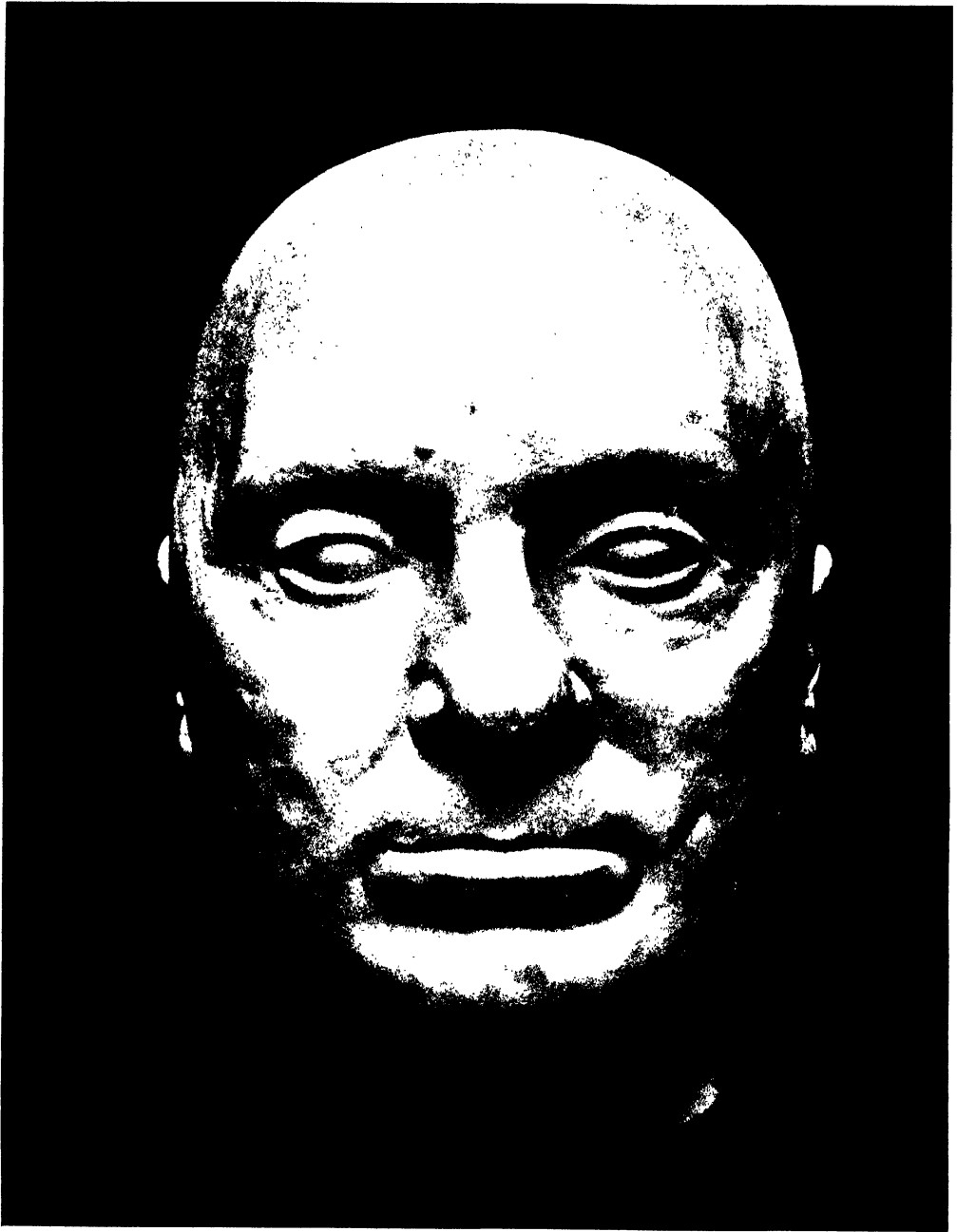


The Bust of Cromwell in the Library, King's College, Cambridge.
Full face.

Pearson and Morant: *The Cromwell Head*



The Bust of Cromwell in the Library,
King's College, Cambridge.
Right Profile.



The Florence Bust of Cromwell, from a cast in the National Portrait Gallery, London.

Full face.



The Florence Bust of Cromwell, from a cast in the National Portrait Gallery, London.
Left Profile.

(G) *The Bust in the Library of King's College, Cambridge.* *A priori*, one might have hoped that this bust had come into the possession of the College during the reign of Benjamin Whichcote, who was installed as a Puritan Provost by the Earl of Manchester in 1645, after ejecting Samuel Collins. Whichcote was in sympathy with Cromwell. But unfortunately the bust has a history more obscure than the Ashmolean. All we know about it is that it came to the College with other portraits as a legacy from George Thackeray, Provost of King's, 1814—1850, who died in the latter year. When and where this antiquarian provost obtained it we do not know. The bust is marble and, if not a great work of art, has something of interest about it. There is in the first place a most characteristic inclination of the head towards the right shoulder, which gives a dominating aspect to the subject. See our Plates LXXX and LXXXI. Although we have come across no portrait clearly showing this aspect, it is difficult to believe that it was hit upon by a non-contemporary artist*. It appears to be either an original, or the copy of an original which so far is unknown to us. All the usual characteristics are present, and the one new feature added is the tilt of the head.

17. *Fit by Superposition of Masks and Busts to the Wilkinson Head.*

Such is the material we have to compare with the Wilkinson Head. Before we actually appeal to numbers, we have to warn our readers of the difficulties attending any measurement of this material, and how subject to error any such measurement must be. Let us turn to the Head itself first. Risky and indefinite as are measurements on the living head, we have here to deal with a skull upon which the flesh has shrunk unequally in different parts. In order to obtain some idea of how much this shrinkage amounts to, the so-called horizontal, transverse and mid-sagittal contours of the Wilkinson Head were drawn showing the cranial boundary and the thickness of the embalmed flesh. Upon the cranial boundary was superposed the thickness of the normal flesh and there resulted the three diagrams given on Plates LXXXV, LXXXVI and LXXXVII. The allowance for flesh has been based upon the frozen sections of human heads made by the late Professor Symington and by William Macewen for *The Atlas of Head Sections*, in the case of one or two individuals of about Cromwell's age, but of course the thickness of the flesh varies widely from individual to individual, and all such contours can achieve is to give general impressions of (i) the shrinkage due to embalment, (ii) the amount to be added to a skull in any given direction in order to approach the correct size, and (iii) the difficulty of taking measurements which are really comparable. Above all we see how widely the profile of a subject may vary with the actually occurring individual differences in the thickness of the flesh.

If we start with the tightly drawn skin on the Wilkinson Head, we are at once met by the extraordinary difficulty first of defining and then of fixing points of measurement, and this in particular applies to lengths in the median sagittal plane. Take in this plane a point corresponding to the gnathion, the lowest point just

* See, however, our remarks as to Viscount Harcourt's miniature on pp. 352—353.

under the central edge of the mandible. This is not difficult to determine on the Head (see Plates XXVIII and XXIX), because the skin is drawn tightly round the mandible, there is not the 6 to 8 mm. of flesh usually found below the border of the mandible. But try to determine it on one of the masks (see Plates LXIII and LXX), or on the busts (Plates LXXIX and LXXXI), or on the coins (Plate LXI), or indeed on the miniatures (Plates XLII and XLVI), and the large possibility of error will be realised! Owing to the protruding under lip and the large amount of flesh under the mandible, Cromwell's chin had a by no means graceful curvature, and it is extremely hard to determine where the "lowest point" should be taken. But there is still another difficulty to be met. Is the mandible of the Wilkinson Head in its normal position when the teeth rest squarely on each other? It is not easy to determine when the lips have perished, whether they would have been in contact and if so where in contact in life.

In the first place we will provide some graphic evidence of how reasonably the Wilkinson Head fits the masks and busts.

(A) We take first the *Life-mask at Chequers Court*. Plates LXII, LXIII and LXIV show this mask full face, right profile and left profile. It was a difficult subject for photography, especially in a difficult environment; it is bronzed plaster. At the top there is a wire loop, but, considering the value of the mask, too slender to venture suspending it by. Thus the mask had to be photographed with a slight inclination, and with an attempt to bring the focal plane of a vertical camera into parallelism. Plate LXXXVIII (*a*) shows the mask fitted with the full face of the Wilkinson Head—not wholly unsuccessfully considering the foreshortening of the mask. The profile (*b*) on the same plate provides an excellent fit; the white line gives the outline in profile of the embalmed head, and the black and white broken line of the probable amount of flesh. This does not differ from the life-mask flesh border more than might be anticipated when we pass from one individual to a second. The Chequers Court life-mask is distinctly in favour of the genuineness of the claims of the Wilkinson Head.

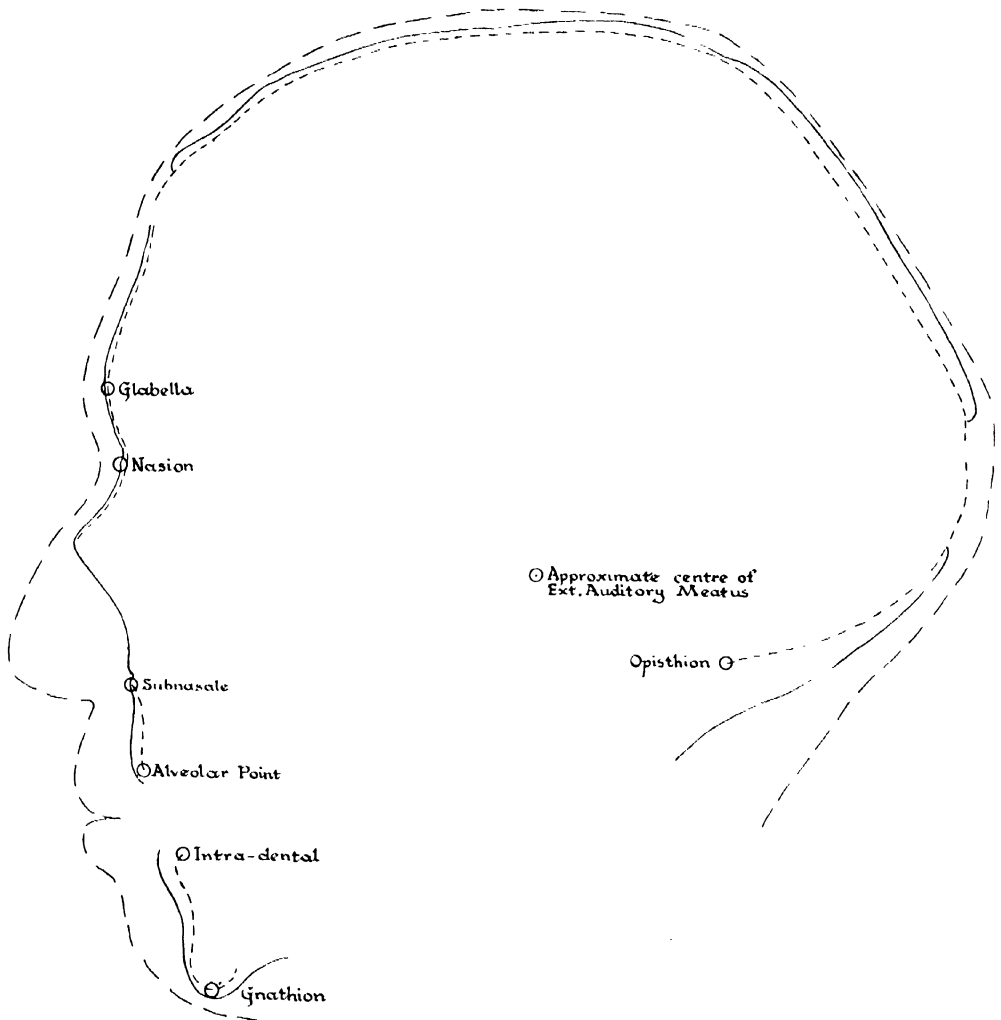
(B) *The Florence Bust*. Plates LXXXII and LXXXIII give full face and profile from the cast in the National Portrait Gallery. Plates XCVI and XCVII show the same reproductions with indications of the positions of the chief features of the Head. Again the fit is with one exception remarkable. In the full face the mandible, broken mouth, nose, eyes and eyebrows, and the cincture are well placed. So it is again with the profile, remembering what has been said about the flesh. But the small circle marking the presumed position of the centre of the auricular orifice is much out of place. We think it may have been marked somewhat too high on the sagittal contour of the Head. But the outline of the ear (see Plate XXVIII) of the Head would never accord with the ear of the Florence bust. An examination of the Florence bust on Plate LXXXIII seems to indicate that the ear of the bust is too far back and too low down to be natural! A separate cast was probably taken of the ear, and the ear afterwards attached to the bust out of its true position!



From a Bust of the Protector
OLIVER CROMWELL,
In the possession of his Grace
the Duke of Grafton.

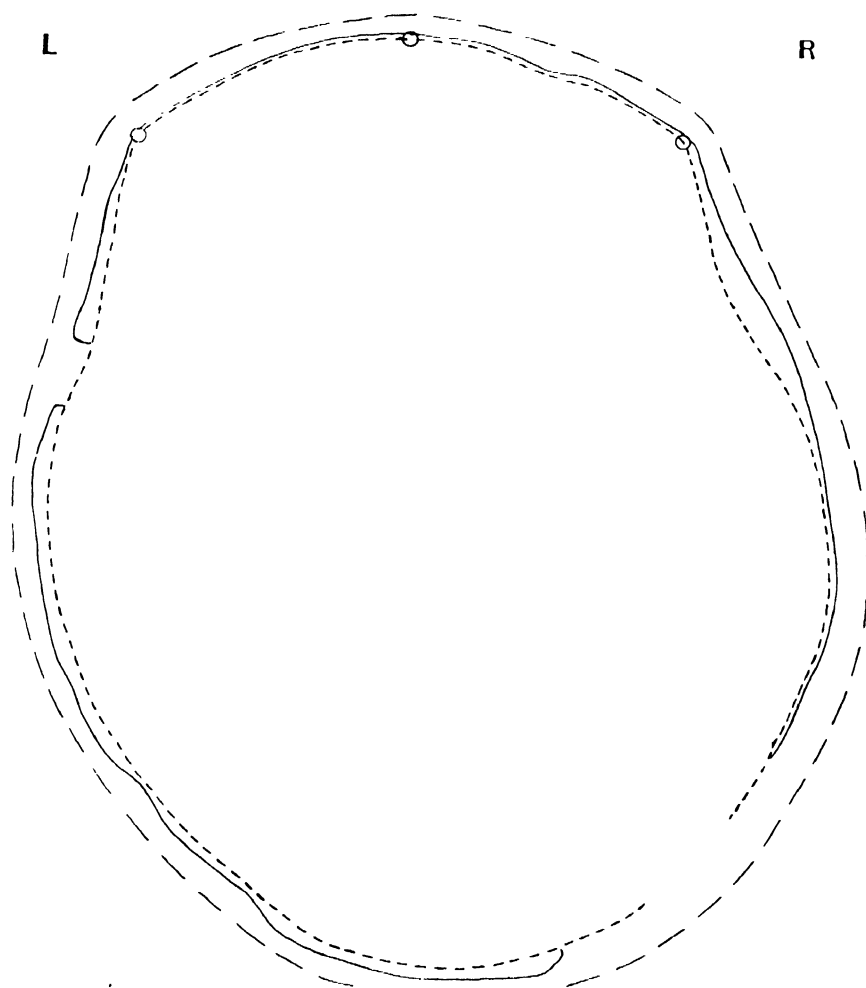
Engraved by R. D. Norton

The Duke of Grafton's Bust of Cromwell. Three-quarter face. The original or another specimen of the Florence Bust.



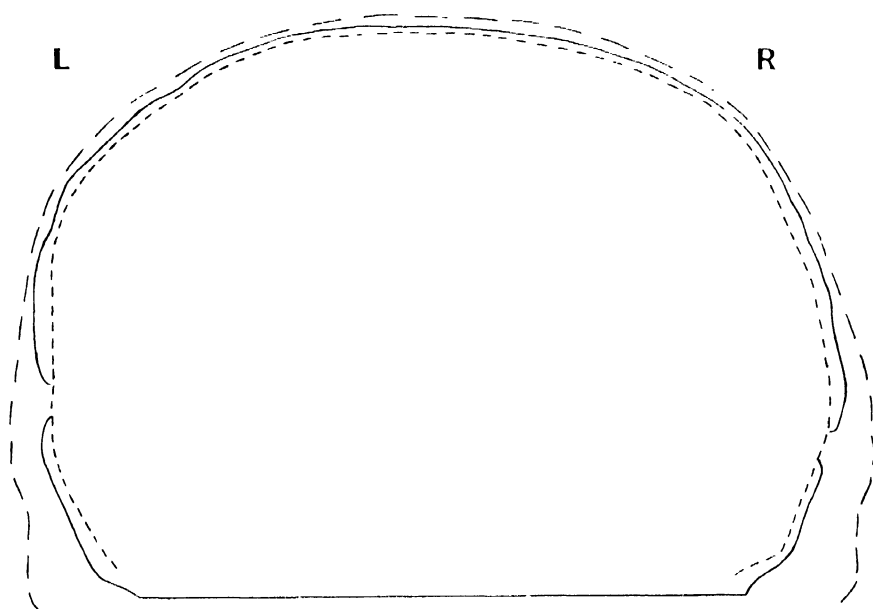
Median Sagittal Section of Head showing ---- bone,
 ——— embalmed head, — — ordinary flesh allowance.

Median Sagittal Section of the Wilkinson Head. $\frac{1}{2}$ natural size.



Horizontal Section of Head showing - - - - bone,
 ——— embained head, — — ordinary flesh allowance.

Horizontal Section of the Wilkinson Head. $\frac{2}{3}$ natural size.



Transverse Section of Head showing - - - - - bone,
——— embalmed head, — — — ordinary flesh allowance.

Transverse Section of the Wilkinson Head. :: natural size.

(C) *British Museum Wax Mask.* The full face and profile are given on Plates LXV and LXVI. The fit of the Wilkinson Head is indicated, also in full face and profile, on Plates LXXXIX and XC. The fit may again be called good. The cincture comes, as it should, under the forehead bandage, the eyebrows, eyes, nose, upper lip and chin are in reasonable position; only the broken edge of the lower lip is too high. On the profile the limits of the embalmed flesh are good, but the allowance for flesh on the lips and chin is too great. This seems to indicate that the Chequers Court life-mask and the British Museum mask show different amounts of flesh on the lower part of the face. There may have been some shrinkage during Cromwell's last year of life and his illness.

(D) *The Ashmolean Death-mask.* This is shown on Plates LXIX and LXX. The chief features of the Wilkinson Head are shown on Plates XCI and XCII by the embalmed flesh boundary. The accordance is distinctly good. We note again, as in the case of the wax mask, the somewhat high broken border of the lower lip. The suggestion may be made that more of the upper lip than of the lower has been broken away. The fit here may be described as excellent.

The Rev. P. Cromwell Bush's death-mask is so close to the Ashmolean that we have not considered it necessary to fit this mask with the Head. On Plate LXXII the nose stopping (see our p. 340) is visible in the left nostril.

(D') *Major A. P. Frankland's Death-mask.* On Plate LXXIII we have given a representation of this. We now give the nearly full face, with the sketch of the Head superposed, on Plate XCIII. It will be seen that when we bring the wart, eyes, mouth and chin into reasonable accordance with the embalmed Head without flesh allowance, the Head projects beyond the mask in a hopeless manner only slightly less unfavourably than in the "Distorted Ashmolean Type." See Plate XCIV. The photograph is not quite full face, and the facial outline might have been that of the Head slightly more rotated, and tilted from right to left. But to make it much smaller would have upset the goodness of fit of wart, eyes, mouth and chin. If we are dealing here with an untouched mask of Cromwell, then we can only conclude that not only the Wilkinson Head, but the Ashmolean and British Museum death-masks are very poor representations of the real Cromwell. We believe that the Frankland mask has been reconstructed from the Ashmolean, and the smaller superficial details worked in; and this again has been from time to time remodelled until the Distorted Ashmolean form of today was reached.

(E) *Distorted Ashmolean Mask.* Full face and right and left profile are given on Plates LXXIV, LXXV and LXXVI. On Plates XCIV and XCV the full face and right profile are fitted with the Wilkinson Head. The profile fit is by no means bad, and might possibly have been bettered. But the full face is no fit at all. Keeping the wart cavity of the Head over the wart on the mask, we might have reduced the vertical scale somewhat, so as to bring the chin and mouth opening higher. The maximum reduction possible is about 17 to 16, but this is not adequate horizontally to bring the embalmed head contour, even *without* allowance

for the natural flesh, within the boundary of the mask. In other words, the length-breadth ratio of the mask is impossible for the Wilkinson Head, and accordingly equally incompatible with those of the Ashmolean death-mask or the Chequers Court life-mask. Quite apart from the Wilkinson Head, the distorted Ashmolean mask should not be looked upon as really representative of Cromwell. Not only has it the wrong proportions for Cromwell's face, but it is in itself crude, and coarsely remodelled, especially in the forehead region, where the difficulty of the cincture has been overcome in a manner which has destroyed all the character of the frontal.

(F) *The Ashmolean Bust by Edward Pierce.* Plates LXXVII, LXXVIII and LXXIX give the full, the three-quarter face and the right profile of this remarkable work. On Plate XCVIII we show the full face fitted with the Wilkinson Head (broken lines), and on Plate XCIX the profile fitted with the bony skeleton of the Head (dotted line) and the probable flesh limit (broken line). The fit is wonderfully good, only the fleshy lips which have been added to the Head appear somewhat too low in the profile. If the Wilkinson Head be Cromwell's, then it seems certain Pierce modelled Cromwell from the life or from some other bust taken from the life. If Pierce could be shown to have modelled from the life, then the Wilkinson Head can hardly be other than Cromwell's.

(G) *The Bust in the Library of King's College, Cambridge.* Plates LXXX and LXXXI give the full face and profile of this bust. Plates C and CI show drawings of the bust in the like position fitted with the outlines of the Head. In the case of the full face the fit is not as good as for the Oxford bust—the wart cavity is too high, and the right eye somewhat too low, but the divergence does not exceed what we may term artistic licence, i.e. the bending of nature to suit the impression of the subject formed by the individual artist: the distortion of actuality—not amounting to caricature—to emphasise the particular characteristic of the subject which has taken possession of the craftsman's mind. In this case it is the dominating head tilt.

If we turn to the profile (Plate CI) we note that there is excellent facial accordance, whether of the bony skeleton, or of the flesh allowance. The upper lip of the flesh allowance protrudes, but in the Cooper miniature the flesh allowance at the upper lip receded behind that of the miniature. It is clear that the Wilkinson Head lies *between* the two artistic presentations. (Cf. Plates LVI and CI.) Again the Wilkinson Head rises above that of the Cooper miniature, but falls inside that of the Cambridge bust, and the real head most probably was between the two.

To sum up this section: we find that even without measurement, but simply by superposition, there exists a very remarkable accordance between the masks and busts of Cromwell and the Wilkinson Head.

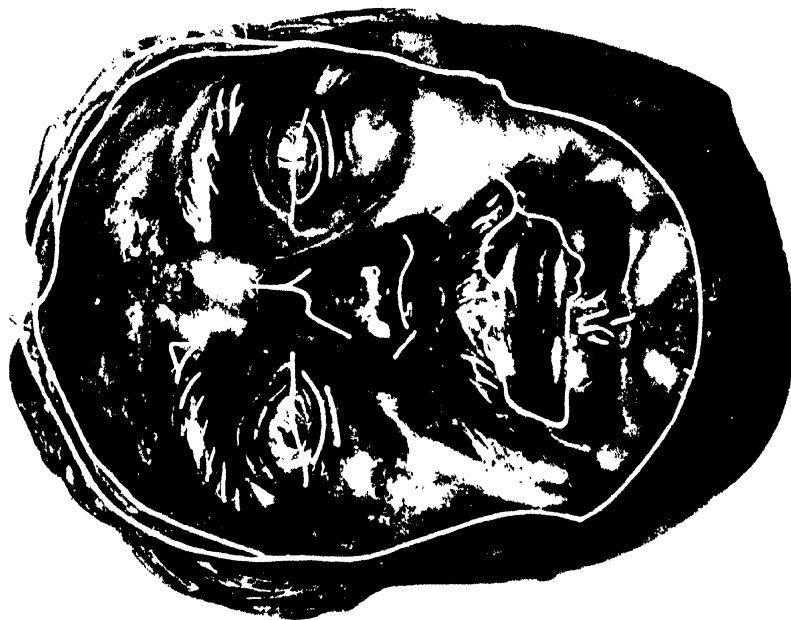
18. *Measurements on the Masks and Busts and on the Wilkinson Head.*

Before we place before the reader the result of our measurements, it is necessary to explain the senses of the terms we have adopted, and recount some of the difficulties of measurement.

Pearson and Morant: *The Cromwell Head*



(b)



(a)

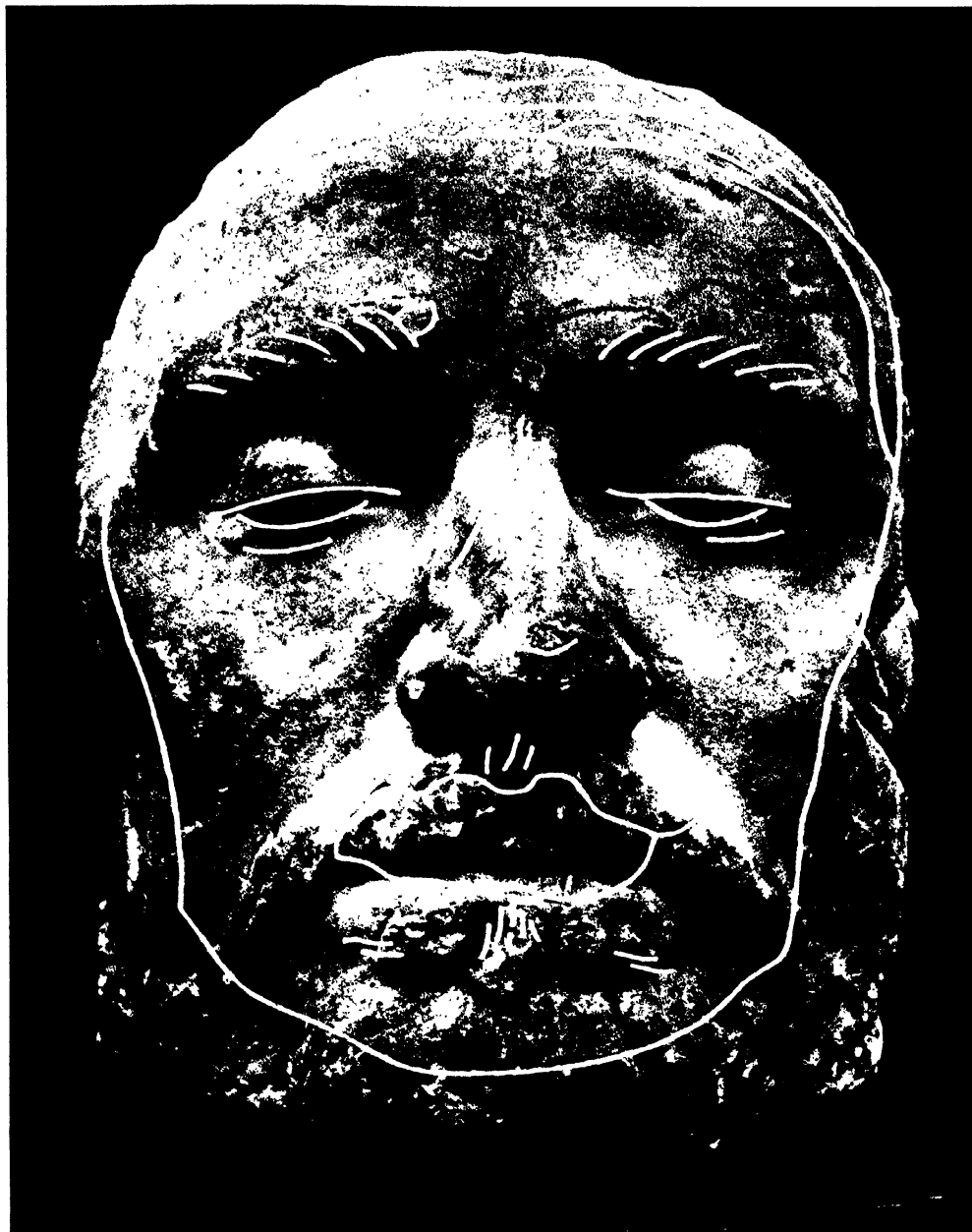
The Chequers Life Mask of Cromwell fitted with (a) the full face facial outlines, and (b) the median sagittal contour with average flesh allowance in profile of the Wilkinson Head.



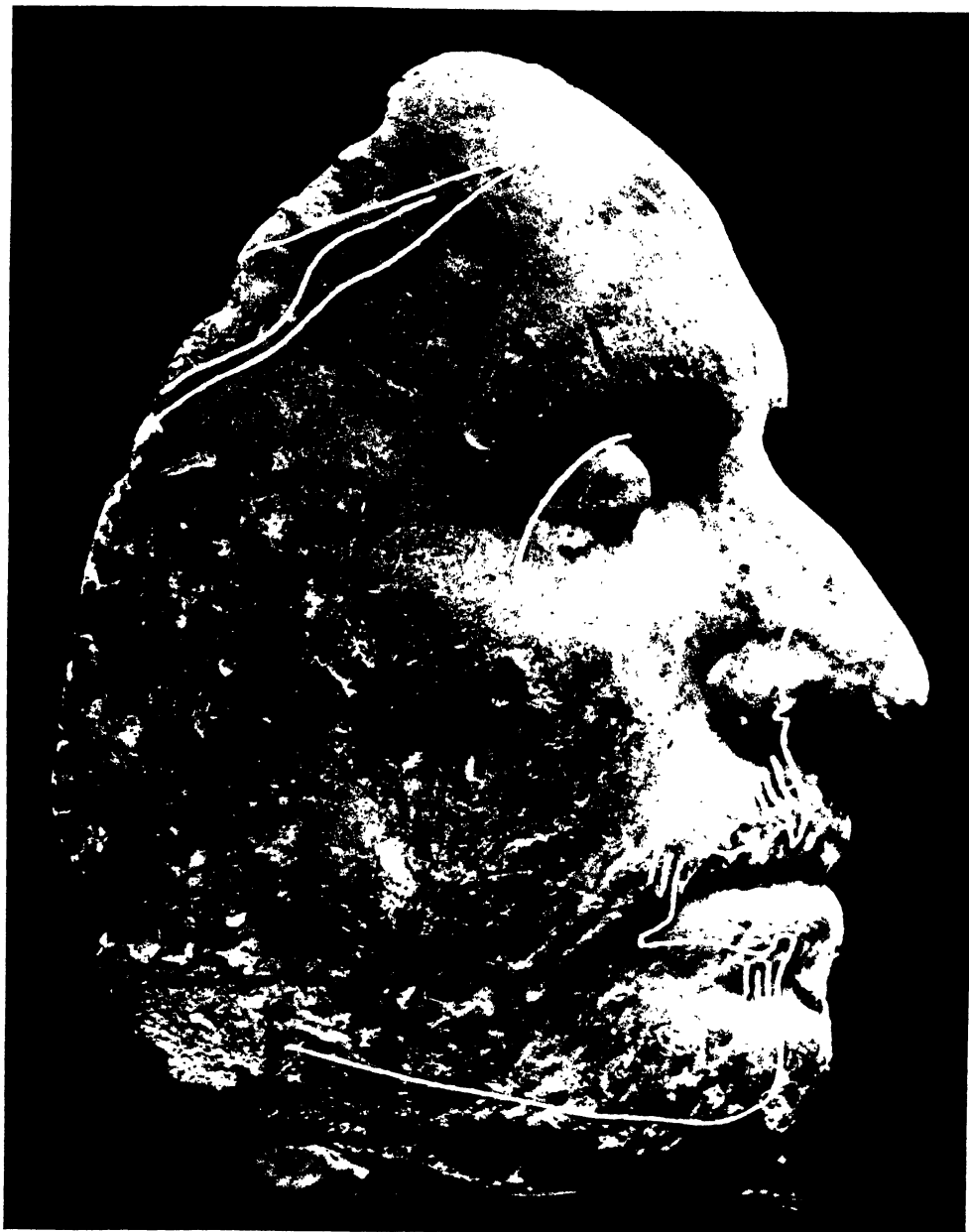
Full face of the British Museum Wax Death Mask fitted with the facial outlines of the Wilkinson Head.



Profile of the British Museum Wax Death Mask fitted with the profile —
and the average flesh allowance -- -- of the Wilkinson Head.



Full face of the Ashmolean Death Mask fitted with the facial outlines of the Wilkinson Head.



Profile of the Ashmolean Death Mask fitted with the profile outline of the Wilkinson Head.

Glabella. We use this term in a special sense. Let a plane be taken parallel to the Frankfurt horizontal plane and tangential to the upper border of the eyebrows. The point in which the trace of this plane on the forehead meets the "mid-sagittal" plane will here be termed the *glabella*. The point thus defined is easily determined on full face pictures or photographs.

Nasion. The nasion on the Wilkinson Head might possibly be ascertained, but it would not be possible to do this on masks, busts or profile portraits. We therefore define the nasion for present purposes as that point on the trace of the "mid-sagittal plane" on the nasal bridge which is nearest to the auricular axis. As the auricular axis is not determinable in a profile portrait, we may say nearest to the tragion as an approximation. But the ear is often not visible, or non-existent in the photographs of death-masks, etc., in profile. We have therefore had to proceed in another way to obtain a point on the nasal bridge. This consisted in drawing a tangent in the "mid-sagittal" plane to nose and forehead and obtaining the point on the nasal bridge most remote from this tangent, which we can for present purposes term the "lowest point" on the nasal bridge, i.e. the point with maximum subtense from the above described tangent. This point has to be transferred from the profile to the full-face portraits as accurately as circumstances permit.

To increase the number of available "points" for measurement, we first considered the external ocular distance, from external lid-meet to lid-meet—but on the Head the eyelids have shrunk so that the lid-meets are not in their natural places, and it seemed desirable to have some further measure at the eye level. We accordingly took the external orbital distance, from outside orbital margin to orbital margin, at the external ocular distance level—fairly possible on the Head, but more a matter of guess-work on the masks and busts.

Length of Mouth. Fairly satisfactory on the masks but doubtful on the Head, where the lip-meet on the left side has been torn.

Lip-line. Easy to determine on busts and masks, but only roughly in the case of the Head, being taken midway between the alveolar borders, i.e. between alveolar point and intradental.

Subnasal Point. Reasonable values were obtainable on most of the masks and busts, but very difficult to determine on the Head; the subnasal point is below the subnasale (which is easily found on the Head), and forward from it (see our "mid-sagittal" contour, Plate LXXXV). This point on the Head and measurements from it are only approximations.

Gnathion. In the case of living subjects with a flat base to the chin, this point is by no means hard to determine by palpation. On the Head the lowest point on the mandible in the "mid-sagittal" plane is capable of fairly accurate determination, but this does not correspond to its position when the shrunken skin of the Head is supposed replaced by living flesh. Further, Cromwell's chin had no flat base to it; it simply curved down to his throat, cloaked in a large amount of flesh (see our Plates LXIII, LXVI, LXXIX, LXXXI, and LXXXIII), so that on the portraits

(see our Plates XLII and XLIII) as well as on the coins (Plate LXI) any determination of the "lowest point of the chin" is likely to be subject to large variation from personal equation.

To obtain another measure in the "mid-sagittal" plane we have taken the "beardlet" which is so conspicuous in all portraits of Cromwell; it starts at the lower border of the under lip and goes down to the horizontal crease in the chin. If, as in some cases, it passes to a fine point beyond that crease (see the Ashmolean bust, Plate LXXVII), we treat that as hair spreading beyond its root. The root of the "beardlet" practically ends with the crease. A good illustration of the normal beardlet is that on the Chequers Court life-mask (see Plate LXII). The British Museum wax mask (Plate LXV) shows the beardlet extending beyond the crease and accompanied by hair from the under lip on either side of it. This appears on all the death-masks, and probably indicates that Cromwell remained unshaven for some time preceding his death.

Interpupillary Distance. This is again a vague measurement, because in the death-masks the position of the pupils and their centres can only be a rough estimate; in life-masks the pupils must be, like the eyelids, put in by the touch-up, and in the Head itself the centres of the pupils can only be taken by a rough approximation as the central points of the openings between the lids.

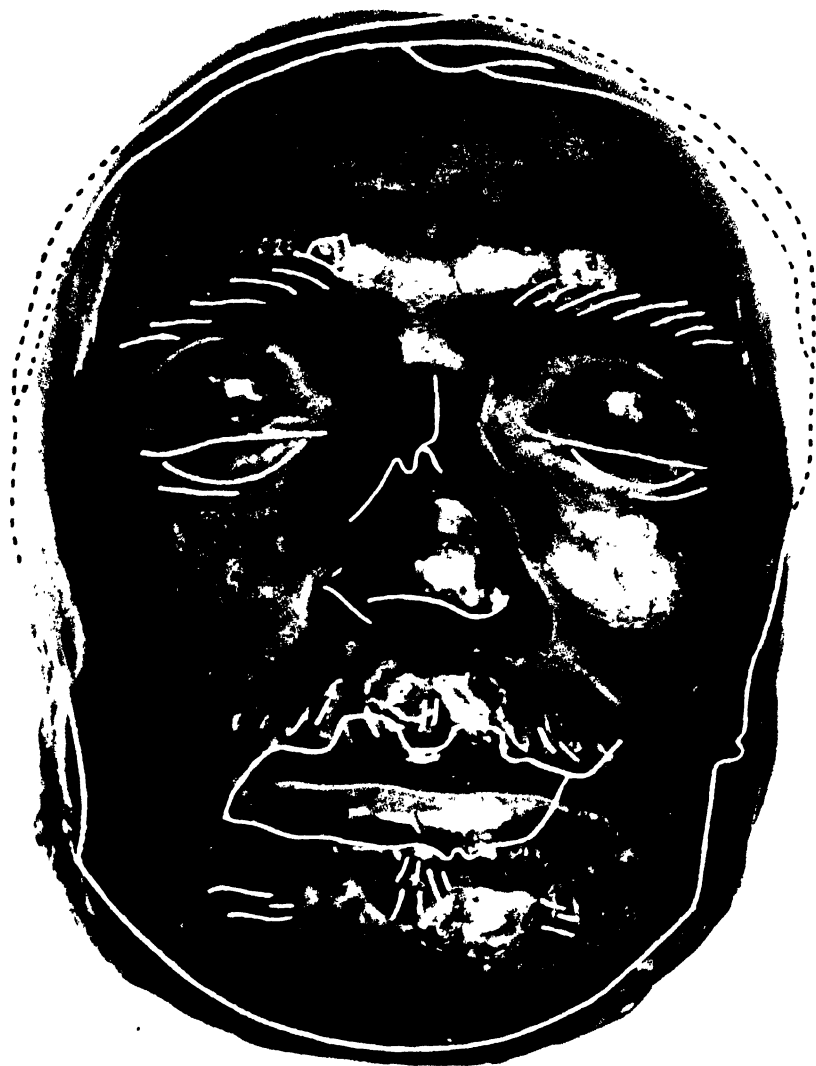
Centre of the Wart. This is a fairly reasonable terminal for measurement, where it exists, as on the three busts (see our Plates LXXVII, LXXX and LXXXII), or on some of the death-masks. Where it has been removed, as on the Ashmolean death-mask or the British Museum wax mask (see our Plates LXIX and LXVII with LXV), it is sometimes fairly easy to determine where it has been by a certain small flat area of the surface. In the case of the Head itself, if it be Cromwell's, then the wart stood where the skin has been broken away over the right eyebrow (see Plates XXVI, XXVII, XXX and XXXI), and we think on that portion of the area nearer to the glabella.

If the reader has had the patience to follow our account of the points used for measurements he will appreciate how difficult is the task of comparison; and there are still further matters to be taken into consideration. In the first place our measurements have often been taken on the photographs of the masks and busts and accordingly the lengths measured should be actually or nearly parallel to the focal plane. If they are not it will not matter if we are comparing photographic measurement with photographic measurement, but we shall be liable to some error if we compare lengths measured on the Head itself or on the busts or masks with those on the photographs. In the next place the busts are not necessarily life size, nor, indeed, the masks if they have been through several moulds. Hence it is needful to reduce (or enlarge) them all to an approximately common scale; in other words, to obtain a multiplying factor for the measurements of each individual portrait. In this matter the Wilkinson Head (for most of our measurements except those to the gnathion) may be taken to be of life size, and the Chequers Court life-mask is so nearly of the same dimensions that these two have been used as guides to what



$\frac{1}{2}$ Scale.

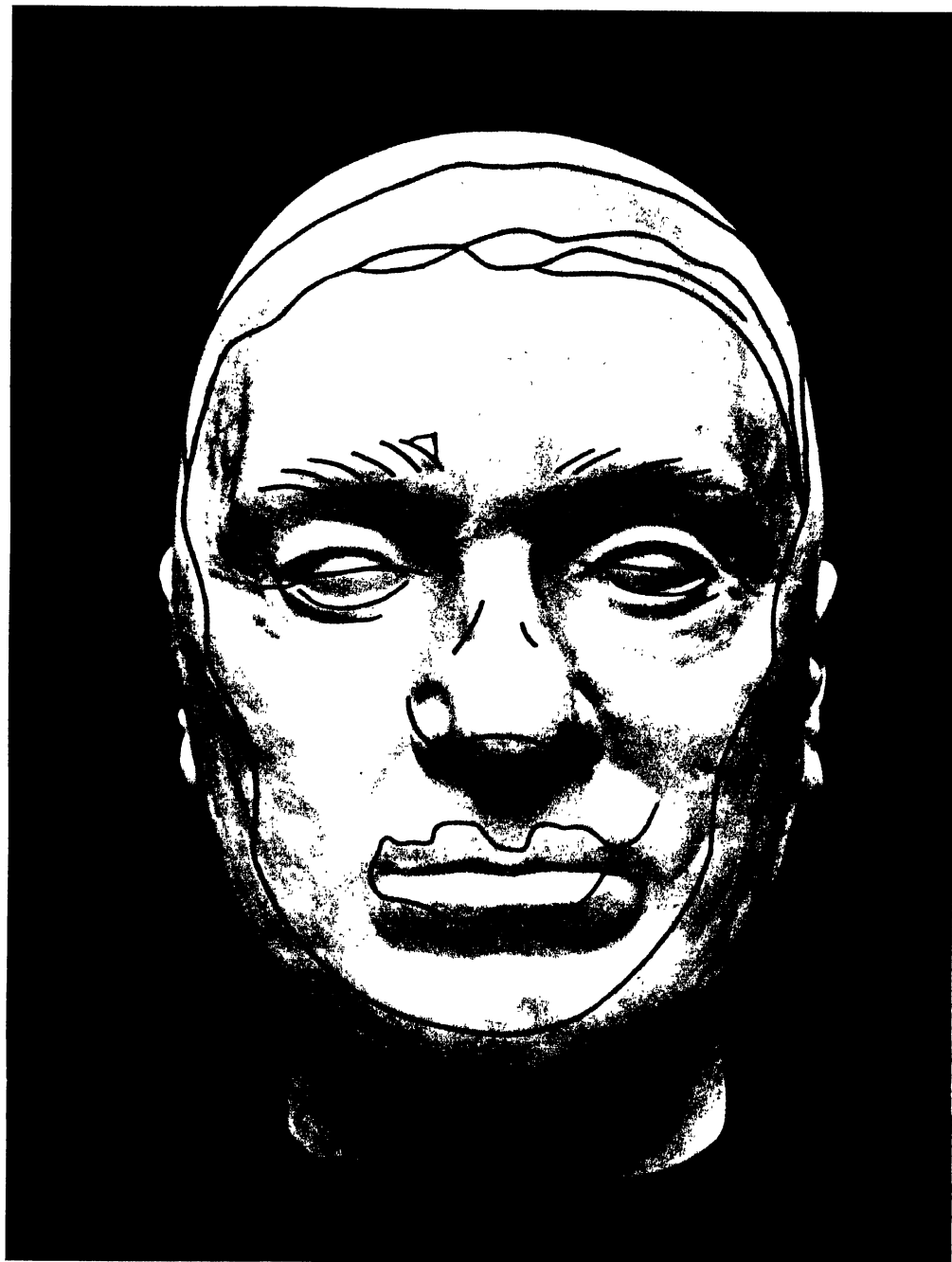
Not quite Full face of Major A. P. Frankland's
Death Mask of Cromwell fitted with the
facial outlines of the Wilkinson Head.



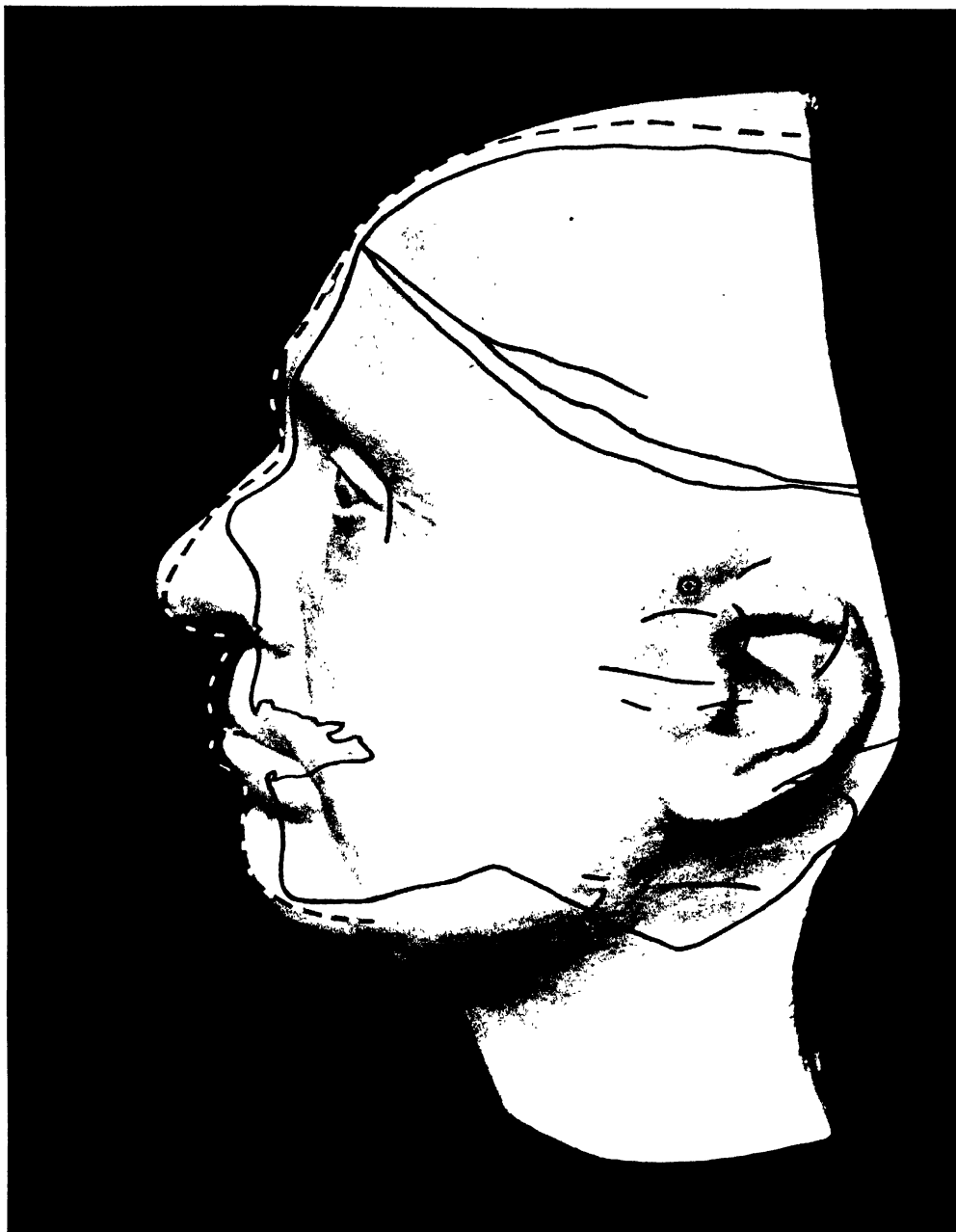
Full face of the Distorted Ashmolean Type of Death Mask
fitted with the facial outlines of the Wilkinson Head.



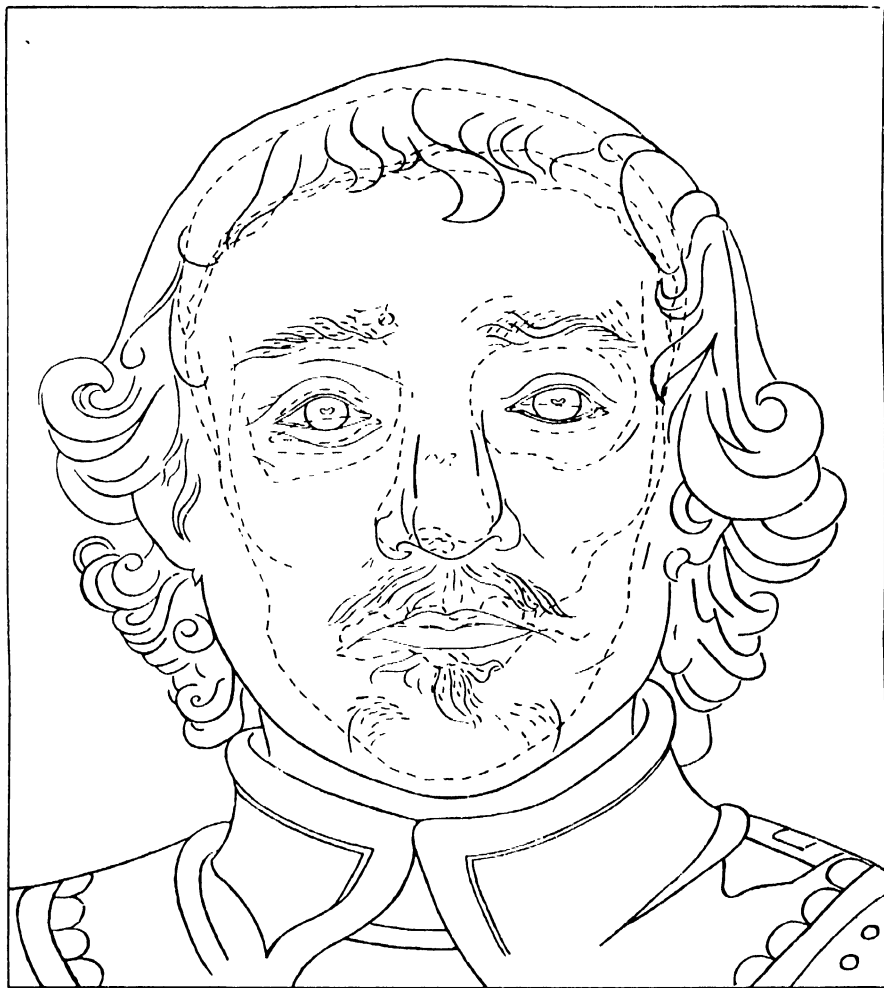
Profile of the Distorted Ashmolean Type of Death Mask fitted with the profile outline of the Wilkinson Head.



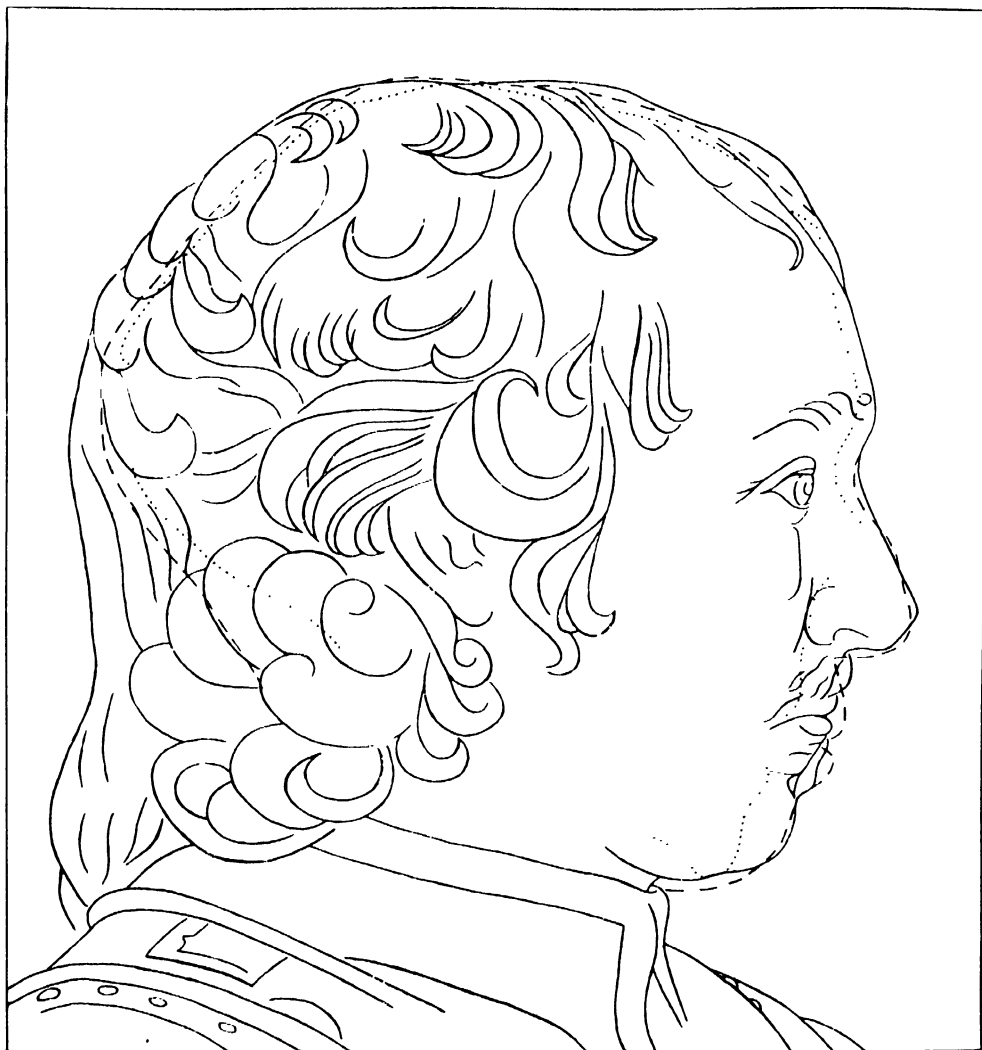
Full face of the Cast of the Florence Bust fitted with the facial outlines of the Wilkinson Head.



Profile of the Cast of the Florence Bust fitted with (a) profile outline - - - and (b) flesh allowance - - - - - of the Wilkinson Head.



Full face of the Ashmolean Bust fitted with the facial outlines
of the Wilkinson Head.

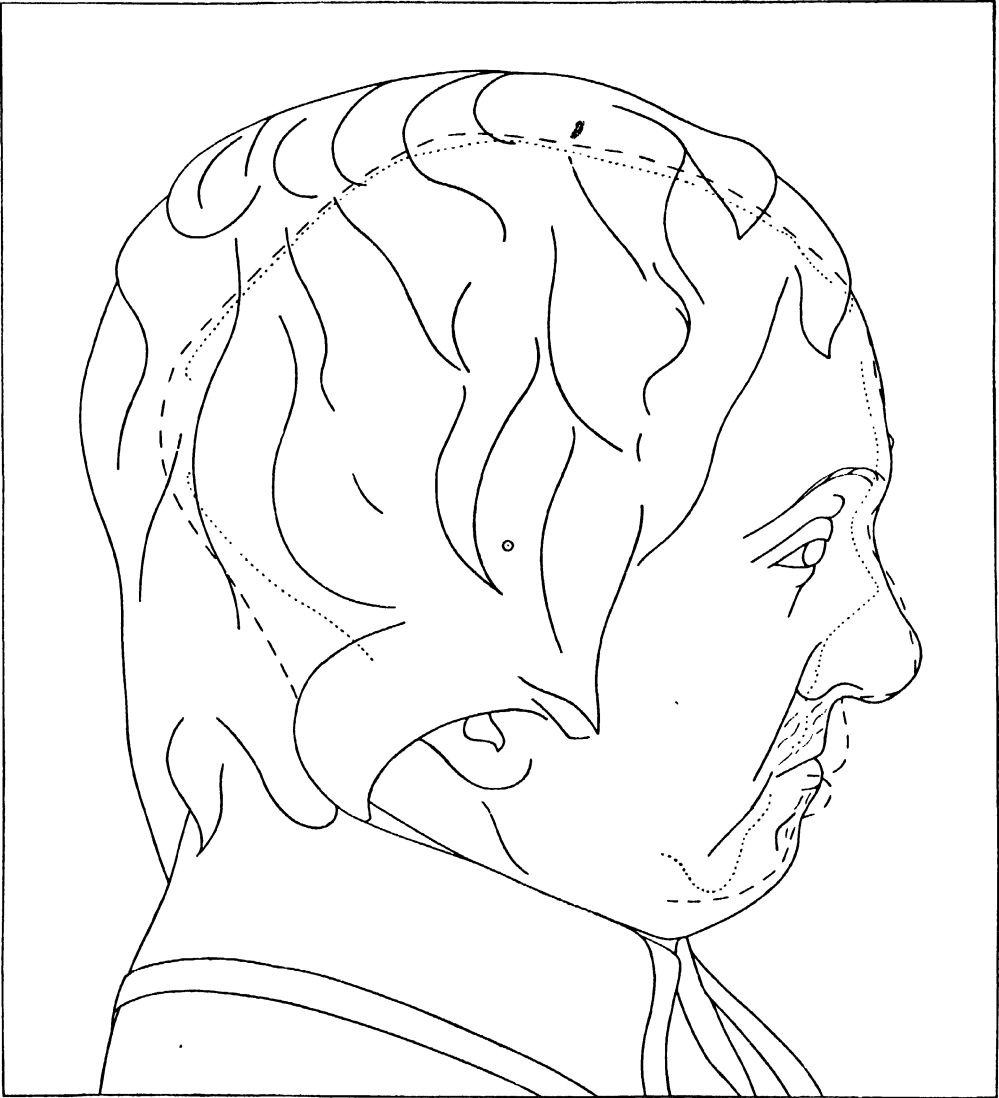


Profile of the Ashmolean Bust fitted with (a) the median sagittal contour and
(b) the flesh allowance of the Wilkinson Head.



Full face of the King's College, Cambridge Bust fitted with
the facial outlines of the Wilkinson Head.

Pearson and Morant: *The Cromwell Head*



Bone boundary

Flesh allowance - - - -

Profile of the King's College, Cambridge Bust fitted with (a) bone profile and
(b) flesh allowance of the Wilkinson Head.

is necessary in bringing up to life size the remaining seven masks and busts. Besides this our photographs were taken with the addition of a scale in the field, but this was found to be only of very occasional service. The method we adopted was to take a vertical length and a horizontal length on any mask, and find what multiplying factors were required to reduce these to the corresponding measurements on the Chequers life-mask. As a rule the multiplying factors thus obtained were close to each other. Their mean was then adopted as the multiplying factor for every measurement to be made on the photograph of that mask. There was only one exception to this procedure, namely the death-mask which is a crude reproduction of the Ashmolean death-mask, or of one akin to it. Here it was found quite impossible to use a single multiplying factor. The multiplying factor for horizontal lengths was found to be 1.3955 and for vertical lengths 1.2686. It was impossible to reconcile these! In the remouldings of this mask by some process or other the customary increase of each such change has been checked in the breadth, while occurring in the height. We have therefore termed this mask (see our Plates LXXIV, LXXV and LXXVI) the Distorted Ashmolean Mask, and placed in the column allotted to its measurements two series of measurements, one due to the horizontal and one to the vertical factor. In the case of the former factor the vertical measurements are enclosed in brackets and in the case of the latter factor the horizontal measurements are in brackets. Thus all values in brackets should be disregarded. We have a striking illustration here of the difficulty of the Cromwell death-masks. It is quite uncertain in the case of any of them that it has been taken from the original mould. Accordingly beside the great room provided for personal equation in taking the photographs and the measurements we cannot deal really with absolute sizes. In fact the Wilkinson Head and the Chequers Court life-mask, upon which we have taken the actual measurements, seem to be the only real relics approaching true life size, so that measurements on photographs are as valid as on the originals, because in both cases it is actually only relative measurements that we are able to consider. With these preliminary statements we place Table I before the reader, giving our attempts at comparing by measurements the Wilkinson Head with the masks and busts. After what has been stated, we feel that the task of taking a really accurate set of measurements on the material is an extremely difficult one; we should prefer to call our values numerical appreciations rather than measurements. As we might anticipate, it is the lengths measured to the gnathion which show the greatest differences between the Wilkinson Head and the masks and busts. Since none of the measures could be guaranteed to one or two millimetres, it appeared worth while taking the means of the mask and bust measurements and placing them alongside the Head measurements. This is done in Table II.

Now if we examine these measurements we see that the most divergent are those to the gnathion. Cromwell, to judge from his portraits, had far greater thickness of flesh below his mandible than the average, say 8 to 9 mm. (instead of the usual 5 mm.) or measured, not from the bone but from the embalmed skin, say about 7 mm. This must be added to gnathion measurements. Now the mandible has been displaced

TABLE I.
Comparison of Facial Measurements on the Wilkinson Head with those on the Death and Life Masks and on the Busts.

Measurement	Wilkinson Head XXIV— XXXI	Chequers Life Mask Plates LXII— LXIV	Ashmolean Death Mask Plates LXX— LXXI	Distorted Ashmolean Mask Plates LXXIV— LXXVI	British Museum Wax Mask Plates LXX— LXXI	Modified Museum Mask Plates LXVII— LXVIII	Florence Bust N.P.G. Cast Plates LXXXII— LXXXIII	Ashmolean Bust Plates LXXXVII— LXXXIX	King's College Bust Plates LXXX— LXXXI
External Ocular Distance (i.e. between Outer Lid-meets)	96.6	104.3	97.1	97.8 (88.9)	103.4	96.7	95.5	97.6	97.1
External Orbital Distance (i.e. between External Margins)	112.3	115.8	112.6	111.2 (101.1)	116.4	110.0	116.3	116.2	111.5
Length of Mouth (i.e. from Lip-join to Lip-join)	59.5?	59.3	59.5	60.8 (55.3)	59.8	59.3	56.3	56.9	59.6
Nasion to Lip-line ...	76.2	74.6	74.9	(82.8) 75.3	76.5	74.6	75.1	75.2	75.2
Nasion to Wart Centre ...	27.2	32.5	31.3	27.9 (25.4)	32.5	28.2	27.2	26.0	26.8
Nasion to Gnathion ...	117.7	124.9	119.2	(135.1) 122.8	122.6	121.2	125.1	122.7	120.0
Nasion to lowest point of "beardlet" root	98.8	94.7	94.4	(106.7) 97.7	95.2	95.6	95.5	97.6	96.5
Nasion to Subnasal Point	53.3	53.8	52.8	(60.3) 54.8	55.1	50.8	53.8	54.0	55.0
Centre of Wart to Right	49.3	52.5	49.1	48.3 (43.9)	52.2	48.6	49.6	47.8	43.1
External Lid-meet
Centre of Wart to Right	51.1	53.2	52.7	53.4 (48.5)	53.0	52.1	51.5	54.3	48.7
External Orbital Margin
Centre of Wart to Left	74.4	76.6	72.3	73.6 (66.9)	77.8	77.4	77.9	71.3	72.3
External Lid-meet
Centre of Wart to Left	81.3	81.1	82.6	81.0 (73.6)	81.7	83.3	81.4	78.1	80.5
External Orbital Margin
Glabella to Subnasal Point	71.7	72.3	71.5	(77.6) 70.5	72.0	70.0	73.2	71.6	74.3
Glabella to Lip-line ...	98.1	97.4	97.5	(105.6) 96.0	97.2	95.5	95.5	97.6	96.5
Glabella to lowest point of "beardlet" root	125.8	122.5	120.4	(130.0) 118.2	125.6	121.6	125.6	121.4	119.9
Glabella to Gnathion ...	135.7	139.9	139.2	(154.9) 140.8	140.2	137.0	143.6	139.6	136.4
Subnasal Point to lowest point of "beardlet" root	48.4	48.0	45.5	(51.7) 47.0	49.3	49.6	48.5	48.8	49.3
Breadth of Nose, without alae	32.0	32.6	31.5	35.8 (32.7)	31.1	30.2	31.7	32.5	32.5
Interpupillary Distance	70.7	77.0	72.0	77.6 (70.5)	70.7	71.5	76.7	69.3	72.0

TABLE II.

Final Comparison of the Wilkinson Head with Masks and Busts.

Characters	External Ocular Distance	External Orbital Distance	Length of Mouth	Nasion to Lip-line	Nasion to Wart Centre	Nasion to Gnathion	Nasion to lowest point of "beardlet" root	Nasion to Subnasal Point	Wart Centre to Right External Lid-meet	Wart Centre to Right External Orbital Margin
Mean Masks and Busts	98.7	113.75	58.9	75.2	28.05	122.3	95.9	53.8	48.9	52.4
Wilkinson Head	96.6	112.3	59.5	76.2	27.2	117.7	98.8	53.3	49.3	51.1

Characters	Wart Centre to Left External Lid-meet	Wart Centre to Left External Orbital Margin	Glabella to Subnasal Point	Glabella to Lip-line	Glabella to lowest point of "beardlet" root	Glabella to Gnathion	Subnasal Point to lowest point of "beardlet" root	Breadth of Nose, without alae	Interpupillary Distance
Mean Masks and Busts	74.9	81.2	71.9	96.65	121.9	139.6	48.25	32.2	73.35
Wilkinson Head	74.4	81.3	71.7	98.1	125.8	135.7	48.4	32.0	70.7

to a small extent by the forcing in of the spike. The third molars on the right side can be separated by about 1.5 mm., and allowing for the perishing of the lining of the dental socket on the upper jaw where the molar is loose, this would signify, perhaps, 1 mm. in life, and some 3 mm. at the alveolar point. We should therefore get an excess of about 4 mm. in the living measurements to the gnathion over those in the embalmed Head: this is in accordance with the observations. Remembering that the "lip-line" is a very vague measure on the Head, we should expect the measurements to the lip-line, taken as half-way between the upper and lower alveolar borders, to be one-half 3 mm. in excess for the Head = 1.5 mm. The glabella to the lip-line is 98.1 for the Head and 1.45 mm. in excess, and the nasion to the lip-line 1 mm. in excess. If the mandible is 3 mm. too low, we should expect any measurement not affected by the shrinkage of the flesh to be also 3 mm. in excess for the Head. The glabella to the lowest point of the beardlet root is 3.9 mm. in excess. The nasion to the beardlet root is 2.9 mm. in excess.

Considering the vagueness of the points between which measurements have to be made and the obvious liability to error, it must be admitted that the accordance between the mean of the masks and busts and the Wilkinson Head is astonishing.

Shall we accordingly assert that the Head must be Cromwell's? We will leave the answer to that question to be reached by the reader who has examined the evidence. We do not know how variable may be the characters we have examined in individuals of the same race. What we do assert, however, is that the measurements of the Head differ less from the means of the corresponding measurements on the masks and busts than these latter differ among themselves. *In other words, there is no measured character of the Head which can be produced as evidence that it is not Cromwell's head.* To those who, without seeing the Head, speak of it as "fraudulent moonshine" we can only retort that their opinion is "idle moonshine"; the problem cannot be treated in that way. To those who have seen the Head and, without thinking of the shrinkage of the flesh, judge by *appearance* that it is quite a different head from that of Cromwell, we reply: Look at the head and skull in Plates LXXV—LXXVII, and note what a difference the flesh makes in the tracings on Plates XCIX and CI. Such judges too often have seen only, or are thinking only, of the Cromwell of Cooper or pseudo-Cooper miniatures (see our Plates XLII, XLV and XLVI), but there is as well the Cromwell of the Walker paintings, and *the same mandible must have been behind the chin of both.* If both are true to the living man, then the "square jaw" of Cromwell was muscle or flesh, not bone, and due to the development of puffiness with oncoming age. If the Walker portraits clothe Cromwell's jaw with the ordinary amount of flesh, then their mandible could certainly be contained inside the square chin of the Cooper miniatures; but if these miniatures show a square mandible covered by an ordinary amount of flesh, then that mandible could not possibly be screened under the chin of Cromwell as painted by Walker! The Walker paintings seem to us to indicate that Cromwell had not a massive square jaw, as far as his facial skeleton was concerned. We do not overlook the personal tendencies of Cooper and Walker. The former may have had some propensity to round his faces and square their chins, the latter to lengthen his faces and narrow their chins. The portraits of Ireton* and Lambert† by Walker prove, however, that he did not narrow *all* his chins. Hence we are bound to conclude that Cooper in the case of Cromwell either painted what was before him‡, or that he was markedly astigmatic. In the former case the apparently massive square jaw of Cromwell is the result of puffiness. In the latter it is an idiosyncrasy of Cooper. Whichever alternative we adopt, it cannot be raised as a fundamental objection to the Wilkinson Head.

Neglecting the inconsistencies of the painted portraits and basing our judgment solely on the evidence of masks and busts, we find nothing improbable in the Wilkinson Head being that of Cromwell; on the contrary, a very considerable probability. Taken in conjunction with the attributes of the Head itself, we appear to have what Bernoulli and Buffon would have termed a "moral certainty" that the Wilkinson Head is the actual head of Oliver Cromwell.

* In the possession of Mrs Polhill-Drabble at Sundridge.

† In the National Portrait Gallery. A further good example is Walker's portrait of Richard Cromwell at Chequers Court.

‡ This alternative seems to have the support of the British Museum wax mask. Cf. our Plates LXV and XLVI.

SECTION V.

19. *Conclusions.*

The following paragraphs will summarise the results which seem to us to flow from our investigations.

(i) There is no reason whatever for doubting that the bodies of Cromwell, Bradshaw and Ireton were buried in Westminster Abbey; that their coffins were taken up and drawn on sledges to Tyburn; that they were there pulled from their coffins, hung on the triple gallows, and, after their heads had been crudely chopped off, the trunks were buried at the foot of the gallows. Whether at any later time they were exhumed could only be determined if the sepulchre at Newburgh Priory were opened and the headless embalmed body of Cromwell found to be there*. The record in the Speaker's Library at the House of Commons (see our Plate X) indicates that the Sheriff of Middlesex far exceeded the order of the Convention Parliament.

(ii) The heads of Cromwell, Bradshaw and Ireton were placed on the south or "farther" end of Westminster Hall. Cromwell's was, we think, towards the east, and Ireton's to the west, Bradshaw's on the pinnacle. This assumes that the heads faced, as they faced Charles I at his trial. The possible western position of Cromwell's head is linked with the position of the tavern called "Heaven."

(iii) Cromwell's head was certainly up in 1681, although Flatman possibly mistook Bradshaw's for Cromwell's. It remained there till 1684, and most probably to the end of James II's reign, when it disappeared, possibly thus creating the superstition which Grossley found prevalent among the English commonalty.

(iv) The next appearance of a head (not skull) asserted to be Cromwell's is in 1710, in the museum of Du Puy, who considered it the most curious of his many curiosities. Von Uffenbach's description of it tallies with what we know of the Wilkinson Head. Du Puy died in 1738. We do not know whether the Head passed from his keeping at that date or earlier.

* A very definite statement is made by Waylen (*The House of Cromwell*, Edn. 1897, p. 224) that two at least of the bodies were recovered by friends and carried off "as proved by Mr Godfrey Meynell's discovery of the coffins of Ireton and Bradshaw in the vault beneath Mugginton Church in Derbyshire." Mugginton is close to Little Ireton whence the Ireton family sprung and near to Marple the seat of the Bradshaws. The Rev. Godfrey Meynell died in 1854. Waylen gives no reference to the source of his information. We have taken some trouble to discover the origin of this statement, but in vain. Cox in his *Churches of Derbyshire*, Vol. III. p. 223, speaks of a legend of this burial, but without reference to Meynell. The great-grandson of the antiquary, General G. Meynell, who is in possession of the papers of his great-grandfather, most kindly looked through them, but could find nothing about the asserted discovery, although one vault was visited in 1817 by the Rev. G. Meynell, and he found therein the body of Sir Thomas Sanders, M.P., of Little Ireton, who was a Colonel of Cromwell's Ironsides. His granddaughter Elizabeth Sanders was the *third* wife of John Mortimer, M.P., F.R.S., whose son, Dr Cromwell Mortimer, was the well-known secretary to the Royal Society (see our p. 298 and Plates IX and XI). His only claim to the name Cromwell was that his father's *first* wife was Dorothy Cromwell, the daughter of the Protector Richard.

We have thus been unable to reach any basis for Waylen's statement. Its confirmation or refutation would be of value, but in the light of our inquiries the former does not appear probable.

(v) What is in all probability the Du Puy head of Cromwell reappears in the possession of an impecunious and bibulous actor Samuel Russell at, or shortly before, the date 1775. His accounts of how he obtained it are inconsistent, and we have not been able to find any evidence for his descent from the Fordham or Chippenham Russells. How he got possession of the Head remains unsettled.

(vi) The tale that it was picked up by a sentinel on guard at Westminster Hall has no element of improbability, but also has nothing to substantiate it except a severe fracture of the occipital, which might have arisen from a fall on to the roof or leads of the buildings at the south end of the Hall.

(vii) From 1775 (or 1773) the history of the Head is clear to the present date.

(viii) The Head does not come in touch with anyone in the least likely to have had the ability to produce a plausible forgery before the time, 1799, when Cox sold it and Cranch embellished its story. It would have required an incredible amount of skill and historical knowledge, which it is certain that even Cranch did not possess, to reproduce so closely a head which in its measurements and external attributes is so like that of Cromwell. Such a forger would also be extraordinarily likely to make a fatal mistake such as embalming the head after decapitation, or removing the skull-cap after embalmment, or severing the head from the trunk in a less crude manner than the executioner did. There would be no sixfold cerecloth to check his blows or screen the neck. We have not the slightest hesitation in asserting that the Head is that of a person who had been embalmed and decapitated after embalmment, and that the Head has been exposed long enough on an iron spike fastened to an oak pole for the pole to rot and the worms to penetrate pole and head. It is what its externals proclaim it to be, a genuine Head carrying much of its past history with it.

(ix) It seems impossible to think of any likely "head of some decapitated man of distinction" fitting the conditions other than those of Henry Ireton or of Oliver Cromwell. The former seems excluded by the frontal breadth and probable age at death of the former owner of the Wilkinson Head.

(x) We can find no external characters, nor any measurements which we have found it possible to take, which contradict the hypothesis that the Head is that of Cromwell.

(xi) The defective history of the Head hinders the *demonstration* that it is Cromwell's, but many a man has been hanged on a smaller amount of circumstantial evidence for his crime than exists for the identity in this case. The probability for the identity is so convincing that any critic need not be considered who cannot produce a higher probability that this Head must be that of another embalmed and decapitated person of the seventeenth century. Who was he, and do his busts or portraits fit to a higher degree this Head?

(xii) We started this inquiry in an agnostic frame of mind, tinged only by scepticism as to whether the positive statements made in the past with regard to it were not based solely on impressions unjustified by any attempt at a scientific

investigation. We finish our inquiry with the conclusion that it is a "moral certainty" drawn from the circumstantial evidence that the Wilkinson Head is the genuine head of Oliver Cromwell, Protector of the Commonwealth.

20. *Acknowledgments of Aid.*

It is impossible to conclude this memoir without expressing our most cordial thanks to the many who have aided us either by answers to our inquiries or by most kindly placing at our disposal their Cromwellian treasures; without their aid our task would have been impossible. In the first place we are indebted to the Director and Staff of the National Portrait Gallery for their ready assistance; to the authorities of the British Museum, especially of the Departments of Prints and Coins, for help; to the Clerk of the Privy Council, to the Staff of the Record Office, and to the Keeper of the Western Manuscripts at the Bodleian for aid in our search for the Proceedings of the Lords of Council during the Protectorate of Richard Cromwell; and to the Marquis of Bath for a vain search for the lost volume after we had located it finally as being formerly in his possession. To the late Keeper of the Ashmolean Museum we are indebted for photographs of Edward Pierce's bust and of the death-mask of Cromwell, as well as for answers to several questions. To the Speaker of the House of Commons for facilities for photographing the original resolution in his Library for the disinterment of the bodies of Cromwell, Ireton and Bradshaw. To the Prime Minister for obtaining permission from the Trustees of Chequers Court for us to photograph some of the Cromwelliana in their keeping, and for his ready personal aid in the actual process of photography. To the Keeper of the London Museum for permission to reproduce some of the pictures and prints in the Tangye Collection. To the Duke of Devonshire, the Marquess of Crewe and Viscount Harcourt for permission to reproduce their miniatures of Cromwell. To the authorities of King's College, Cambridge, for permission to photograph the bust of Cromwell in their Library. To the custodians of Warwick Castle for information as to the Cromwelliana there. To Messrs J. B. Oldham and Pilcher, for photographs and measurements on the death-mask of Cromwell in the Library of Shrewsbury School. To Dr J. Bell for numerous searches in the British Museum Library. To Major A. P. Frankland, D.S.O., for permission to reproduce his death-mask of Cromwell. To Mr and Mrs R. B. Polhill-Drabble for much information concerning their portraits of Cromwell and Ireton. To Sir John and Lady Payne-Gallwey for particulars as to their picture and miniatures of Cromwell. We have also to thank for very special help the Rev. Paul Cromwell Bush, who took very great pains to give us information with regard to his Cromwelliana. We owe our thanks to Mr A. H. Gerrard for information with regard to the state and preparation of death-masks. We further acknowledge the aid of Dr A. Davin, of the Linen Institute, Belfast, in preparing microphotographs of cotton and flax fibres; of Dr Dudley Buxton, of the Anatomical Department Oxford, for work connected with skull-caps and Pacchionian depressions; and of Sir Arthur Keith for a letter regarding the skull-cap of the Wilkinson Head. To Major Charles ffoulkes, Curator of the Armouries in the Tower, for information as to pikes.

To Messrs P. and D. Colnaghi and Co. for permission to reproduce their engraving of the Duke of Buccleuch's miniature of Cromwell. To the Town Clerk of the City of Westminster for permission to examine the old ratebooks of the City. To the Rev. G. A. Sykes, Rector of Mugginton, General Godfrey Meynell of Meynell Langley Mr Williamson, Curator of the Derby Museum and Mrs Hermione Money for answering various questions as to the statement that the Rev. Godfrey Meynell had discovered the coffins of Bradshaw and Ireton in a vault of Mugginton Church*. We have to thank Dr Laurence and Miss L. M. Tildesley, of the Royal College of Surgeons, Dr Thomas E. Anderson and Dr G. M. Duncan, of the Aberdeen Royal Infirmary, Professor Sir Flinders Petrie and Dr M. Murray for aid in a variety of ways. Lastly we owe a great debt to the able assistance of Miss Ida McLearn (Mrs Fraser Larmor) and Miss Mary Kirby for their drawing work in connection with the fitting of the Head.

Appendix. See Footnote, p. 348. The following reference to a Florence portrait of Cromwell occurs in *An Essay towards an English School with the Lives and Characters of 100 Painters*, which is appended to the English translation of Roger de Piles' *The Art of Painting*, 1706:

MR. ROBERT WALKER was an *English Face-Painter* contemporary with Van-Dyck, and whose Works by the *Life*, best speak their own praises. He lived in *Oliver Cromwell's* Days and drew the Portraits of that Usurper, and all his Officers, both by Sea and Land. The Great Duke of *Tuscany* bought an original of *Oliver* by this master the manner thus. Having sent over some agent here to purchase such a Picture for him, the Person could light on none to his mind for a long while, till at length hearing of a Woman, a relation of the Usurper's that had one, he went to see it, and found it, in all respects, so well performed, that he bid her a good Price for it. She not wanting money, told him, since she had the Honour to be related to the Protector, she would by no means part with his Picture; but the gentleman still insisting upon having it, and desiring her to set what Price she pleased upon it, she thinking to get rid of his Importunity by exorbitant Demand, asked him 500*L.* for it, when, contrary to her Expectation, he told her she should have it, and accordingly paid down the money immediately, which she being bound by her Word to take, parted with her Picture even with regret, tho' at so great a Rate. This is to be understood to have happen'd in the Protector's Life-Time.

Thus we have evidence of a portrait of Cromwell being in the possession of the Grand Duke of Tuscany in 1706. Unfortunately the writer attributes it to Walker, not Lely, to whom the "ancient" catalogue of the Pitti collection and the style agree in attributing the Florence portrait.

* We still believe that historical research demands that the vault under the chancel of this church, and probably entered from outside the church below the visible base of the south wall, should be investigated, and with the same end in view the truth of the legend that Cromwell's trunk was taken out of the Tyburn pit and conveyed to Newburgh Priory should be settled by examining the vault in that house. In the latter case it might turn out that the vault contained not the trunk, but the *ejecta* including the heart of Cromwell (see our p. 295). A letter to Newburgh Priory, asking whether an investigation could not now be made, met with the same courteous refusal that the late Sir George Wombwell gave to King Edward VII. The tradition kindly provided of how Cromwell's trunk came to be in the vault was, however, clearly erroneous.

ERRATA

Page 379, après la ligne 3, *ajouter* Univ. Litoral (Argentina)

„ 381, ligne 14, *lire* $\sum_{i=1}^n p_i F_k(x_i) \dots$

„ 382, note au bas de la page, *lire* E. Pascal

„ 385, ligne 8 du bas de la page, *lire* Hermite

„ 389, ligne 8, *lire* $\alpha_n = \frac{1}{H_n} \sum_{s=0}^n (-1)^{n+s} \frac{\binom{n}{s} \gamma^{n-s}}{\prod_{j=n+s+1}^{2n} (p-j)} \mu_s$ et.....

„ 394, ligne 4, *lire* $\dots = \int_{-\infty}^{+\infty} \rho(y) y [1 + \dots$

„ 394, ligne 13, *pour* p. 388 *lire* p. 387

„ 396, ligne 4, *lire* $\mu_{1,j}$

„ 396, ligne 8, *lire* $\mu_{i,1} \neq 0$ ou $\mu_{1,j} \neq 0$

„ 396, ligne 11, *lire* $\mu_{i,1} = \mu_{1,j} = 0$

„ 396, ligne 27, *lire* $\dots [y - y_x]^2 dx dy$

„ 398, la première équation, *lire* $\frac{\beta_{s+1}}{\beta_s} \beta_3'$

„ 398, ligne 4 du bas de la page, *lire* $\mu_3(y)_x = \sum_{s=0} N_s \dots$

„ 400, près du haut de la page, *pour* N_s *lire* N_s

„ 400, la dernière ligne, *lire* $\mu_3(y)_x = -y_x^3 - \dots$

„ 402, au milieu de la page, *lire* $d = \frac{\mu_{0,2}^3 \mu_{0,3} \mu_{0,5} (\mu_{0,2} - 1) (\mu_{0,3}^3 + \mu_{0,4} \mu_{0,5})}{\beta_3' \beta_4'}$

CONTRIBUTION A L'ÉTUDE DE LA THÉORIE DE LA CORRÉLATION.

PAR CARLOS E. DIEULEFAIT.

I.

Introduction. La corrélation est une relation intermédiaire entre la dépendance stochastique et la correspondance fonctionnelle.

Quand la corrélation existe, au sens de Galton-Pearson, elle entraîne la dépendance stochastique et elle s'approche, plus ou moins, à une relation fonctionnelle.

La dépendance stochastique est la base pour la description de la corrélation et la relation fonctionnelle, l'étalon de sa mesure.

Toutes ces données (description et mesure) sont déterminées quand on connaît la fonction de la surface de corrélation.

Mais les surfaces de corrélation connues sont en nombre assez limitées, ayant échoué les procédés parallèles à ceux avec lesquels K. Pearson a trouvé son répertoire des fonctions des fréquences d'une seule variable.

Par cette raison, on remplace l'étude des surfaces par des calculs directs sur les tableaux ou le domaine des éléments. Les équations de régression, de sélasticité, de symétrie et de normalité liées (elitics et kurtics) sont, en somme, un ensemble de relations signalétiques de la dépendance stochastique entraînées par la corrélation, mesurée celle-ci au moyen des deux coefficients de Pearson: η_{yx}^2 et η_{xy}^2 .

On sait que η^2 est compris entre 0 et 1.

Pour une forme déterminée de dépendance stochastique, la variation de η^2 nous donne une famille de surfaces. A mesure que η^2 s'approche de l'unité, les surfaces individuelles correspondantes tendent à une surface cylindrique qui est l'image d'une correspondance fonctionnelle de forme biunivoque.

Mais la connaissance de la valeur de η^2 ne suffit pas pour déterminer la forme des surfaces correspondantes qui ne forment plus une famille au sens mathématique, sinon un ensemble de familles.

Si l'étude que l'on fait a seulement pour objet la prédiction des valeurs des autres éléments quand l'un d'eux a été fixé, la valeur de η^2 suffit presque toujours, et dans tous les cas semblables on suppose que l'on se rapporte à la surface bien connue de Gauss-Bravais.

Mais dans les questions biométriques dans lesquelles aux erreurs accidentelles s'ajoutent des variations individuelles, la description du phénomène est aussi importante que la mesure de l'un de ces facteurs. Nous devons souligner que cette description n'est pas faite dans le sens qualitatif ni non plus dans le sens métrique, sinon dans le sens fonctionnel.

Décrire une corrélation signifie donner (en défaut de la fonction de sa surface) un ensemble de relations fonctionnelles : équations sédastiques, kurtics, etc.

Habituellement pour la détermination des fonctions descriptives de la corrélation, on doit opérer avec des calculs très pénibles sans l'aide de formules simplificatrices. Mais on peut se demander si l'on ne pourrait pas faire un classement des surfaces au moyen de quelques éléments caractéristiques des mêmes, étant donné que l'étude directe des surfaces présente de grandes difficultés.

C'est justement cette question qui fait le point de départ de ce travail au cours duquel nous croyons pouvoir donner les bases pour obtenir un répertoire de surfaces de corrélation.

L'élément systématique que nous avons utilisé pour cette étude a été le développement en série des polynômes orthogonaux. On a choisi comme les éléments plus caractéristiques des surfaces les fonctions des fréquences marginales, obtenant ainsi l'expression formelle de la surface au moyen d'une série double avec des coefficients que l'on calcule très facilement par la méthode de Fourier. Ces développements, d'ailleurs, sont connus dans deux cas particuliers, c'est à dire, pour les surfaces du Type A-A et B-B, mais la façon dont nous les introduisons est tout à fait générale et comprend ceux-ci.

Dans la seconde partie de ce travail nous traiterons ces développements en général, c'est à dire, sans spécifier la forme des fonctions de fréquences marginales. C'est ce procédé que K. Pearson a suivi, le premier, quand il donna les formules des régressions paraboliques jusqu'au quatrième degré, en utilisant les méthodes de générations orthogonales de J. P. Gram. Nous montrerons aussi que les développements donnés pour les lignes de régression d'un degré quelconque par J. Neyman sont de la même nature et qu'il a fait appel aux polynômes orthogonaux, quoiqu'il ait déclaré qu'il se passerait d'eux.

Comme résultat de notre méthode il sera très facile (tenant compte des généralisations des courbes de Pearson, faites par Romanovsky et par nous avec des procédés différents) de donner un répertoire des surfaces de corrélation. Il ne faudra pour cela que substituer aux fonctions marginales indéterminées les fonctions de Pearson.

Mais il est nécessaire, dès maintenant, faire une observation importante. Nous croyons, avec le professeur K. Pearson, que la généralisation des fonctions de fréquence est très discutable, au double point de vue de la légitimité mathématique et de l'avantage statistique de nous conduire à une plus grande exactitude. Ces généralisations dans la méthode que j'exposerai ne prétendent pas contredire ces principes qui remontent déjà aux objections que Cauchy opposa à Bienaymé au sujet de la méthode des moindres carrés.

Les généralisations des courbes de Pearson sont un auxiliaire utile pour l'étude des surfaces de corrélation et de ce point de vue ces courbes montrent leur supériorité par rapport au développement en série de la statistique à une dimension du type de Gram-Charlier et Poisson-Charlier.

En effet, on pourrait faire la classification des surfaces au moyen de fréquences marginales, ces fréquences étant représentées par des séries du type A ou B de Charlier, mais alors le développement pour la surface paraîtrait assez compliqué et moins riche en conséquences.

Les développements des fonctions de fréquences en séries de polynômes orthogonaux.

Cette partie de notre travail a pour objet de faire un bref exposé que nous utiliserons dans les calculs de corrélation.

Occupons-nous d'abord de la méthode de Tschébysschew. Soit les variables $x_1, x_2, x_3, \dots, x_n$ auxquelles correspondent les fréquences $y_1, y_2, y_3, \dots, y_n$ avec les poids $p_1, p_2, p_3, \dots, p_n$ respectivement.

Si l'on écrit:

$$F(x) = a_0 F_0(x) + a_1 F_1(x) + \dots + a_k F_k(x) \quad k < n-1,$$

les $F_i(x)$ étant des polynômes orthogonaux en x de degré i , c'est à dire tel que

$$\sum_{i=1}^n F_h(x_i) F_j(x_i) \begin{cases} = 0 & \text{si } h \neq j \\ \neq 0 & \text{si } h = j \end{cases},$$

alors la condition pour que:

$$S_n^2 = \sum_{i=1}^n [y_i - F(x_i)]^2 p_i \dots\dots\dots (1)$$

soit un minimum, est accomplie si l'on prend pour a_j la valeur qui fait:

$$\frac{\partial S_n^2}{\partial a_j} = 0,$$

ce qui conduit à:

$$a_j = \frac{\sum_{i=1}^n y_i p_i F_j(x_i)}{\sum_{i=1}^n p_i F_j^2(x_i)} \dots\dots\dots (1').$$

Pour obtenir alors les polynômes $F_j(x)$, Tschébysschew fait appel aux propriétés des fractions continues*, trouvant pour eux la suivante formule récurrente:

$$F_{\lambda+1}(x) = F_{\lambda}(x) \left[x - \frac{(\lambda; \lambda+1)}{(\lambda; \lambda)} + \frac{(\lambda-1; \lambda)}{(\lambda-1; \lambda-1)} \right] - \frac{(\lambda; \lambda)}{(\lambda-1; \lambda-1)} F_{\lambda-1}(x),$$

bien entendu que l'on ait:

$$(\lambda; \lambda') = \sum_{i=1}^n p_i F_{\lambda}(x_i) F_{\lambda'}(x_i).$$

Si la variable x était continue avec les valeurs correspondantes de y et les poids $p(x)$, alors on pourrait conserver les idées de Tschébysschew simplement en remplaçant l'antérieure condition des moindres carrés† par cette autre que:

$$S_n^2 = \int [y - F(x)]^2 p(x) dx$$

soit un minimum; l'intégrale étant prise dans l'intervalle de variation de x .

* Consulter à ce sujet la note de Isserlis dans *Biometrika*, Vol. xix. p. 87: "On Chebyshev's Interpolation Formula."

† V. Camille Jordan, *Cours d'Analyse* (1918), Tome II. p. 301.

Pour trouver les polynômes $F_i(x)$ il suffirait d'étendre au champ continu le raisonnement fait par Tschébyshew dans le cas antérieur.

D'ailleurs ces résultats découlent du fait bien connu de l'analyse que si l'on développe en fractions continues la fonction :

$$\int_a^b \frac{p(x)}{z-x} dx,$$

les dénominateurs des réduites successives pourront être calculés directement en donnant des polynômes qui seront justement ceux que l'on cherche, parce qu'ils vérifieront la condition d'orthogonalité pondérée :

$$\int_a^b p(x) F_n(x) F_h(x) dx \begin{cases} = 0 & \text{si } n \neq h \\ \neq 0 & \text{si } n = h \end{cases} \dots\dots\dots (2),$$

les $F_n(x)$ étant précisément les dénominateurs des réduites.

Comme expression de ces polynômes $F_n(x)$ on aura :

$$F_n(x) = \begin{vmatrix} 1, & x, & x^2, & \dots, & x^n \\ m_0, & m_1, & m_2, & \dots, & m_n \\ \vdots & & & & \\ m_{n-1}, & m_n, & m_{n+1}, & \dots, & m_{2n-1} \end{vmatrix} \dots\dots\dots (3),$$

où :

$$m_s = \int_a^b p(x) x^s dx \dots\dots\dots (3'),$$

qui est le moment d'ordre s .

On peut vérifier que les déterminants (3) étudiés par Jacobi, qui a démontré leur formule de récurrence*, accomplissent la condition (2), car si l'on multiplie les deux membres de la formule (3) par $p(x)x^s$; ($s = 0, 1, 2, \dots, n-1$) et l'on intègre entre a et b on trouve :

$$\int_a^b p(x) F_n(x) x^s dx = \begin{vmatrix} m_s, & m_{s+1}, & m_{s+2}, & \dots, & m_{s+n} \\ m_0, & m_1, & m_2, & \dots, & m_n \\ & & & & \\ & & & & \\ m_{n-1}, & m_n, & m_{n+1}, & \dots, & m_{2n-1} \end{vmatrix} \dots\dots\dots (4),$$

et ce déterminant ayant deux lignes égales sera nul.

Alors si nous supposons

$$F_h(x) = \alpha_{h,0} + \alpha_{h,1}x + \alpha_{h,2}x^2 + \dots + \alpha_{h,h}x^h,$$

et l'on donne à s dans la formule (4) les valeurs $0, 1, 2, \dots, h$, en multipliant chacune de ces égalités par $\alpha_{h,0}, \alpha_{h,1}, \alpha_{h,2}, \dots, \alpha_{h,h}$ respectivement et ajoutant on arrive à la formule (2).

On pourrait remarquer aussi, en passant, que si les polynômes

$$X_0(x), X_1(x), \dots, X_n(x)$$

sont orthogonaux avec un pôle $p(x)$, ces polynômes pourront toujours être mis sous la forme des déterminants (3) multipliés par une constante.

* Voir, par exemple, F. Pascal, *I Determinanti*, Milano, 1922.

Réciproquement, si les polynômes $X_0(x)$, $X_1(x)$, ..., $X_n(x)$ peuvent se mettre sous la forme des déterminants (3), ces polynômes déterminent une fonction $p(x)$ avec laquelle ils seront orthogonaux.

Pour démontrer la première de ces propositions, supposons que :

$$I_{h,j} = \int_a^b p(x) X_h X_j dx \begin{cases} = 0 & \text{si } h \neq j \\ \neq 0 & \text{si } h = j \end{cases},$$

où $X_j(x) = \omega_{j,0} + \omega_{j,1}x + \omega_{j,2}x^2 + \dots + \omega_{j,j}x^j \dots\dots\dots(5).$

On aura alors :

$$I_{0,j} = \omega_{0,0} \int_a^b p(x) X_j dx = 0,$$

$$I_{1,j} = \omega_{1,0} \int_a^b p(x) X_j dx + \omega_{1,1} \int_a^b p(x) X_j x dx = 0,$$

$$I_{2,j} = \omega_{2,0} \int_a^b p(x) X_j dx + \omega_{2,1} \int_a^b p(x) X_j x dx + \omega_{2,2} \int_a^b p(x) X_j x^2 dx = 0, \text{ etc.,}$$

d'où l'on déduit que :

$$\int_a^b p(x) X_j(x) x^s dx = 0 \quad (s = 0, 1, 2, \dots, j-1) \dots(6).$$

Alors si nous multiplions les deux membres de la formule (5) par $p(x) x^s dx$ et nous intégrons, nous aurons, en tenant compte des formules (6) et (3') :

$$\begin{aligned} \omega_{j,0} m_0 + \omega_{j,1} m_1 + \dots + \omega_{j,j} m_j &= 0 && \text{pour } s = 0, \\ \omega_{j,0} m_1 + \omega_{j,1} m_2 + \dots + \omega_{j,j} m_{j+1} &= 0 && \text{pour } s = 1, \\ \dots\dots\dots \\ \omega_{j,0} m_{j-1} + \omega_{j,1} m_j + \dots + \omega_{j,j} m_{2j-1} &= 0 && \text{pour } s = j-1. \end{aligned}$$

Et si l'on ajoute à ce système l'équation formée avec le polynôme $X_j(x)$ de la formule (5) multipliée par une constante k_j , en éliminant les $\omega_{s,j}$, cette équation pourra s'écrire :

$$X_j(x) = k_j \begin{vmatrix} 1, & x, & x^2, & \dots, & x^j \\ m_0, & m_1, & m_2, & \dots, & m_j \\ \vdots & & & & \\ m_{j-1}, & m_j, & m_{j+1}, & \dots, & m_{2j-1} \end{vmatrix} = 0.$$

Si dans le polynôme (5) $\omega_{j,j} = 1$ alors k_j sera le mineur complémentaire de X_j .

Dans le second théorème la succession $X_0, X_1, X_2, \dots, X_n(x), \dots$, admettait la forme des déterminants (3). On connaît alors ces moments m_s par rapport à un pôle $p(x)$ jusqu'ici inconnu, mais les données des moments déterminent le pôle; donc, le théorème est vrai*.

Alors, au fond, toute la question de l'orthogonalité des polynômes $X_0, X_1, X_2, \dots, X_n, \dots$, se réduit à la possibilité ou non pour que ces X_s puissent s'écrire sous forme des déterminants du type (3). On sait que ces questions pourraient

* Voir G. Castelnuovo, *Calcolo delle Probabilità*, 1919. [This statement is not quite correct, real moments are subject to a series of inequalities: see *Biometrika*, Vol. xxi. (1929), pp. 361—375. Ed.]

aussi se rattacher à la théorie des équations différentielles et intégrales, mais nous devons nous contenter des remarques déjà faites, ne désirant pas sortir du nécessaire pour notre exposé purement statistique.

La méthode de Tschébysschew étant donnée, on a voulu présenter les résultats de son application d'un point de vue élémentaire, c'est à dire, pouvoir construire effectivement des polynômes orthogonaux sans faire appel à la théorie des fractions continues. Telle est par exemple la position du Prof. K. Pearson* quand il utilise la génération de J. P. Gram et aussi—d'autre côté—celle du Prof. Romanovsky†.

Nous allons nous occuper des calculs de Gram. Nous suivons pour cela le célèbre mémoire de K. Pearson que nous exposerons uniquement dans l'essentiel.

Soit la fonction $\psi(x)$ et (a, b) son intervalle. On veut déterminer la succession des polynômes $B_0(x)$, $B_1(x)$, $B_2(x)$, ..., tels qu'ils soient

$$\int_a^b \psi(x) B_s(x) B_{s'}(x) dx \begin{cases} = 0 & \text{pour } s \neq s' \\ \neq 0 & \text{,, } s = s' \end{cases} \dots\dots\dots(7).$$

On pourra prendre $B_0(x) = 1$ pour satisfaire ainsi la condition de probabilités:

$$\int_a^b \psi(x) dx = 1$$

et nous écrirons:

$$B_s(x) = \alpha_{s,0} B_0 + \alpha_{s,1} B_1 + \dots + \alpha_{s,s-1} B_{s-1} + x^s \dots\dots\dots(8)$$

si l'on veut définir des polynômes, ou bien en changeant x^s par $\mu_s(x)$ comme a fait Romanovsky, $\mu_s(x)$ étant des fonctions univoques, par exemple $\mu_s(x) = \sin x$, etc.

En multipliant la formule (8) par $\psi(x) B_{s'}(x) dx$ et en intégrant on aura, en tenant compte de la formule (7) et posant:

$$\int_a^b \psi(x) B_s(x) B_j(x) dx = (B_s B_j),$$

$$\text{pour } s' = 0 \qquad 0 = \alpha_{s,0} (B_0 B_0) + (x^s B_0),$$

$$\text{pour } s' = 1 \qquad 0 = \alpha_{s,1} (B_1 B_1) + (x^s B_1),$$

$$\text{pour } s' = s-1 \qquad 0 = \alpha_{s,s-1} (B_{s-1} B_{s-1}) + (x^s B_{s-1}).$$

On obtient alors:

$$\alpha_{s,i} = - \frac{(x^s B_i)}{(B_i B_i)},$$

et en substituant dans la formule (8):

$$B_s(x) = - \sum_{i=0}^{s-1} \frac{(x^s B_i)}{(B_i B_i)} B_i + x^s \dots\dots\dots(9).$$

Observons en passant que K. Pearson en adoptant cette méthode pour les calculs des lignes de régression parabolique retient la fonction $\psi(x)$ comme la

* Voir Karl Pearson, "On a General Method of determining the successive terms in a Skew Regression Line," *Biometrika*, Vol. XIII. July, 1921, p. 296.

† Voir V. Romanovsky, "Note on Orthogonalising Series of Functions and Interpolation," *Biometrika*, Vol. XIX. July, 1925, p. 93.

fonction marginale des fréquences des x et qu'il aboutit à des lignes de régression jusqu'au quatrième degré. Si Pearson avait précisé, même en hypothèse, des formes spéciales pour ces fonctions marginales, qu'il aurait pu prendre de son classique répertoire des courbes, ou bien s'il les avait exprimées faisant recours aux déterminants du type (3), alors les fonctions orthogonales auraient acquis une grande souplesse et ses calculs auraient gagné au point de vue pratique et théorique. Nous reviendrons sur cette remarque en faisant aussi référence à un autre mémoire, celui de J. Neyman, dont nous nous occuperons en particulier.

Quand, étant donnée une fonction $\psi(x)$, un intervalle (a, b) et une succession de polynômes $P_0, P_1, P_2, \dots, P_n(x), \dots$ orthogonaux avec le noyau $\psi(x)$, on se propose en analyse de développer une certaine fonction arbitraire $f(x)$, on part du développement:

$$f(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \dots + \alpha_n P_n(x) + \dots \quad \dots\dots\dots(10),$$

et en lui supposant un sens légitime, on calcule les α_s par la méthode de Fourier, et l'on obtient:

$$\alpha_s = \frac{\int_a^b f(x) \psi(x) P_s(x) dx}{\int_a^b \psi(x) P_s^2(x) dx}$$

En statistique les $f(x)$ sont des fonctions de fréquence expérimentales et les noyaux $\psi(x)$ des lois de probabilité; c'est à dire que souvent on peut, par ajustement des $f(x)$, obtenir des $\psi(x)$ qui pourront nous donner des valeurs approchées.

C'est ainsi que J. P. Gram et Charlier ont choisi, pour représenter certaines fonctions de fréquence $f(x)$, le noyau $\psi_0(x)$ donné par la formule de Laplace, et au lieu du développement (10) ils sont partis de:

$$f(x) = \alpha_0 \psi_0(x) + \alpha_1 \psi_0^{(1)}(x) + \alpha_2 \psi_0^{(2)}(x) + \dots + \alpha_s \psi_0^{(s)}(x) + \dots \quad \dots\dots\dots(11),$$

où:
$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \begin{matrix} (-\infty \\ +\infty \end{matrix}$$

et
$$\psi_0^{(s)}(x) = \psi_0(x) H_s(x) \quad \dots\dots\dots(12),$$

les $H_s(x)$ sont les polynômes d'Hermitte multipliés par $(-1)^s$.

Grâce à la formule (12) le développement antérieur peut s'écrire:

$$f(x) = \psi_0(x) [\alpha_0 + \alpha_1 H_1 + \alpha_2 H_2 + \dots]$$

et comme l'on sait que:

$$\int_{-\infty}^{+\infty} \psi_0(x) H_s(x) H_{s'}(x) dx \begin{cases} = 0 & \text{pour } s \neq s' \\ \neq 0 & \text{pour } s = s' \end{cases},$$

un simple recours à la méthode de Fourier conduit à la détermination des coefficients, par la formule:

$$\alpha_s = \frac{\int_{-\infty}^{+\infty} f(x) H_s(x) dx}{\int_{-\infty}^{+\infty} \psi_0(x) H_s^2(x) dx}.$$

Ce développement, comme tous les autres développements donnés par V. Romanovsky ayant pour point de départ les fonctions de Pearson, s'appuie essentiellement sur le théorème que nous allons établir. Auparavant, rappelons en passant les procédés par lesquels Romanovsky a fait connaître les développements des courbes de Types I et IV de Pearson.

Essentiellement la fonction de Type I peut s'écrire:

$$\psi(x) = (a+x)^\alpha (b-x)^\beta.$$

Si l'on pose: $U_h(x) = D^{(h)} \{ \psi(x) (a+x)^h (b-x)^h \} = \psi(x) P_h(x)$,

on démontre que:

$$\int_{-a}^b \psi(x) P_h(x) P_{h'}(x) dx \begin{cases} = 0 & \text{pour } h \neq h' \\ \neq 0 & \text{pour } h = h' \end{cases}.$$

Pour le Type IV, c'est à dire:

$$\psi(x) = (a^2 + x^2)^{-m} e^{-\nu \arctan \frac{x}{a}},$$

on pose:

$$U_h(x) = D^{(h)} \left\{ (a^2 + x^2)^{-m+h} e^{-\nu \arctan \frac{x}{a}} \right\} = \psi(x) P_h(x),$$

et l'on démontre la condition d'orthogonalité des $P_h(x)$ par rapport à la fonction de Type IV.

Cette remarque suffit pour faire voir comment tous ces procédés ne sont au fond qu'une application du théorème suivant:

Soit la fonction de probabilité $\psi(x)$ donnée dans l'intervalle $(a; b)$.

Si $\psi(a) = \psi(b) = 0$,

alors cette hypothèse suppose que a et b sont finis. Dans le cas où $a = -\infty$ et $b = +\infty$ il faudra la changer par la suivante: Limite $\psi(x) x^n = 0$ avec $n = 0, 1, 2, \dots$ pour $x = -\infty$ et pour $x = +\infty$.

Si de plus il est possible de déterminer une succession des fonctions $F_h(x)$ telle que:

$$U_h(x) = D^{(h)} \{ \psi(x) F_h(x) \} = \psi(x) P_h(x) \dots\dots\dots(13),$$

$P_h(x)$ étant un polynôme de degré h , on aura:

$$\int_a^b \psi(x) P_h(x) P_{h'}(x) dx \begin{cases} = 0 & \text{pour } h \neq h' \\ \neq 0 & \text{pour } h = h' \end{cases},$$

ce que l'on démontre facilement avec la méthode d'intégration par parties.

Le procédé de génération des polynômes $P_h(x)$ orthogonaux avec $\psi(x)$ s'accomplit alors par dérivation des fonctions de probabilité.

C'est la méthode suivie jusqu'ici.

Si l'on part du développement

$$f(x) = \sum \alpha_\nu U_\nu(x),$$

qui peut s'écrire grâce à la formule (13):

$$f(x) = \psi(x) \sum_{\nu} \alpha_{\nu} P_{\nu}(x),$$

la méthode de Fourier conduit facilement à:

$$\frac{\int_a^b f(x) P_{\nu}(x) dx}{\int_a^b \psi(x) P_{\nu}^2(x) dx} \dots\dots\dots(14)$$

et avec le coefficient α_{ν} ainsi calculé on a fait un minimum de l'intégrale:

$$I = \int_a^b \frac{1}{\psi(x)} [f(x) - \sum_{\nu} \alpha_{\nu} P_{\nu}(x)]^2 dx \dots\dots\dots(15),$$

l'approximation du développement pouvant ainsi être mesurée par la valeur de:

$$\epsilon_n^2 = \int_a^b \frac{[f(x)]^2 dx}{\psi(x)} - \sum_{\nu=0}^n \alpha_{\nu}^2 \int_a^b \psi(x) P_{\nu}^2(x) dx \dots\dots\dots(16).$$

La comparaison de la formule (14) avec la formule (1') de la page 381 montre suffisamment qu'il ne s'agit plus dans ce développement de la méthode de Tschébyschew. Rappelons que la méthode suppose qu'il soit $\psi(a) = \psi(b) = 0$. Son domaine d'application est alors restreint.

Mais on peut modifier la façon conceptuelle pour arriver à ces développements, en les faisant applicables pour le développement de toute fonction de fréquences. Supposons que la fonction des fréquences ajustée par la méthode des moments de Pearson nous conduit à la fonction théorique $\psi(x)$.

On aura alors:

$$\int_a^b f(x) x^s dx = \int_a^b \psi(x) x^s dx \quad \text{pour } s = 0, 1, 2, 3, \text{ et } 4 \dots\dots\dots(17).$$

Pour transformer l'ajustement en interpolation posons $f(x) = \psi(x) \eta(x)$, la fonction corrective $\eta(x)$ se trouvant ainsi précisée.

Si nous écrivons $\eta(x) = \sum_{\nu} \alpha_{\nu} P_{\nu}(x)$, où les $P_{\nu}(x)$ sont des polynômes orthogonaux avec $\psi(x)$, nous aurons:

$$f(x) = \psi(x) \sum_{\nu} \alpha_{\nu} P_{\nu}(x) \dots\dots\dots(18).$$

Avec cela il n'est plus nécessaire d'avoir recours à la dérivation de $\psi(x)$ pour générer, selon les procédés déjà connus, les polynômes $P_h(x)$. A côté de cette méthode on pourra disposer d'autres procédés, par exemple déterminants du type (3), ou bien la formule récurrente:

$$P_{\nu}(x) = - \sum_{h=0}^{\nu-1} \frac{\int_a^b \psi(x) P_h(x) u_{\nu}(x) dx}{\int_a^b \psi(x) P_h(x) u_h(x) dx} P_h(x) + u_{\nu}(x)$$

de Gram-Romanovsky, déjà trouvée.

Maintenant, si par la méthode de Fourier on calcule les coefficients du développement (18), on aura fait avec ces valeurs un minimum pour l'intégrale:

$$\Sigma^2 = \int_a^b \left[\frac{f(x)}{\sqrt{\psi(x)}} - \sqrt{\psi(x)} \sum_v \alpha_v P_v(x) \right]^2 dx,$$

une forme en général différente de la formule (15) mais qui nous a conduit à la même expression de la (16) pour la mesure de l'approximation. Remarquons que si $P_n(x) \psi(x) = U_n(x)$ alors $s^2 = \Sigma^2$.

Cependant on n'a pas suivi cette voie plus générale. C'est en 1924 que Romanovsky* trouva le développement pour les fonctions de Types I, II et III de K. Pearson suivant le procédé de la méthode dérivative déjà montrée.

C'est au mois de juillet de 1932 que nous avons cru obtenir pour la première fois les généralisations des fonctions de Types V et VI en employant la seconde voie directe et au mois de juillet de 1933† nous avons lu la communication que le professeur Romanovsky nous envoya alors et où il trouva les généralisations des courbes de Pearson qui faisaient défaut. Remarquons que dans l'hypothèse (17), le développement (18) se réduit. En effet, on a:

$$\alpha_s = \frac{\int_{-\infty}^{+\infty} f(x) P_s(x) dx}{\int_0^{\infty} \psi(x) P_s^2(x) dx}$$

mais pour $s = 1, 2, 3$ et 4 , on a grâce à la formule (17)

$$\int_{-\infty}^{+\infty} f(x) P_s(x) dx = \int_{-\infty}^{+\infty} \psi(x) P_s(x) dx = 0.$$

Et alors on a effectivement:

$$f(x) = \psi(x) [1 + \alpha_5 P_5(x) + \alpha_6 P_6(x) + \dots] \dots\dots\dots(19).$$

Rien qu'à titre d'exercice, nous montrerons comment on arrive au développement concret pour la fonction de Type V de K. Pearson. Dans ce cas la fonction théorique est:

$$\psi(x) = \frac{\gamma^{p-1}}{\Gamma(p-1)} x^p e^{-\frac{\gamma}{x}} \quad \text{avec } x \begin{cases} 0 \\ \infty \end{cases}.$$

En appliquant la formule de Gram-Romanovsky, étendue au champ continu:

$$P_n(x) = - \sum_{\nu=0}^{+\infty} \frac{\int_0^{\infty} \psi(x) P_\nu(x) x^n dx}{\int_0^{\infty} \psi(x) P_\nu(x) x^\nu dx} P_\nu(x) + x^n,$$

* V. Romanovsky, "Generalisation of some Types of the Frequency Curves of Prof. Pearson," *Biometrika*, Vol. xvi. p. 106.

† Après notre communication aux *Annales argentines* nous avons présenté un article détaillé à *Metron* que nous annoncions depuis le mois de juillet de 1932, à l'occasion d'avoir présenté à cette revue l'article sur "Le développement des fonctions des fréquences au moyen des fonctions orthogonales," où l'on trouvera exposée dans l'essentiel la méthode suivie ici. Voir aussi *Anales de la Sociedad Científica Argentina*, mois d'avril de 1933. Les articles où M. V. Romanovsky a fait connaître ses généralisations des courbes de K. Pearson ont paru dans *Biometrika*, Vol. xvi. 1924, p. 106, et dans les *Atti del Congresso dei matematici di Bologna*, 1928.

on trouve en particulier:

$$P_0 = 1, \quad P_1 = -\frac{\gamma}{p-2} + x,$$

$$P_2 = \frac{\gamma^2}{(p-3)(p-4)} - \frac{2\gamma}{p-4}x + x^2,$$

et en général:

$$P_n(x) = \sum_{s=0}^n (-1)^{n+s} \frac{\gamma^{n-s}}{\prod_{j=n+s+1}^{2n} (p-j)} x^s.$$

On en déduit facilement:

$$a_n = \frac{1}{H_n} \sum_{s=0}^n (-1)^{n+s} \frac{\binom{n}{s} \gamma^{n-s}}{\prod_{j=n+s+1}^{2n} (p-j)} \quad \text{et} \quad \mu_s = \int_0^\infty f(x) x^s dx,$$

ce qui est le moment expérimental, et:

$$H_n = \int_0^\infty \psi(x) P_n^2(x) dx = \frac{n! \gamma^{2n}}{\prod_{s=2}^{2n+1} (p-s) \prod_{l=n+1}^{2n} (p-l)}$$

et pour les moments théoriques on a:

$$\lambda_s = \int_0^\infty \psi(x) x^s dx = \frac{\gamma^s}{\prod_{l=2}^{s+1} (p-l)}.$$

Les intégrations intermédiaires nécessaires pour arriver à ces résultats n'offrent aucune difficulté. La méthode est donc tout à fait élémentaire.

À la rigueur il reste à étudier la question de convergence de ces développements. C'est une question d'analyse délicate qui pourra attirer l'attention des théoriciens et de laquelle nous ne nous occuperons pas dans ce travail, croyant pouvoir y revenir plus tard.

Il faut faire une remarque. Si le noyau choisi est la fonction du Type V, la série est terminée et le dernier terme sera:

$$\alpha_{p-1} P_{p-1}(x).$$

Cela vient de ce que les moments,

$$\mu_n = y_0 \int_0^\infty e^{-\frac{\gamma}{x}} x^p x^n dx$$

qui peuvent s'écrire en faisant $\frac{\gamma}{x} = z$

$$\mu_n = y_0 \gamma^{n+1-p} \int_0^\infty e^{-z} z^{p-n-2} dz = y_0 \gamma^{n+1-p} \Gamma(p-n-1),$$

ne sont déterminés que pour l'argument $p-n-1 > 0$, d'où $n < p-1$.

Si l'on avait suivi le procédé de généralisation de M. Romanovsky qui consiste à prendre:

$$\psi(x) P_n(x) = D^{(h)} [\psi(x) x^{2h}],$$

on serait arrivé aussi à un développement fini, mais moins étendu, car il arriverait seulement jusqu'au terme d'ordre $< \frac{1}{2}(p-1)$. Il est vrai que Romanovsky a donné un autre développement infini pour cette même fonction, mais là les fonctions orthogonales ne se rapportent plus à la variable naturelle x mais à $1/x$, changement peu pratique pour les applications.

Nous avons abordé ces points comme introduction nécessaire à l'étude que nous ferons de la corrélation, car c'est là qu'ils trouvent une application intéressante et utile.

II.

Nous avons parlé auparavant du mémoire de K. Pearson* où il étudie les lignes de régression paraboliques jusqu'au quatrième degré. C'est plus tard, en 1926, que le professeur J. Neyman a généralisé ces résultats†. Nous ferons remarquer premièrement comme M. J. Neyman se trompe quand il déclare dans la page 261: "In this way we shall reach the expressions for successive regression-parabolas, without any appeal to the theory of continued fractions and of orthogonal functions."

En vérité, il ne fait pas appel aux fonctions orthogonales, mais il les introduit sans s'en apercevoir‡. Nous montrerons aussi comme c'est simple de substituer les développements de Neyman en obtenant ces mêmes résultats directement et en donnant des cas spéciaux pour la mise en pratique des mêmes dans les applications à la statistique réelle.

Reprenons le sujet pour notre compte. Soit $z = f(xy)$ la fonction de fréquence des x et des y . On aurait, quelque soit leur domaine de variation:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) dx dy = 1,$$

car en dehors de ce domaine $f(xy) = 0$.

Pour la ligne de régression on a, d'après sa définition:

$$\bar{y}_x = \frac{\int_{-\infty}^{+\infty} f(xy) y dy}{\int_{-\infty}^{+\infty} f(xy) dy} = \frac{\phi(x)}{\psi(x)} \dots \dots \dots (20),$$

$\psi(x)$ étant la fonction de fréquences marginale des x :

$$\psi(x) dx = dx \int_{-\infty}^{+\infty} f(xy) dy.$$

Au lieu de la valeur exacte (20) prenons le développement:

$$\bar{y}_x \sim P_n(x) = a_0 B_0 + a_1 B_1 + \dots + a_n B_n(x) \dots \dots \dots (21),$$

avec les polynômes $B_s(x)$ orthogonaux avec la fonction marginale $\psi(x)$.

* Voir "On a General Method of determining the Successive Terms in a Skew Regression Line," *Biometrika*, Vol. XIII. p. 296.

† Voir "Further Notes on Non-Linear Regression," *Biometrika*, Vol. XVIII. p. 257.

‡ [There is no evidence in Dr Neyman's paper that he was not aware that the polynomials he was dealing with were orthogonal. On the contrary, his statements, e.g. "For $n \leq 4$ these expressions have been given by Prof. K. Pearson..." who deduced them as orthogonal polynomials, suggest that he was. Ed.]

On imposera pour déterminer les α_s la condition que

$$s_n^2 = \int_{-\infty}^{+\infty} \left[\frac{\phi(x)}{\psi(x)} - P_n(x) \right]^2 \psi(x) dx$$

soit minimum.

Cela revient à fixer les α_s de façon que :

$$\frac{\partial s_n^2}{\partial \alpha_s} = 0 \quad \text{pour } s = 0, 1, 2, \dots, n,$$

d'où :

$$\alpha_s = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) y B_s(x) dx dy}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) B_s^2(x) dx dy} \dots\dots\dots(22).$$

Le problème est résolu. Nous ne faisons pas encore des hypothèses sur la nature de $\psi(x)$; parce que nous avons vu auparavant qu'on peut définir les $B_s(x)$ par les déterminants :

$$B_s(x) = \begin{vmatrix} 1, & x, & \dots, & x^s \\ \mu_{0,0}, & \mu_{1,0}, & \dots, & \mu_{s,0} \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix}$$

où :

$$\mu_{s,0} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) x^s dx dy = \int_{-\infty}^{+\infty} \psi(x) x^s dx, —$$

c'est à dire le s -moment de l' x marginal; mais pour plus de simplicité, mettons :

$$B_s(x) = \frac{(-1)^s}{\beta_s} \begin{vmatrix} 1, & x, & \dots, & x^s \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix} \dots\dots\dots(23),$$

β_s étant le mineur de x^s dans (23). Ces polynômes sont bien connus* intervenant dans le problème fondamental des moments, abordés après Laplace et Poisson par Tschébysschew, Markoff, Liaponoïff et Stieltjes. Du reste, il est très facile de démontrer que :

$$\int_{-\infty}^{+\infty} \psi(x) B_s^2(x) dx = \frac{\beta_{s+1}}{\beta_s}.$$

Alors la formule (22) peut s'écrire :

$$\alpha_s = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) y (-1)^s \begin{vmatrix} y, & (xy), & \dots, & (x^s y) \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix} dx dy}{\beta_{s+1}}$$

c'est à dire :

$$\alpha_s = \frac{(-1)^s}{\beta_{s+1}} \begin{vmatrix} \mu_{0,1}, & \mu_{1,1}, & \dots, & \mu_{s,1} \\ \mu_{0,0}, & \mu_{1,0}, & \dots, & \mu_{s,0} \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix}$$

* Voir G. Castelnuovo, *Calcolo delle Probabilità*, 1919.

d'où, en remplaçant a_s dans la formule (21) et tenant compte de la formule (23), on a :

$$P_n(x) = \sum_{s=0}^n \begin{vmatrix} \mu_{0,1} & \dots & \mu_{s,1} \\ \mu_{0,0} & \dots & \mu_{s,0} \\ \vdots & & \vdots \\ \mu_{s-1,0} & \dots & \mu_{2s-1,0} \\ \mu_{0,0} & \mu_{1,0} & \dots & \mu_{s,0} \\ \vdots & & & \vdots \\ \mu_{s,0} & \mu_{s+1,0} & \dots & \mu_{2s,0} \end{vmatrix} \cdot \begin{vmatrix} 1 & x & \dots & x^s \\ \mu_{0,0} & \mu_{1,0} & \dots & \mu_{s,0} \\ \vdots & & & \vdots \\ \mu_{s-1,0} & \mu_{s,0} & \dots & \mu_{2s-1,0} \\ \mu_{0,0} & \mu_{1,0} & \dots & \mu_{s-1,0} \\ \vdots & & & \vdots \\ \mu_{s-1,0} & \mu_{s,0} & \dots & \mu_{2s-2,0} \end{vmatrix}$$

qui coïncide entièrement avec le développement donné par J. Neyman dans son mémoire (voir page 262). Les déterminants où les x figurent dans la première ligne sont ceux que Neyman désigne par $V_s(x)$ de nature orthogonale avec la fonction $\psi(x)$ qui détermine les moments $\mu_{i,0}$.

On a retrouvé donc, par des procédés directs, le développement de Neyman. Mais à vrai dire, la solution était déjà atteinte avec la formule (22) car on savait d'avance comment les $B_s(x)$ pouvaient être déterminés.

Le chemin est alors tout naturellement ouvert aux hypothèses. La fonction de fréquence $\psi(x)$ pourra être prise d'une certaine forme, par exemple une des courbes de Pearson.

On connaît alors effectivement les polynômes $B_s(x)$ qui lui sont orthogonaux, d'où les formules pour le calcul des a_s , avec quoi $P_n(x)$ est déterminé. S'il s'agit d'un cas réel, on ajuste la marginale des x et des y selon la ligne de régression considérée et tout le reste s'écoule simplement de ce que nous avons dit. Mais faisons une remarque. Soit :

$$\int_{-\infty}^{+\infty} f(x, y) dy = f_x,$$

f_x étant la vraie fonction de fréquences des x , et supposons que $\psi(x)$ soit la formule des probabilités des x , obtenues par l'ajustement de f_x . Pour la théorie on pourra considérer $\psi(x) = f_x$, mais il est bon de tenir compte que pour la pratique cette identité n'est qu'approchée. Si l'on fait, comme il est plus commode et généralement permis, la supposition que l'identité est suffisamment accomplie, toute la méthode est directe. Mais aussi on pourra conserver des indices en mesurant les écarts que l'on pourra commettre et si l'on veut, revenir aussi facilement sur les formules théoriques approchées jusqu'à les rendre rigoureuses. On aura l'occasion de traiter avec quelques détails ces diverses questions; mais pour le moment nous allons nous occuper des surfaces de corrélation, desquelles nous allons obtenir les lignes de régression, coefficients de relation de corrélation et équation sédistique, etc.

Occupons-nous maintenant de l'étude des surfaces de corrélation en conservant la méthode jusqu'ici exposée. Le lecteur désireux d'être au courant des principaux problèmes qui se posent dans ces questions, peut avoir recours à l'excellente étude de Dr S. J. Pretorius* qui contient des informations très détaillées au point de vue actuel et technique de notre sujet.

* Voir "Skew Bivariate Frequency Surfaces, examined in the light of Numerical Illustrations," *Biometrika*, Vol. xxii. p. 109.

Soit $z = f(x, y)$ la surface de fréquence et soient :

$$\int_{-\infty}^{+\infty} z dy = \psi(x), \quad \int_{-\infty}^{+\infty} z dx = \rho(y),$$

les distributions marginales des x et des y respectivement. Si, et uniquement si, ces variables sont indépendantes, on aura :

$$f(x, y) = \psi(x) \rho(y) \dots\dots\dots(24).$$

Pour qu'il existe une dépendance stochastique, il est nécessaire que l'égalité (24) ne se vérifie pas. La condition de dépendance stochastique est nécessaire mais pas suffisante pour qu'il existe la corrélation.

Dans le cas de dépendance stochastique on aura :

$$f(x, y) = \psi(x) \phi_x(y) = \rho(y) R_y(x) \dots\dots\dots(25),$$

$\phi_x(y)$ étant la fonction des probabilités des y liées aux x et $R_y(x)$ inversement. Mais au lieu de (25) nous écrirons :

$$f(x, y) = \psi(x) \rho(y) \eta(x, y) \dots\dots\dots(26),$$

et puis nous adopterons pour $\eta(x, y)$ le développement :

$$\eta(x, y) = \sum_s \sum_j \omega_{s,j} X_s(x) Y_j(y),$$

où les $X_s(x)$ et $Y_j(y)$ sont des polynômes en x et en y de degrés s et j , et orthogonaux avec les fonctions $\psi(x)$ et $\rho(y)$ respectivement. On aura alors :

$$f(x, y) = \psi(x) \rho(y) \sum_s \sum_j \omega_{s,j} X_s Y_j \dots\dots\dots(27),$$

d'où, par la méthode de Fourier, on déduit facilement :

$$\omega_{s,j} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) X_s Y_j dx dy}{\int_{-\infty}^{+\infty} \psi(x) X_s^2 dx \int_{-\infty}^{+\infty} \rho(y) Y_j^2 dy} \dots\dots\dots(28),$$

et en particulier :

$$\omega_{0,0} = 1, \quad \omega_{s,0} = \omega_{0,j} = 0,$$

pour $s = 1, 2, 3, \dots$, et $j = 1, 2, 3, \dots$.

Faisons voir qu'effectivement $\omega_{s,0} = 0$. On a, par la formule (28),

$$\omega_{s,0} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) X_s dx dy}{\int_{-\infty}^{+\infty} \psi(x) X_s^2 dx},$$

mais le numérateur peut aussi s'écrire :

$$\int_{-\infty}^{+\infty} X_s(x) dx \int_{-\infty}^{+\infty} f(x, y) dy = \int_{-\infty}^{+\infty} \psi(x) X_s(x) dx = 0.$$

La formule (27) peut alors aussi s'écrire :

$$f(x, y) = \psi(x) \rho(y) [1 + \sum_s \sum_j \omega_{s,j} X_s Y_j] \dots\dots\dots(29),$$

d'où, pour toutes les $\omega_{s,j} = 0$ on a la formule (24) en particulier, de sorte que ce sont les coefficients qui mesureront, comme nous le ferons voir, la forme et le degré de dépendance entre les x et les y .

En vertu de la formule (25) on a :

$$\phi_x(y) = \rho(y) [1 + \sum_s \sum_j \omega_{s,j} X_s Y_j],$$

d'où pour la ligne de régression des y en x on aura :

$$y_x = \int_{-\infty}^{+\infty} \phi_x(y) y dy = \int_{-\infty}^{+\infty} \rho(y) [1 + \sum_s \sum_j \omega_{s,j} X_s Y_j] dy,$$

$$\text{c'est à dire: } \bar{y}_x = \int_{-\infty}^{+\infty} \rho(y) y dy + \sum_s X_s \sum_j \omega_{s,j} \int_{-\infty}^{+\infty} \rho(y) y Y_j dy.$$

Mais si l'on mesure les variables x et y à partir de leur moyenne arithmétique comme origine, alors on trouvera que la première intégrale est nulle. Pour simplifier les calculs des secondes intégrales il suffit de choisir les Y_j (et on pourra faire de même avec les X_s) de façon que si, étant :

$$Y_j(y) = \alpha_{j,0} + \alpha_{j,1}y + \alpha_{j,2}y^2 + \dots + \alpha_{j,j}y^j,$$

on ait toujours :

$$\alpha_{j,j} = 1.$$

Pour générer ainsi ces $Y_j(y)$ il suffira de prendre le déterminant (23) de la page 391 ou la formule de Gram-Romanovsky de la page 388 avec la base $u_\nu(y) = y^\nu$. Ces remarques faites, on aura alors : $Y_1(y) = y$, d'où :

$$\sum_j \omega_{s,j} \int_{-\infty}^{+\infty} \rho(y) y Y_j dy = \omega_{s,1} \int_{-\infty}^{+\infty} \rho(y) y^2 dy = \omega_{s,1} \sigma_y^2,$$

$$\text{d'où: } \bar{y}_x = \sigma_y^2 \sum_s \omega_{s,1} X_s(x) \dots \dots \dots (30).$$

D'une façon tout analogue pour l'autre ligne de régression des x en y , on trouvera* :

$$\bar{x}_y = \sigma_x^2 \sum_j \omega_{1,j} Y_j(y).$$

* Il est facile de s'apercevoir que ces équations coïncident avec celle que l'on a trouvée préalablement (pp. 391 - 392) et qui satisfait la condition des moindres carrés. En effet on a :

$$\omega_{s,1} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s Y_1 dx dy}{\int_{-\infty}^{+\infty} \psi(x) X_s^2 dx \int_{-\infty}^{+\infty} \rho(y) Y_1^2 dy} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \frac{(-1)^s}{\beta_s} y \begin{vmatrix} 1, & x, & \dots, & x^n \\ \mu_{0,0}, & \mu_{1,0}, & \dots, & \mu_{s,0} \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix} dx dy}{\frac{\beta_{s+1}}{\beta_s} \frac{\beta_2'}{\beta_1'}}.$$

$$\text{étant } \beta_2' = \int_{-\infty}^{+\infty} \rho(y) y^2 dy = \sigma_y^2; \quad \omega_{s,1} = \frac{(-1)^s}{\beta_{s+1} \sigma_y^2} \begin{vmatrix} \mu_{0,1}, & \mu_{1,1}, & \dots, & \mu_{s,1} \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix},$$

et alors :

$$\bar{y}_x = \sigma_y^2 \sum_s \omega_{s,1} X_s = \sigma_y^2 \sum_s \frac{(-1)^s}{\beta_{s+1} \sigma_y^2} \begin{vmatrix} \mu_{0,1}, & \mu_{1,1}, & \dots, & \mu_{s,1} \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix} \frac{(-1)^s}{\beta_s} \begin{vmatrix} 1 & \dots & x^s \\ \vdots & & \\ \mu_{s-1,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix},$$

$$\text{c'est à dire: } \bar{y}_x = \sum_s \frac{\begin{vmatrix} \mu_{0,1}, & \dots, & \mu_{s,1} \\ \vdots & & \\ \mu_{s-1,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix} \begin{vmatrix} 1, & \dots, & x^s \\ \vdots & & \\ \mu_{s-1,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix}}{\beta_s \beta_{s+1}},$$

et alors les α_s de la formule (22) de la page 391 satisfont l'égalité :

$$\alpha_s = \omega_{s,1} \sigma_y^2.$$

Relations analogues existent pour l'autre ligne de régression.

D'où, si dans la formule (27) $\omega_{s,j} \neq 0$ pour quelques s et j il y a dépendance stochastique et pour que cette dépendance entraîne corrélation de y en x il est nécessaire qu'il soit $\omega_{s,1} \neq 0$ pour quelques s et $\omega_{1,j} \neq 0$ pour quelques j pour qu'il existe corrélation de x en y . Ces deux dernières conditions entraînent la bicorrélation.

Par exemple, on pourrait avoir $\omega_{s,j} \neq 0$ pour quelques s et j et alors même en existant dépendance stochastique, cela n'entraînerait pas la corrélation parce que à côté des antérieures conditions pourrait se trouver $\omega_{s,1} = 0$ où $\omega_{1,j} = 0$. Mais l'inverse n'est pas vrai, car s'il y a corrélation, cela entraîne $\omega_{s,1} \neq 0$ pour quelques s , où $\omega_{1,j} \neq 0$ pour quelques j .

On peut aussi transférer ces considérations, grâce à la formule (28), aux moments doubles qui sont:

$$\mu_{s,j} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) x^s y^j dx dy.$$

Les conditions nécessaires et suffisantes pour qu'il existe une corrélation de y en x ou de x en y , sont:

$$\left. \begin{array}{l} \mu_{s,1} \neq 0 \text{ pour quelques } s, \text{ et} \\ \mu_{1,j} \neq 0 \text{ " " } j, \text{ respectivement} \end{array} \right\} \dots\dots\dots (I).$$

Démontrons que la condition est nécessaire. Le numérateur de $\omega_{s,1}$ selon l'égalité (28) pourra s'écrire:

$$N = \gamma_{s,0} \mu_{0,1} + \gamma_{s,1} \mu_{1,1} + \dots + \mu_{s,1} \quad \text{en prenant } Y_1 = y,$$

$$\text{et} \quad X_s = \gamma_{s,0} + \gamma_{s,1}x + \gamma_{s,2}x^2 + \dots + x^s \quad \text{étant } \gamma_{s,s} = 1.$$

Donc, si $\omega_{s,1} \neq 0$ il en sera de même pour N et alors nécessairement on aura $\mu_{i,1} \neq 0$ pour quelque i . La condition est donc nécessaire.

Démontrons qu'elle est aussi suffisante. Supposons pour cela que $\mu_{i,1}$ est le premier moment double de la succession $\mu_{1,1}, \mu_{2,1}, \mu_{3,1}, \dots$ qui ne soit nul. Alors, on aura, en tenant compte de la formule (28) et de la définition des X_s et des Y_j , sous forme de déterminant (23):

$$\omega_{i,1} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(xy) y \frac{(-1)^i}{\beta_i} \begin{vmatrix} 1, & x, & \dots, & x^i \\ \vdots & \vdots & & \vdots \end{vmatrix} dx dy}{\begin{vmatrix} \mu_{i-1,0}, & \mu_{i,0}, & \dots, & \mu_{2i-1,0} \\ \beta_{i+1} & \beta_2' & & \\ \beta_i & \beta_1' & & \\ 0, & 0, & \dots, & \mu_{i,1} \\ \frac{(-1)^i}{\beta_{i+1} \sigma} & \mu_{0,0}, & \mu_{1,0}, & \dots, & \mu_{i,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{i-1,0}, & \mu_{i,0}, & \dots, & \mu_{2i-1,0} \end{vmatrix}}$$

$$\text{d'où:} \quad \omega_{i,1} = (-1)^{2(1+i)} \mu_{i,1} \frac{\beta_i}{\beta_{i+1} \sigma_y^2},$$

$$\text{donc si } \mu_{i,1} \neq 0 \text{ on a:} \quad \omega_{i,1} \neq 0.$$

Remarquons que, dans les conditions de notre étude, les β_i sont toujours non nulles*.

Les conditions (I) au-dessus obtenues si facilement nous montrent le rôle fondamental des moments $\mu_{s,1}$ et $\mu_{i,j}$ pour l'existence ou non de la corrélation.

Avant de quitter le développement (27), insistons sur une remarque importante. Le calcul de r se fait sur la base de $\mu_{1,1}$. Bien; il ne suffit pas d'avoir trouvé $r = 0$ pour nier l'existence de corrélation, car celle-ci existera du moment qu'il soit $\mu_{i,1} \neq 0$ ou $\mu_{1,j} \neq 0$ pour quelque i ou j . Les jugements portés uniquement sur la valeur de r supposent les lignes de régression droites; cela exige que $\omega_{i,1} = \omega_{1,j} = 0$ pour $i = 2, 3, \dots$ et $j = 2, 3, \dots$; mais pour que ces égalités se vérifient on aura ou bien $\mu_{i,1} = \mu_{1,j} = 0$ pour $i = 2, 3, \dots$ et $j = 2, 3, \dots$, ou bien—la valeur N de la page 395 devant être nulle en dehors de ces cas—il faudra que:

$$\gamma_{s,0} \mu_{0,1} + \gamma_{s,1} \mu_{1,1} + \dots + \mu_{s,1} = 0 \quad \text{avec } s = 2, 3, \dots \quad (31),$$

ainsi qu'un quelconque de ces $\mu_{i,1}$ soit une combinaison directe de $\mu_{1,1}, \mu_{2,1}, \dots$. Mais s étant égale à 2, 3, etc. la formule (31) donnera, en remarquant que $\mu_{0,1} = 0$:

$$\gamma_{s,1} \mu_{1,1} + \gamma_{s,2} \mu_{2,1} + \dots + \mu_{s,1} = 0,$$

pour $s = 2$: $\gamma_{2,1} \mu_{1,1} + \mu_{2,1} = 0,$

pour $s = 3$: $\gamma_{3,1} \mu_{1,1} + \gamma_{3,2} \mu_{2,1} + \mu_{3,1} = 0$, etc.

D'où $\mu_{s,1}$ doit être une combinaison de $\mu_{1,1}$, c'est à dire, une expression contenant $\mu_{1,1}$ comme facteur. C'est précisément ce qu'il arrive pour la fonction de Bravais, qui est à régression droite†.

Les lignes de régression paraboliques étant déjà trouvées, occupons-nous du calcul des coefficients de relation de corrélation η_{yx}, η_{xy} .

D'après la définition de Karl Pearson on a:

$$\eta_{yx}^2 = 1 - \frac{s_y^2}{\sigma_y^2},$$

s_y^2 étant la dispersion par rapport à la ligne de régression de y en x . On aura:

$$s_y^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) [y - \bar{y}]^2 dx dy,$$

mais:

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y^2 dx dy &= \int_{-\infty}^{+\infty} y^2 dy \int_{-\infty}^{+\infty} f(x, y) dx = \int_{-\infty}^{+\infty} \rho(y) y^2 dy = \sigma_y^2; \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y \bar{y}_x dx dy &= \sigma_y^2 \sum_s \omega_{s,1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y X_s dx dy; \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \bar{y}_x^2 dx dy &= \sigma_y^4 \sum_s \omega_{s,1}^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) X_s^2 dx dy \\ &= \sigma_y^4 \sum_s \omega_{s,1}^2 \int_{-\infty}^{+\infty} \psi(x) X_s^2 dx = \sigma_y^2 \sum_s \omega_{s,1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y X_s dx dy. \end{aligned}$$

* Voir G. Castelnuovo, *loc. cit.* p. 323.

† Voir Tschowprow, p. 58, dans son *Grundbegriffe und Grundprobleme der Korrelationstheorie*, Berlin, 1925.

Ces derniers résultats s'obtiennent en substituant la valeur d'un $\omega_{s,1}$, et—les doubles produits disparaissant—il résulte alors:

$$\begin{aligned} s_y^2 &= \sigma_y^2 - \sigma_y^2 \sum_s \omega_{s,1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y X_s dx dy \\ &= \sigma_y^2 \left[1 - \sum_s \omega_{s,1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y X_s dx dy \right], \end{aligned}$$

d'où:
$$\eta_{yx}^2 = \sum_s \omega_{s,1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y X_s dx dy,$$

ou si l'on veut:

$$\sum_s \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x) X_s dx}{\sigma_y^2 \int_{-\infty}^{+\infty} \psi(x) X_s dx},$$

et aussi:

$$\eta_{yx}^2 = \frac{1}{\sigma_y^2} \sum_s \frac{\left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) y \frac{(-1)^s}{\beta_s} \begin{vmatrix} 1, & x, & \dots, & x^s, \\ \vdots & & & \end{vmatrix} dx dy \right]}{\beta_{s+1}/\beta_s}$$

$$\text{d'où: } \eta^2 = \frac{1}{\sigma_y^2} \sum_s \frac{\begin{vmatrix} \mu_{0,1}, & \mu_{1,1}, & \dots, & \mu_{s,1} \\ \mu_{0,0}, & \mu_{1,0}, & \dots, & \mu_{s,0} \\ \vdots & & & \\ \mu_{s-1,0}, & \mu_{s,0}, & \dots, & \mu_{2s-1,0} \end{vmatrix}^2}{\beta_s \beta_{s+1}}.$$

Déterminons maintenant l'équation sédastique. On a:

$$\sigma_{yx}^2 = \int_{-\infty}^{+\infty} \phi_x(y) [y - \bar{y}_x]^2 dy,$$

d'où:
$$\sigma_{yx}^2 = \sum_{s=0} X_s \sum_{j=0} \omega_{s,j} \int_{-\infty}^{+\infty} \rho(y) Y_j [y^2 - 2y\bar{y}_x + \bar{y}_x^2] dy,$$

puisque:
$$\phi_x(y) = \rho(y) \sum_{s=0} \sum_{j=0} \omega_{s,j} X_s Y_j,$$

alors:
$$\sigma_{yx}^2 = \sum_{s=0} X_s \left[\omega_{s,0} \{\sigma_y^2 + \bar{y}_x^2\} + \omega_{s,1} \{\mu_{0,3} - 2\bar{y}_x \sigma_y^2\} + \omega_{s,2} \frac{\beta_3'}{\beta_2'} \right],$$

où:
$$\frac{\beta_3'}{\beta_2'} = \int_{-\infty}^{+\infty} \rho(y) Y_2 y^2 dy \quad \text{et } \beta_s' \text{ est le mineur de } y_s$$

dans le déterminant des Y_s . Mais comme $\omega_{0,s} = \omega_{s,0} = 0$ pour $s = 1, 2, 3, \dots$ et $\omega_{0,0} = 1$ on aura:

$$\begin{aligned} \sigma_{yx}^2 &= \sigma_y^2 + \bar{y}_x^2 + \sum_{s=1} X_s \omega_{s,1} \{\mu_{0,3} - 2\bar{y}_x \sigma_y^2\} + \frac{\beta_3'}{\beta_2'} \sum_{s=1} \omega_{s,2} X_s, \\ \sigma_{yx}^2 &= \sigma_y^2 + \bar{y}_x^2 + \mu_{0,3} \sum_{s=1} \omega_{s,1} X_s - 2\bar{y}_x \sigma_y^2 \sum_{s=1} \omega_{s,1} X_s + \frac{\beta_3'}{\beta_2'} \sum_{s=1} \omega_{s,2} X_s, \\ \sigma_{yx}^2 &= \sigma_y^2 + \bar{y}_x^2 + \mu_{0,3} \frac{\bar{y}_x}{\sigma_y^2} - 2\bar{y}_x^2 + \frac{\beta_3'}{\beta_2'} \sum_{s=1} \omega_{s,2} X_s, \\ \sigma_{yx}^2 &= \sigma_y^2 - \bar{y}_x^2 + \mu_{0,3} \frac{\bar{y}_x}{\sigma_y^2} + \frac{\beta_3'}{\beta_2'} \sum_{s=1} \omega_{s,2} X_s \dots\dots\dots (32), \end{aligned}$$

mais:
$$\omega_{s,2} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s Y_2 dx dy}{\frac{\beta_{s+1} \beta_3'}{\beta_s \beta_2'}}$$

$$\frac{1, \quad y, \quad y^2}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s \cdot \mu_{0,0}, \quad \mu_{0,1}, \quad \mu_{0,2} \quad dx dy}$$

$$\frac{\mu_{0,1}, \quad \mu_{0,2}, \quad \mu_{0,3}}{\frac{\beta_{s+1} \beta_3'}{\beta_s \beta_2'}}$$

$$\omega_{s,2} = \frac{\beta_s}{\beta_{s+1} \beta_3'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s [-\mu_{0,2}^2 - y \mu_{0,3} + y^2 \mu_{0,2}] dx dy,$$

$$\omega_{s,2} = \frac{\beta_s}{\beta_{s+1} \beta_3'} \left[-\mu_{0,3} \omega_{s,1} \frac{\beta_{s+1} \beta_2'}{\beta_s \beta_1'} + \mu_{0,2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s y^2 dx dy \right],$$

$$\omega_{s,2} = -\mu_{0,3} \frac{\beta_2'}{\beta_3'} \omega_{s,1} + \frac{\beta_s \mu_{0,2}}{\beta_3' \beta_{s+1}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s y^2 dx dy,$$

et alors substituant $\omega_{s,2}$ dans la formule (32), β_2' étant égal à $\mu_{0,2}$, on a:

$$\sigma_{yx}^2 = \sigma_y^2 - \bar{y}_x^2 + \mu_{0,3} \frac{\bar{y}_x}{\sigma_y^2} - \mu_{0,3} \sum_{s=1} \omega_{s,1} X_s + \sum_{s=1} \frac{\beta_s}{\beta_{s+1}} X_s \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s y^2 dx dy;$$

mais alors comme :

$$\sum_{s=1} \omega_{s,1} X_s = \frac{\bar{y}_x}{\sigma_y^2},$$

on a :

$$\sigma_{yx}^2 = \sigma_y^2 - \bar{y}_x^2 + \sum_{s=1} \frac{\beta_s}{\beta_{s+1}} X_s \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s y^2 dx dy$$

$$= \sigma_y^2 - \bar{y}_x^2 + \sum_{s=1} \frac{\beta_s}{\beta_{s+1}} X_s \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \frac{(-1)^{1, x, \dots, x^s}}{\beta_s} y^2 dx dy$$

d'où :

$$\sigma_{yx}^2 = -\bar{y}_x^2 + \sum_{s=0} \frac{\begin{vmatrix} 1, & x, & \dots, & x^s \\ \vdots & & & \end{vmatrix} \begin{vmatrix} \mu_{0,2}, & \mu_{1,2}, & \dots, & \mu_{s,2} \\ \vdots & & & \end{vmatrix}}{\beta_s \beta_{s+1}}$$

Cherchons maintenant les équations "clitics" et "kurtics," c'est à dire :

$$\sqrt{\beta_1(y)_x} = \frac{\mu_3(y)_x}{\mu_2^{\frac{3}{2}}(y)_x} \quad \text{et} \quad \beta_2(y)_x - 3 = \frac{\mu_4(y)_x}{\mu_2^2(y)_x} - 3$$

que nous appellerons de symétrie et de normalité partielle de y en x . On a :

$$\mu_3(y)_x = \int_{-\infty}^{+\infty} \phi_x(y) [y - \bar{y}_x]^3 dy = \sum_{s=0} X_s \sum_{j=0} \omega_{s,j} \int_{-\infty}^{+\infty} \rho(y) Y_j [y - \bar{y}_x]^3 dy,$$

$$\mu_3(y)_x = \sum_{s=0} X_s \left\{ \omega_{s,0} (\mu_{0,3} - 3\bar{y}_x \mu_{0,2} - y_x^3) + \omega_{s,1} (\mu_{0,4} - 3\bar{y}_x \mu_{0,3} + 3\bar{y}_x^2 \mu_{0,2}) \right.$$

$$+ \omega_{s,2} \left(\int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy - 3\bar{y}_x \frac{\beta_3'}{\beta_2'} \right) + \omega_{s,3} \frac{\beta_4'}{\beta_3'} \left. \right\} = \mu_{0,3} - 3\bar{y}_x \mu_{0,2} - \bar{y}_x^3$$

$$+ (\mu_{0,4} - 3\bar{y}_x \mu_{0,3} + 3\bar{y}_x^2 \mu_{0,2}) \sum_{s=1} \omega_{s,1} X_s + \left[\int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy - 3\bar{y}_x \frac{\beta_3'}{\beta_2'} \right]$$

$$\times \sum_{s=1} \omega_{s,2} X_s + \frac{\beta_4'}{\beta_3'} \sum_{s=1} \omega_{s,3} X_s \dots \dots \dots (33).$$

Nous pouvons simplifier ce résultat en substituant les valeurs de $\omega_{s,2}$ et $\omega_{s,3}$. On a :

$$\begin{aligned}\omega_{s,3} &= \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s Y_3 dx dy}{\frac{\beta_{s+1}}{\beta_s} \frac{\beta_4'}{\beta_3'}} \\ &= -\frac{\beta_s}{\beta_{s+1} \beta_4'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s \begin{vmatrix} 1, & y, & y^2, & y^3 \\ \mu_{0,0}, & \mu_{0,1}, & \mu_{0,2}, & \mu_{0,3} \\ \mu_{0,1}, & \mu_{0,2}, & \mu_{0,3}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,3}, & \mu_{0,4}, & \mu_{0,5} \end{vmatrix} dx dy, \\ \omega_{s,3} &= -\frac{\beta_s}{\beta_{s+1} \beta_4'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s \left[-y \begin{vmatrix} \mu_{0,0}, & \mu_{0,2}, & \mu_{0,3} \\ \mu_{0,1}, & \mu_{0,3}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,4}, & \mu_{0,5} \end{vmatrix} \right. \\ &\quad \left. + y^2 \begin{vmatrix} \mu_{0,0}, & \mu_{0,1}, & \mu_{0,3} \\ \mu_{0,1}, & \mu_{0,2}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,3}, & \mu_{0,5} \end{vmatrix} - y^3 \beta_3' \right] dx dy \quad \text{pour } s = 1, 2, 3, \text{ etc.}, \\ \omega_{s,3} &= -\frac{\beta_s}{\beta_{s+1} \beta_4'} \left[- \begin{vmatrix} \mu_{0,0}, & \mu_{0,2}, & \mu_{0,3} \\ 0, & \mu_{0,3}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,4}, & \mu_{0,5} \end{vmatrix} \omega_{s,1} \frac{\beta_{s+1}}{\beta_s} \frac{\beta_2'}{\beta_1'} \right. \\ &\quad \left. + \begin{vmatrix} \mu_{0,0}, & 0, & \mu_{0,3} \\ 0, & \mu_{0,2}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,3}, & \mu_{0,5} \end{vmatrix} \frac{(-1)^s}{\beta_s} \begin{vmatrix} \mu_{0,2}, & \dots, & \mu_{s,2} \\ \vdots \\ \mu_{s-1,0} \end{vmatrix} \right. \\ &\quad \left. - \frac{\beta_3' (-1)^s}{\beta_s} \begin{vmatrix} \mu_{0,3}, & \dots, & \mu_{s,3} \\ \vdots \\ \mu_{s-1,0} \end{vmatrix} \right],\end{aligned}$$

mais aussi :

$$\omega_{s,2} = -\mu_{0,3} \frac{\beta_2'}{\beta_3'} \omega_{s,1} + (-1)^s \mu_{0,2} \frac{1}{\beta_3' \beta_{s+1}} \begin{vmatrix} \mu_{0,3}, & \dots, & \mu_{s,2} \\ \vdots \\ \mu_{s-1,0} \end{vmatrix}$$

pour $s = 1, 2, \dots$

Remplaçant $\omega_{s,2}$ dans la formule (33), on a :

$$\begin{aligned}\mu_3(y)_x &= \mu_{0,3} - 3\mu_{0,2} \bar{y}_x - \bar{y}_x^3 + \left[\mu_{0,4} - 3\mu_{0,3} \bar{y}_x + 3\mu_{0,2} \bar{y}_x^2 \right. \\ &\quad \left. - \left(\int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy - 3 \frac{\beta_3'}{\beta_2'} \bar{y}_x \right) \frac{\mu_{0,3} \beta_2'}{\beta_3'} + \frac{\beta_2'}{\beta_3'} \begin{vmatrix} 1, & \mu_{0,2}, & \mu_{0,3} \\ 0, & \mu_{0,3}, & \mu_{0,4} \\ \mu_{0,2}, & \mu_{0,4}, & \mu_{0,5} \end{vmatrix} \right] \sum_{s=1} \omega_{s,1} X_s\end{aligned}$$

$$\begin{aligned}
 & + \left(\int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy - 3 \frac{\beta_3'}{\beta_2'} \bar{y}_x \right) \frac{\mu_{0,2}}{\beta_3'} \sum_{s=1} (-1)^s \frac{1}{\beta_{s+1}} \left| \begin{array}{ccc} \mu_{0,2} & \dots & \mu_{s,2} \\ \vdots & & \\ \mu_{s-1,0} & & \end{array} \right| X_s \\
 & - \frac{1}{\beta_3'} \left| \begin{array}{ccc} 1 & 0 & \mu_{0,3} \\ 0 & \mu_{0,2} & \mu_{0,4} \\ \mu_{0,2} & \mu_{0,3} & \mu_{0,5} \end{array} \right| \sum_{s=1} \frac{(-1)^s}{\beta_{s+1}} \left| \begin{array}{ccc} \mu_{0,2} & \dots & \mu_{s,2} \\ \vdots & & \\ \mu_{s-1,0} & & \end{array} \right| X_s \\
 & + \sum_{s=1} \frac{(-1)^s}{\beta_{s+1}} \left| \begin{array}{ccc} \mu_{0,3} & \dots & \mu_{s,3} \\ \vdots & & \\ \mu_{s-1,0} & & \end{array} \right| X_s.
 \end{aligned}$$

Mais:

$$\begin{aligned}
 \int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy &= \frac{1}{\beta_2'} \int_{-\infty}^{+\infty} \rho(y) \left| \begin{array}{ccc} 1 & y & y^2 \\ \mu_{0,0} & \mu_{0,1} & \mu_{0,2} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} \end{array} \right| y^3 dy \\
 &= \frac{\left| \begin{array}{ccc} \mu_{0,3} & \mu_{0,4} & \mu_{0,5} \\ 1 & 0 & \mu_{0,2} \\ 0 & \mu_{0,2} & \mu_{0,3} \end{array} \right|}{\mu_{0,2}} = \frac{\mu_{0,2} \mu_{0,5} - \mu_{0,2}^2 \mu_{0,3} - \mu_{0,3} \mu_{0,4}}{\mu_{0,2}},
 \end{aligned}$$

et aussi:

$$\beta_3' = \left| \begin{array}{ccc} \mu_{0,0} & \mu_{0,1} & \mu_{0,2} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} \\ \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \end{array} \right| = \left| \begin{array}{ccc} 1 & 0 & \mu_{0,2} \\ 0 & \mu_{0,2} & \mu_{0,3} \\ \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \end{array} \right| = \mu_{0,2} \mu_{0,4} - \mu_{0,2}^3 - \mu_{0,3}^2;$$

après quelques simplifications on arrive finalement à:

$$\begin{aligned}
 \mu_3(y)_x &= 2\bar{y}_x^3 - 3\bar{y}_x \sum_{s=0} \frac{\left| \begin{array}{ccc} 1 & \dots & x^s \\ \vdots & & \end{array} \right| \cdot \left| \begin{array}{ccc} \mu_{0,2} & \dots & \mu_{s,2} \\ \vdots & & \end{array} \right|}{\beta_s \beta_{s+1}} \\
 &+ \sum_{s=0} \frac{\left| \begin{array}{ccc} 1 & \dots & x^s \\ \vdots & & \end{array} \right| \cdot \left| \begin{array}{ccc} \mu_{0,3} & \dots & \mu_{s,3} \\ \vdots & & \end{array} \right|}{\beta_s \beta_{s+1}}.
 \end{aligned}$$

Nous devons calculer maintenant $\mu_2(y)_x$, mais:

$$\mu_2(y)_x = \sigma_{y_x}^2 = \int_{-\infty}^{+\infty} \phi_x(y) [y - \bar{y}_x]^2 dy,$$

* Si l'on veut tenir compte de l'équation sédastique on aura:

$$\mu_3(y)_x = 2\bar{y}_x^3 - [\sigma_{y_x}^2 + \bar{y}_x^2] 3\bar{y}_x + \sum_{s=0} \frac{\left| \begin{array}{ccc} 1 & \dots & x^s \\ \vdots & & \end{array} \right| \cdot \left| \begin{array}{ccc} \mu_{0,3} & \dots & \mu_{s,3} \\ \vdots & & \end{array} \right|}{\beta_s \beta_{s+1}},$$

d'où:

$$\mu_2(y)_x = -\bar{y}_x - 3\bar{y}_x \sigma_{y_x}^2 + \sum_{s=0} \frac{\left| \begin{array}{ccc} 1 & \dots & x^s \\ \vdots & & \end{array} \right| \cdot \left| \begin{array}{ccc} \mu_{0,3} & \dots & \mu_{s,3} \\ \vdots & & \end{array} \right|}{\beta_s \beta_{s+1}}.$$

et l'on a :

$$\mu_2(y)_x = \sigma_{yx}^2 = -\bar{y}_x^2 + \sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,2}, \dots, \mu_{s,2} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}},$$

mais :

$$\mu_{\frac{3}{2}}(y)_x = \sigma_{yx}^3 = \left[\sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,2}, \dots, \mu_{s,2} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}} - \bar{y}_x^2 \right]^{\frac{3}{2}}.$$

Et alors, pour l'équation de symétrie partielle, on a :

$$\begin{aligned} \sqrt{\beta_1(y)_x} &= \frac{\mu_3(y)_x}{\sigma_{yx}^{\frac{3}{2}}} = \frac{\mu_3(y)_x}{\sigma_{yx}^3} \\ &= \frac{2\bar{y}_x^3 - 3\bar{y}_x \sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,2}, \dots, \mu_{s,2} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}} + \sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,3}, \dots, \mu_{s,3} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}}}{\left[\sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,2}, \dots, \mu_{s,2} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}} - \bar{y}_x^2 \right]^{\frac{3}{2}}}, \end{aligned}$$

que l'on pourra écrire, la dispersion partielle une fois connue :

$$\begin{aligned} \sqrt{\beta_1(y)_x} &= \frac{1}{\sigma_{yx}^3} \left[2\bar{y}_x^3 - 3\bar{y}_x \sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,2}, \dots, \mu_{s,2} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}} \right. \\ &\quad \left. + \sum_{s=0} \frac{\left| \begin{array}{c} 1, \dots, x^s \\ \vdots \end{array} \right| \cdot \left| \begin{array}{c} \mu_{0,3}, \dots, \mu_{s,3} \\ \vdots \end{array} \right|}{\beta_s \beta_{s+1}} \right] \dots (34). \end{aligned}$$

Déterminons l'équation de normalité partielle :

$$\beta_2(y)_x - 3 = \frac{\mu_4(y)_x}{\mu_2^2(y)_x} - 3.$$

Il nous faut calculer : $\mu_4(y)_x$,

$$\begin{aligned} \mu_4(y)_x &= \int_{-\infty}^{+\infty} \phi_x(y) [y - \bar{y}_x]^4 dy \\ &= \sum_{s=0} X_s \sum_{j=0} \omega_{s,j} \int_{-\infty}^{+\infty} \rho(y) Y_j [y^4 - 4y^3 \bar{y}_x + 6y^2 \bar{y}_x^2 - 4y \bar{y}_x^3 + \bar{y}_x^4] dy \\ \mu_4(y)_x &= \mu_{0,4} - 4\mu_{0,3} \bar{y}_x + 6\mu_{0,2} \bar{y}_x^2 + \bar{y}_x^4 + (\mu_{0,5} - 4\mu_{0,4} \bar{y}_x + 6\mu_{0,3} \bar{y}_x^2 - 4\mu_{0,2} \bar{y}_x^3) \sum_{s=1} \omega_{s,1} X_s \\ &\quad + \left(\int_{-\infty}^{+\infty} \rho(y) Y_2 y^4 dy - 4\bar{y}_x \int_{-\infty}^{+\infty} \rho(y) Y_2 y^3 dy + 6\bar{y}_x^2 \frac{\beta_3'}{\beta_2'} \right) \sum_{s=1} \omega_{s,2} X_s \\ &\quad + \left(\int_{-\infty}^{+\infty} \rho(y) Y_3 y^4 dy - 4\frac{\beta_4'}{\beta_3'} \bar{y}_x \right) \sum_{s=1} \omega_{s,3} X_s + \frac{\beta_5'}{\beta_4'} \sum_{s=1} \omega_{s,4} X_s. \end{aligned}$$

Pour simplifier cette formule il suffira de calculer $\omega_{s,4}$, car nous connaissons les autres :

$$\omega_{s,4} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s Y_4 dx dy}{\frac{\beta_{s+1} \beta_5'}{\beta_s \beta_4'}} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s \begin{vmatrix} 1, & y, & y^2, & y^3, & y^4 \\ \vdots & & & & \end{vmatrix} dx dy}{\beta_5' \frac{\beta_{s+1}}{\beta_s}},$$

et après quelques simplifications, on a :

$$\beta_2(y)_x = \left\{ \begin{array}{l} -3\bar{y}_x^4 + d \sum_{s=1} \omega_{s,1} X_s + 6\bar{y}_x^2 \sum_{s=0} \begin{vmatrix} 1, & \dots, & x^s \\ \vdots & & \end{vmatrix} \cdot \begin{vmatrix} \mu_{0,2}, & \dots, & \mu_{s,2} \\ \vdots & & \end{vmatrix} \\ -4\bar{y}_x \sum_{s=0} \begin{vmatrix} 1, & \dots, & x^s \\ \vdots & & \end{vmatrix} \cdot \begin{vmatrix} \mu_{0,3}, & \dots, & \mu_{s,3} \\ \vdots & & \end{vmatrix} + \sum_{s=0} \begin{vmatrix} 1, & \dots, & x^s \\ \vdots & & \end{vmatrix} \cdot \begin{vmatrix} \mu_{0,4}, & \dots, & \mu_{s,4} \\ \vdots & & \end{vmatrix} \end{array} \right\} \\ = \frac{\sigma_{\mu_x}^4}{\beta_s \beta_{s+1}},$$

avec :

$$d = \frac{\mu_{0,2}^3 \mu_{0,3} \mu_{0,5} (\mu_{0,2})^{-1} (\mu_{0,3}^3 + \mu_{0,4} \mu_{0,5})}{\beta_3' \beta_4'}.$$

III.

Jusqu'ici nous n'avons pas précisé des formes particulières pour les fonctions marginales $\psi(x)$ et $\rho(y)$, et nos développements ont un caractère générale. Cependant il est facile de se convaincre que selon le choix que l'on fait pour ces fonctions en adoptant diverses courbes du répertoire de Pearson, l'on aboutira à des formules concrètes qui constitueront un ensemble très complet pour les applications. Nous n'allons pas accomplir cette tâche dans cet article, la laissant pour une occasion prochaine qui se présentera lors d'une communication dans laquelle nous pensons pouvoir le faire, en donnant un groupe d'applications numériques.

Pour terminer nous allons nous occuper des changements qu'il faudrait faire aux formules données pour pouvoir les conserver en toute leur rigueur dans les cas quand on traite des surfaces de corrélation avec des marginales ajustées par les fonctions de Pearson, mais sans qu'il s'agisse d'une véritable interpolation, comme dans la théorie l'on a dû supposer.

Au moyen de la série double (27) le calcul des coefficients $\omega_{s,j}$ fait par la méthode de Fourier donnait un minimum à la valeur :

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{f(x,y)}{\sqrt{\psi(x)\rho(y)}} - \sqrt{\psi(x)\rho(y)} \sum_s \sum_j \omega_{s,j} X_s Y_j \right]^2 dx dy,$$

c'est à dire alors que si l'on étendait la série double jusqu'aux termes tels que $s+j \leq k$, le degré de l'approximation serait mesuré par la valeur de :

$$\epsilon_k = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{[f(x,y)]^2}{\psi(x)\rho(y)} dx dy - \sum_s \sum_j^{s+j \leq k} \omega_{s,j} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) X_s(x) Y_j(y) dx dy.$$

Occupons-nous maintenant des corrections annoncées.

Soit $f(x, y)$ la surface de corrélation et:

$$\int_{-\infty}^{+\infty} f(x, y) dy = f_x; \quad \int_{-\infty}^{+\infty} f(x, y) dx = f_y,$$

f_x et f_y étant les fréquences marginales expérimentales. Supposons que—en les ajustant par la méthode des moments—on arrive aux fonctions:

$$\psi(x) \text{ et } \rho(y).$$

On aura alors:

$$\int_{-\infty}^{+\infty} f_x x^s dx = \int_{-\infty}^{+\infty} \psi(x) x^s dx,$$

$$\int_{-\infty}^{+\infty} f_y y^j dy = \int_{-\infty}^{+\infty} \rho(y) y^j dy,$$

pour s et $j = 0, 1, 2, 3$ et 4 .

Supposons que nous conservons le développement déjà étudié pour la surface $f(x, y)$. Alors, comme expression de la ligne de régression, on aura:

$$\bar{y}_x = \frac{\int_{-\infty}^{+\infty} f(x, y) y dy}{\int_{-\infty}^{+\infty} f(x, y) dy},$$

mais:

$$\int_{-\infty}^{+\infty} f(x, y) y dy = \psi(x) \sum_s X_s \sum_j \omega_{s,j} \int \rho(y) Y_j y dy,$$

et, tenant compte des avertissements déjà faits, on aura:

$$\int_{-\infty}^{+\infty} f(x, y) y dy = \sigma_y^2 \psi(x) \sum_{s=0} \omega_{s,1} X_s.$$

Analoguement on aura:

$$\int_{-\infty}^{+\infty} f(x, y) dy = \psi(x) \sum_{s=0} \omega_{s,0} X_s(x),$$

d'où:

$$\bar{y}_x = \sigma_y^2 \frac{\sum_{s=0} \omega_{s,1} X_s(x)}{\sum_{s=0} \omega_{s,0} X_s(x)}.$$

On voit que le cas particulier de ce développement rigoureux pour la ligne de régression arrive quand le dénominateur se réduit à l'unité, c'est à dire, quand $\omega_{s,0} = 0$ pour $s = 1, 2, 3, 4, \dots$

De semblables corrections sont faciles à donner pour les autres équations.

THE USE OF CONFIDENCE OR FIDUCIAL LIMITS ILLUSTRATED IN THE CASE OF THE BINOMIAL.

BY C. J. CLOPPER, B.Sc., AND E. S. PEARSON, D.Sc.

(1) *General Discussion.*

In facing the problem of statistical estimation it may often be desirable to obtain from a random sample a single estimate, say a , of the value of an unknown parameter, α , in the population sampled. It has always, however, been realised that this single value is of little use unless associated with a measure of its reliability and the traditional practice has been to give with a its probable error (or more recently its standard error), in the form

$$a \pm p.e(a) \dots\dots\dots(1).$$

From this information it was possible, if the sample was not too small, to draw the conclusion that the unknown value of α lay within the limits

$$\alpha_1 = a - 3 \times p.e(a) \text{ and } \alpha_2 = a + 3 \times p.e(a) \dots\dots\dots(2)$$

with a high degree of probability. But it was neither easy to give any precise definition of this measure of probability nor to assess the extent of error involved in estimating the value of $p.e(a)$ from the sample.

The recent work of R. A. Fisher introducing the conception of the fiducial interval has made it possible under certain conditions to treat this problem of estimation in a simple yet powerful manner*. It is proposed in the present paper to illustrate on the following problem the ideas involved in this method of approach.

A sample of n units is randomly drawn from a very large population in which the proportion of units bearing a certain character, A , is p . In the sample x individuals bear the character A and $n - x$ do not. p is unknown and the problem is to obtain limits p_1 and p_2 such that we may feel with a given degree of confidence that

$$p_1 < p < p_2 \dots\dots\dots(3).$$

In the first place, how is this degree of confidence to be defined? The underlying conception involved in all problems of this type is extremely simple. In our statistical experience it is likely that we shall meet many values of n and of x ; a rule must be laid down for determining p_1 and p_2 given n and x . Our confidence that p lies within the interval (p_1, p_2) will depend upon the proportion of times that this prediction is correct in the long run of statistical experience, and this

* R. A. Fisher, *Proc. Camb. Phil. Soc.* 26 (1930), p. 528; *Proc. Roy. Soc.* A 139 (1933), p. 343. References to the discussion of these concepts in lectures may also be found in papers published by students of J. Neyman. See for instance pp. 28--29 of a paper by W. Pytkowski written in 1929--30 and published at Warsaw in 1932, entitled, "The dependence of the income in small farms upon their area, the outlay and the capital invested in cows."

may be termed the *confidence coefficient*. Thus subject to certain approximations discussed below, arising from the fact that x can assume only discrete integral values in this particular problem, it is possible to choose the fiducial or confidence limits p_1 and p_2 in such a manner that, for example, the prediction

(1) will be correct in 95% of cases met with in the long run of experience, and wrong in 5%, in 2.5% because $p \leq p_1$, and 2.5% because $p \geq p_2$.

Or again,

(2) will be correct in 99% of cases and wrong in 1%, in 0.5% because $p \leq p_1$, and in 0.5% because $p \geq p_2$.

These intervals (p_1, p_2) may be termed either the central* confidence or

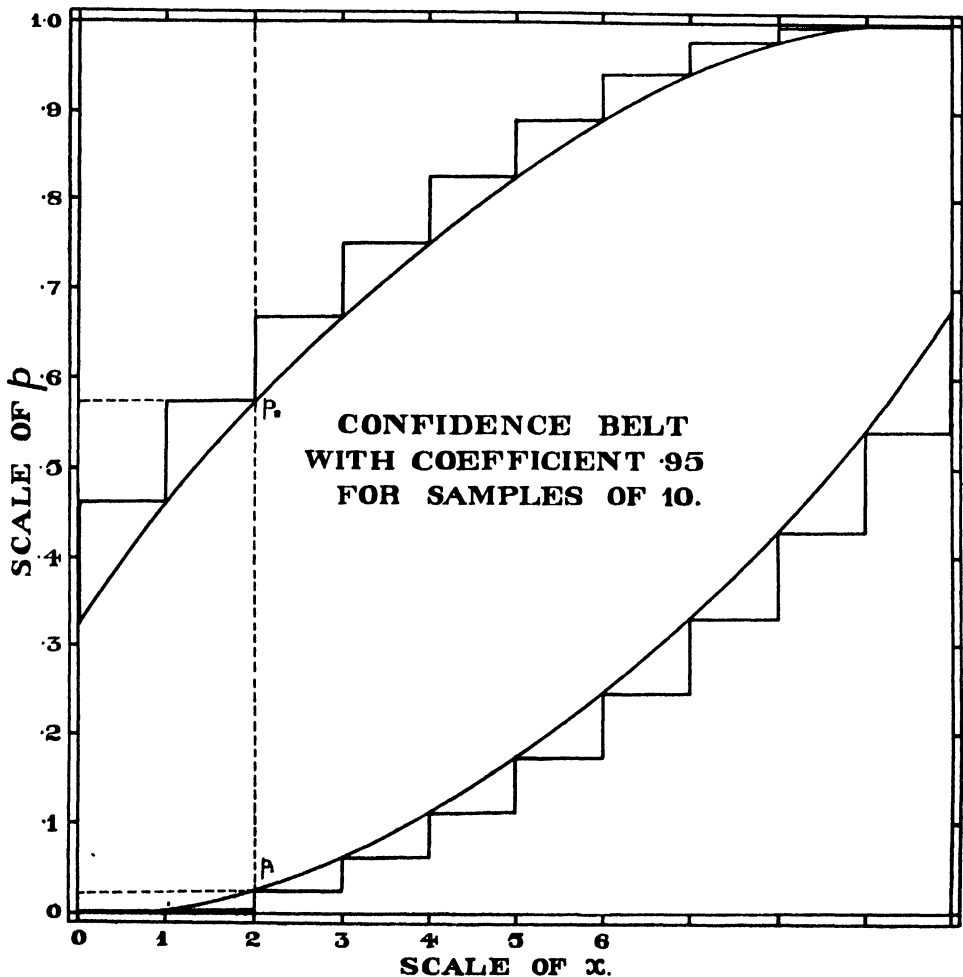


FIG. 1

* In the charts described below the coefficients .95 and .99 were chosen as giving two useful pairs of limits. It is not essential that the intervals chosen should be "central," but for many purposes this appears to be the most convenient arrangement.

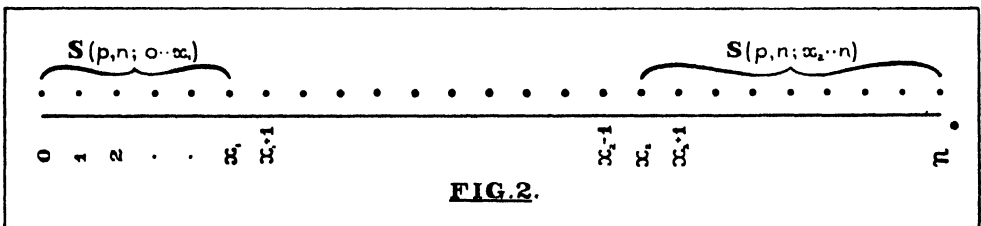
central fiducial intervals and are associated with confidence coefficients of .95 and .99 respectively. In his development of the subject, R. A. Fisher has used the term "fiducial probability" to describe the chance that in the long run a correct prediction will be made of the limits within which the unknown parameter falls. The concept of fiducial probability cannot, it appears, be distinguished from that of ordinary probability, and it seems possible that the use of this term may lead to some misunderstanding, especially when associated with a "fiducial distribution." We are inclined therefore to adopt the terminology suggested by J. Neyman, and to convey what is fundamentally the same notion by specifying the confidence coefficient associated with an interval. Thus the confidence coefficient may be regarded as a particular value of the fiducial probability selected to form the basis of the calculation, to be employed in repeated experience, of the confidence interval*.

The method of solution of the problem may be illustrated with the help of Fig. 1, in which $n = 10$; p and x have been taken as coordinate axes, so that p may lie between 0 and 1, while x may assume any of the integral values 0, 1, ... 10. In our experience with samples of 10 individuals, no point (x, p) can lie outside the square of the diagram. For a given value of p , the chance of occurrence of different values of x will be given by the terms of the binomial expansion $(q + p)^{10}$. Let (a) $S(p, n; 0 \dots x)$, and (b) $S(p, n; x \dots n)$, denote the sum of (a) the 1st $x + 1$, and (b) the last $n - x + 1$ terms. Then while it will not in general be possible to choose values of x_1 and x_2 so that both $S(p, n; 0 \dots x_1)$ and $S(p, n; x_2 \dots n)$ equal exactly some selected value, say .025, it will be possible to choose x_1 and x_2 so that

$$S(p, n; 0 \dots x_1) \leq .025 < S(p, n; 0 \dots x_1 + 1) \dots \dots \dots (4),$$

$$S(p, n; x_2 \dots n) \leq .025 < S(p, n; (x_2 - 1) \dots n) \dots \dots \dots (5).$$

The position is illustrated diagrammatically below :



If it is supposed that such values are determined for x_1 and x_2 throughout the whole range, $p = 0$ to 1, we shall obtain two series of stepped lines running across the diagram as shown in Fig. 1, all points on which satisfy conditions (4) and (5) respectively. It follows that in the long run of our statistical experience from whatever populations random samples of 10 are drawn, we may expect at least 95% of the points (x, p) will lie inside the lozenge shaped belt, not more than

* J. Neyman, "On the two different aspects of the representative method: the method of stratified sampling and the method of purposive selection," *Journal of Royal Statistical Society*, xcvi. pp. 558—606, 1934.

$2\frac{1}{2}\%$ on or above the upper boundary and not more than $2\frac{1}{2}\%$ on or below the lower boundary. If then as a general rule, when x alone is known these boundaries are used to determine points (x, p_1) and (x, p_2) , we may have confidence that we shall be correct in the estimate $p_1 < p < p_2$ in about 95% of cases. If greater confidence is desired, we may determine wider limits leading to a higher value of the expected percentage accuracy, e.g. 99%. In the diagram, values of p_1 and p_2 corresponding to $x = 2$ are shown.

This plan has been carried out below with the following modifications adopted for practical convenience:

(1) The charts prepared are entered with p and $\theta = x/n$, so that $0 \leq \theta \leq 1$, and boundaries for a number of values of n can be drawn on the same chart.

(2) Instead of the stepped boundaries, curves have been drawn as in Fig. 1 passing through the inner "corner" points, i.e. the points (x_1, p) and (x_2, p) for which p is such that $S(p, n; 0 \dots x_1)$ and $S(p, n; x_2 \dots n)$ are exactly equal to the desired chance (in the cases chosen, these chances are .025 and .005). These curves are more convenient than the stepped lines for interpolation for intermediate values of n . Since no possible point (x, p) can fall inside the area between a curve and the steps, no error is involved in using the curves.

(3) While the "corner" points could have been calculated precisely and the curves drawn through them, it was considered sufficiently accurate for the purpose to obtain the curves by an approximate method of interpolation described below.

Before describing the charts and illustrating their use, it may be well to make clear the sense in which this method of estimation in terms of a confidence or fiducial interval does not depend on any *a priori* knowledge regarding possible values of p . Consider the following situation. Suppose that in the course of our experience samples of 30 are continually drawn, and that although we are not aware of the fact, these are taken from populations in which p has three different values only, namely $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$. Further that the proportions of times these three cases are met with are as $\frac{1}{10} : \frac{1}{10} : \frac{1}{10}$ respectively.

The expectation, on a basis of 10,000 draws, is shown in Fig. 3, in which the axes of p and x have been reversed for convenience. For example, for $p = \frac{1}{3}$, $x = 12$, the expectation is 881, while for $p = \frac{2}{3}$, $x = 28$, it is 1. The chart of Fig. 4 described below will provide for each of the 31 possible values of x the limits for the confidence interval, with coefficient .95, for p . Thus when $x = 15$, we find $p_1 = .31$, $p_2 = .69$. Taken over the whole experience, these intervals include the true population value of p in 9676 out of the 10,000 cases, and in the remaining 324 do not, that is to say we are wrong in less than 5% of cases. This is the risk of error that we have accepted, and it is quite independent of the particular set of three values of p introduced, or the relative frequency with which they are encountered in our experience.

It will be noticed, however, from the figures in the margin, that the percentage of wrong judgments differs according to the value of x , from 100 to 0. We cannot

THE CONFIDENCE BELT AND A PRIORI PROBABILITY.

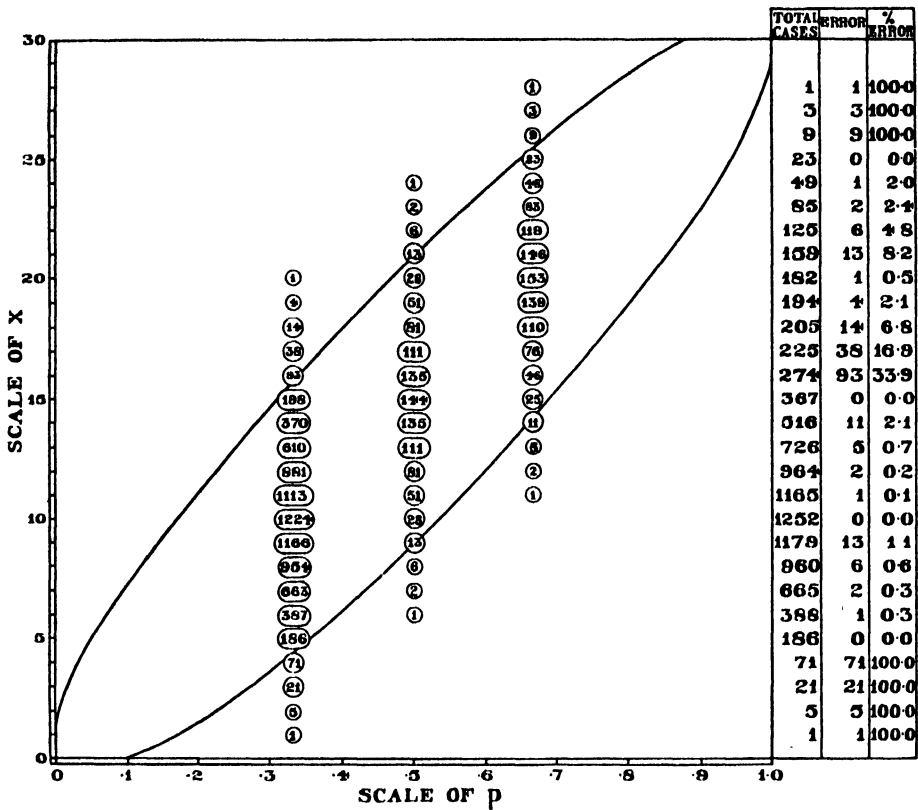


FIG. 3.

therefore say that for any specified value of x the probability that the confidence interval will include p is .95 or more. The probability must be associated with the whole belt, that is to say with the result of the continued application of a method of procedure to all values of x met with in our statistical experience.

Indeed it will be clear that *if* we had information *a priori* regarding the values of p likely to be met in our experience, and if this information could be expressed in precise numerical form, it would be possible to shift the confidence belt and so narrow the limits of uncertainty while retaining the same risk of error. For instance, if we knew that $\frac{1}{3} \leq p \leq \frac{2}{3}$, we should certainly cut off the two points of the lozenge by lines at $p = \frac{1}{3}$ and $p = \frac{2}{3}$.

In practice, however, it is rare

(1) for the *a priori* information to be expressed in exact form,

(2) even when it appears so expressible, for the working statistician to have time to calculate suitable modification for the limits.

Under general conditions, therefore, the statistician will usually be satisfied with limits which are "safe" in the sense that they give an expectation of long run

accuracy which is precisely known*, and thus avoid the uncertain risk of error involved in an attempt to introduce *a priori* information.

(2) *Calculation and use of the charts.*

The following method was employed in obtaining points from which to draw the curves in Figs. 4 and 5.

Samples with $n = 10, 15, 20, 30$.

Use was made of the tables giving the continued sum of the binomial terms, published in one of the Medical Research Council's Reports†. From these tables it is possible to find the sum of any number of binomial terms for $p = .025, .05, .075, .10, .15, .20, \dots, .85, .90, .925, .95, .975$. It will happen only rarely that for these values of p , $S(p, n; 0 \dots x_1)$ or $S(p, n; x_2 \dots n)$ approach the desired values of $.025$ or $.005$, i.e. that we can obtain directly the inside "corners" of the steps of Fig. 1. For the purpose of the charts, however, it was considered that sufficient accuracy would be obtained by interpolation for x in the tables. Take for example the case of $n = 20$ and consider the sums of the binomial terms for $p = 0.45$ given below. At what points should the two curves associated with $n = 20$ cut the lines

x	$S(.45, 20; 0 \dots x)$	x	$S(.45, 20; 0 \dots x)$
0	.0000	8	.4143
1	.0001	9	.5914
2	.0009	10	.7507
3	.0049	11	.8692
4	.0189	12	.9420
5	.0553	13	.9786
6	.1299	14	.9936
7	.2520	15	.9985

$p = .45$ in the charts? The point $x = 3$ ($x/n = .150$) is approximately a "corner" point, since the sum of the first 4 terms equals almost exactly $.005$, but the other points must be obtained by interpolation. Thus we argue:

(a) *The lower .025 point.* The sum of the terms $0 \dots 4$ is $.0189 (< .025)$, and the sum of the terms $0 \dots 5$ is $.0553 (> .025)$; a linear interpolation‡ gives for x , 4.17 ($x/n = .208$).

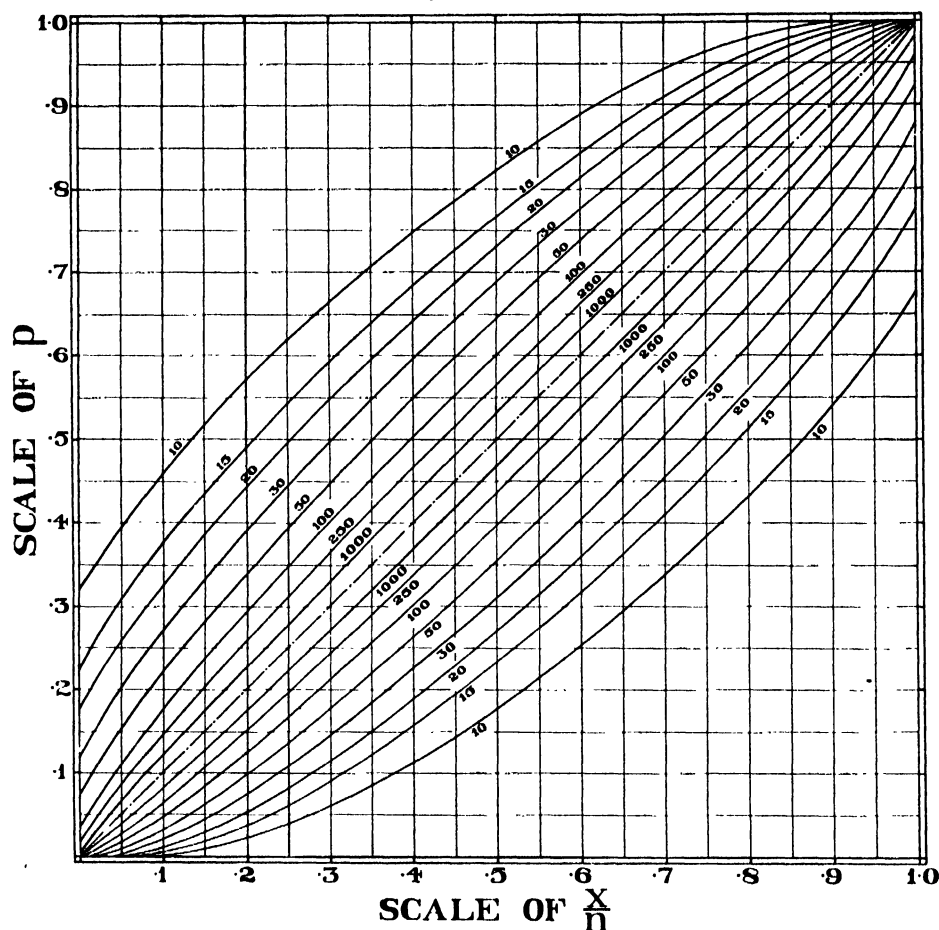
(b) *The upper .025 point.* The sum of the terms $13 \dots 20$ is $1 - .9420 = .0580$, and the sum $14 \dots 20$ is $1 - .9786 = .0214$. Take $x = 13.90$ ($x/n = .695$).

(c) *The upper .005 point.* The sum of the terms $15 \dots 20$ is $1 - .9936 = .0064$, and the sum $16 \dots 20$ is $1 - .9985 = .0015$. Take $x = 15.29$ ($x/n = .764$).

* This is not strictly true of course, since only an upper limit to the error is known, owing to the fact that x can assume discrete values only. As n increases, however, the true risk will rapidly approach the limiting value. In cases where the coefficient, x , is a continuous variable such as a sample mean or standard deviation, this difficulty does not arise.

† Reports on Biological Standards, "II. Toxicity Tests for Novarsenobenzene," by Durham, Gaddum and Marchal.

‡ This form of interpolation, if crude, appeared adequate for the curve drawing.

CONFIDENCE BELTS FOR p (CONFIDENCE COEFFICIENT $\cdot 95$)**FIG. 4.**

Consequently in Fig. 4 the curves marked $n=20$ cut the line $p=.45$ at $x/n=.208$ and $.695$, and in Fig. 5 at $x/n=.150$ and $.764$.

Fresh calculations of binomial terms were made for $n=50$, 100 and 250, while the limits were obtained from the normal curve in the case $n=1000$.

It will be noted that the curves cut the axis $x/n=0$ at points at some distance from $p=0$ when n is small. The points of intersection correspond in the two diagrams to those values of p for which the first term of the binomial $q^n = (1-p)^n$ equals $\cdot 025$ and $\cdot 005$ respectively. On the other side, the end points on the axis $x/n=1$ correspond to values of p for which the last term, p^n , equals $\cdot 025$ and $\cdot 005$.

The charts have been prepared to give rapid answers in problems such as the following:

(1) A sample has been drawn (n and x known), to obtain the confidence or fiducial interval for p .

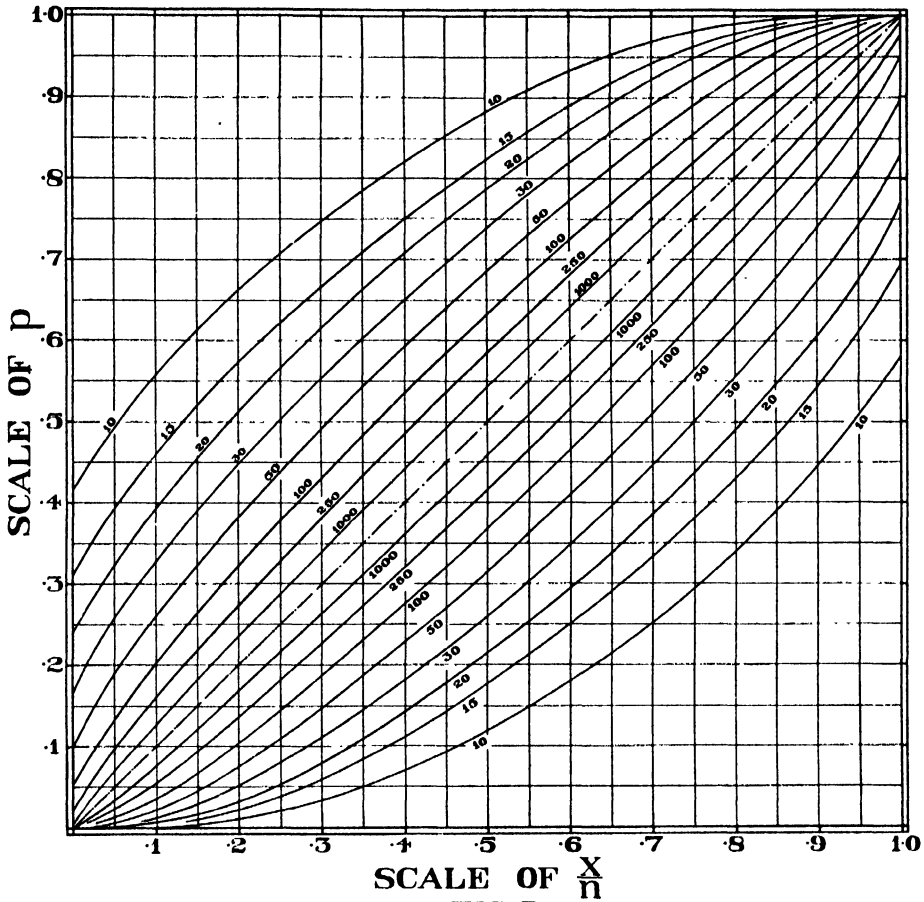
CONFIDENCE BELTS FOR p (CONFIDENCE COEFFICIENT .99)

FIG. 5.

Example A. The toxicity of a drug may be measured by the proportion, p , of mice in a standard laboratory population that will die after injection with a dose of given strength. Out of a sample of 30 mice randomly selected from the population, 8 die after injection; within what limits may we expect that p lies? Turning to Fig. 4, and taking $n = 30$, $x/n = 8/30 = .267$, it will be seen that we may say that $.12 < p < .46$, if we are prepared to accept a risk of error of not more than 1 in 20. To obtain greater confidence in prediction (risk of error 1 in 100) we must turn to Fig. 5 and obtain $.09 < p < .52$.

(2) To plan in advance the size of sample necessary to provide a desired degree of accuracy in estimation.

Example B. In a manufacturing process a crude index of quality, P , has been the percentage of articles which pass a certain test. This index has fluctuated in

the past round $P = 60$, but it is proposed to make an intensive effort to improve quality (which will mean the raising of this percentage) by tightening the control of manufacture. Improvement is to be judged by studying the changes in the proportion of articles (x/n) passing the test in a random sample of n articles. How large would n need to be to obtain from the sample an estimate of P , with a range of uncertainty of not more than 5?

At the start, the value of $p = P/100$ in the material sampled is not more than .60, and we wish to determine n so that the confidence belt will be of breadth about .05. On the assumption that a confidence coefficient of .95 is adequate, we may use Fig. 4. It will be seen that for x/n having values between .6 and .8, n must be more than 1000 for the interval $p_2 - p_1$ to be as small as .05*. In many cases the testing of so large a sample would be quite out of the question, and this result points to the fact that an index of this type is not an efficient measure of quality. Much more information of changes could probably be drawn from a smaller sample, if the index could be based on the mean value of some measured character determined for each article of the sample.

(3) To determine the limits of sampling variation that may be expected in x when p is known, and so determine the size of sample needed.

Example C. There are two alternative hypotheses regarding the chance of an individual in a certain population bearing a given character; the alternatives are that $p = \frac{1}{4}$ or $p = \frac{1}{2}$. Such might be the case in some genetic investigation. How large a sample must be planned to make it practically certain that we can discriminate between the two hypotheses?

In this case we are concerned with the sampling variation of x for $p = \frac{1}{4}$ and $p = \frac{1}{2}$, and n should be chosen so large that there is no "overlap" of any consequence between the two distributions. Suppose we choose n so that the upper .005 point of the x distribution for $p = \frac{1}{4}$, as judged from the curves of Fig. 5, corresponds to the lower .005 point of the distribution for $p = \frac{1}{2}$. This will occur when n is slightly over 100, say 110†.

* Since for large values of n the upper and lower bounds of the confidence belt are very nearly parallel lines making an angle of 45° with the axes, and the binomial may be represented by a normal curve, the breadth of the belt is approximately $4\sqrt{p(1-p)/n}$, which if equated to .05 gives $n = 1600$ for $p = .60$ and $n = 1000$ for $p = .80$.

† [It is interesting to consider what the solution would be, if the two binomials, $(p_1 + q_1)^n$ and $(p_2 + q_2)^n$, were replaced by normal curves. The means of these curves will be np_1 and np_2 ($p_2 > p_1$) while their standard deviations will be $\sigma_1 = \sqrt{np_1q_1}$ and $\sigma_2 = \sqrt{np_2q_2}$. Let l represent the overlap and x_1 represent the distance from mean to overlap in first curve and x_2 represent distance from overlap to mean in second curve. Accordingly

$$n(p_2 - p_1) = x_1 + x_2 \dots\dots\dots(i).$$

If l be the overlap, and $\frac{1}{2}(1 - \alpha_1)$ be the area cut off from the first curve and $\frac{1}{2}(1 - \alpha_2)$ from the second curve, it will be reasonable to take

$$\frac{1}{2}(1 - \alpha_1) = \frac{1}{2}(1 - \alpha_2) = \frac{1}{2}l.$$

Thus x_1/σ_1 and x_2/σ_2 must be obtained from the tables of the normal probability integral with $\frac{1}{2}(1 + \alpha) = 1 - \frac{1}{2}l$, or say they have the value ξ .

If we were prepared to accept a greater risk of an inconclusive result, which we might well be prepared to do if the sample could be readily increased in size in a doubtful case, then we might choose n so that the upper and lower .025 points of the x distributions correspond. Turning to Fig. 4, it is found that this occurs when n is about 65.

It follows from (i) that

$$n(p_2 - p_1) = \xi(\sqrt{np_1q_1} + \sqrt{np_2q_2})$$

or

$$\sqrt{n} = \frac{\xi(\sqrt{p_1q_1} + \sqrt{p_2q_2})}{p_2 - p_1} \dots \dots \dots (ii).$$

In the case in the text $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $l = .005$, $1 - \frac{1}{2}l = .9975$, which corresponds to $\xi = 2.81$ nearly. Thus $\sqrt{n} = 2.81(\sqrt{3} + 2) = 2.81 \times 3.7205 = 10.455$ and $n = 109.3$, according well with the value in the text. If we desire to alter the overlap, the first factor ξ only is changed in (ii). If $l = .025$, then $\frac{1}{2}(1 + \alpha) = .9875$ and $\xi = 2.2416$, $\sqrt{n} = 2.2416 \times 3.7205 = 8.34$ and $n = 69.6$, or 70 as against 65 of text above. ED.]

THE ROUMANIAN SILHOUETTE.

By MARIOARA PERTIA AND OTHERS.

THE Editor of *Biometrika* is glad to have the opportunity of presenting to his readers two further racial silhouettes—those of the Roumanians from Old Roumania and from Transylvania, which latter before the Great War was part of Hungary.

SKETCH MAP OF OLD AND NEW ROUMANIA.



The circumstances under which these silhouettes have been prepared are the following. Mlle Marioara Pertia came in 1930 to the Biometric Laboratory, as a post-graduate student for special statistical training. The then Director of the Laboratory was able to interest her in anthropological statistics, and in particular made inquiries as to the possibility of obtaining a series of Roumanian silhouettes, on which to base a type silhouette. Miss Pertia took great interest in the proposal, and the nature of the silhouetting apparatus in the Biometric Laboratory as well as the system of coordinates employed for the measurement of the life-sized silhouettes were very fully explained to her, and training in taking the measure-

ments was given by Dr G. M. Morant. On returning to Cluj in Roumania Mlle Pertia was able to interest Professor Victor Papilian, Director of the Department of Anatomy in the Faculty of Medicine in the University of Cluj, in the proposal. He most generously placed at Mlle Pertia's disposal not only a room for the work, and apparatus built on the Biometric Laboratory model, but all material needs, and invited the students of his department in their first and second years to have their silhouettes taken. Thus most of the hundred silhouettes taken were those from students registered in the Faculty of Medicine at Cluj. Without Professor Papilian's active interest in the scheme the work could not have been carried out. His claim to an ample share in the production of the silhouettes is here most fully acknowledged*. Fifty of the silhouetted students were from Old Roumania, pure Roumanians for three generations, parents, grandparents and great-grandparents. A second fifty students were from Transylvania including the Banat; these also had three generations of pure Roumanians behind them. One of the points of interest to be considered was whether the Roumanians from either side of the Transylvanian Alps were sensibly differentiated. Of course the term "pure Roumanian" is of doubtful significance. Independent Roumania is of very recent growth, and we should anticipate considerable Slav, Greek, Turk and Magyar influences on the population in various districts of the Roumania of today. All we understand by the words "pure Roumanian" is that there has been among their direct ancestors no intermarriage with persons of Slav, Greek, etc. blood for at least three generations. Until we have silhouettes of the adjacent and intermingled racial groups, it will not be possible to measure the degree of divergence between the Roumanians and these groups.

The entire work of measuring the hundred silhouettes and taking the means of their numerous coordinates was carried out by Miss Pertia and from her measurements Figs. 1 and 2, with the silhouettes themselves have been constructed. The original life-size silhouettes were prepared by Miss Ida McLearn (Mrs Fraser Larmor) and accurately reduced to half-scale.

In order to remind the reader of the terms used in our silhouette analysis the figure from the initial paper on this subject in *Biometrika*† is reproduced here.

Fig. 1 gives for Old Roumanians the mean values of the various lengths upon which the plotting of the type contour is based. The full description of the manner in which this coordinate system is arranged is given, together with the names selected for various points of the silhouette, in *Biometrika*, Vol. xx^B (1928), pp. 389—397. The Frankfurt Horizontal makes an angle of $12^{\circ}6'$ with the line AN , joining the auricular point A to the hyperrhinion. In the silhouette formed from this index figure the

* With the apparatus installed at Cluj it would be of much interest if the members of other Balkan races could be silhouetted, e.g. Magyars in Transylvania, and Bulgars in Dobrogea.

† "On the Importance of the Type Silhouette for Racial Characterisation in Anthropology." *Biometrika*, Vol. xx^B, pp. 389—400. The figure is on p. 390. A further paper used below is "The Albanians of the North and South," by Miriam Tildesley and others, *Biometrika*, Vol. xxv, pp. 21—51. Both these papers like the present one contain tissues of the silhouettes, admitting of easy superposition. These are the bases of the comparisons made in this paper.

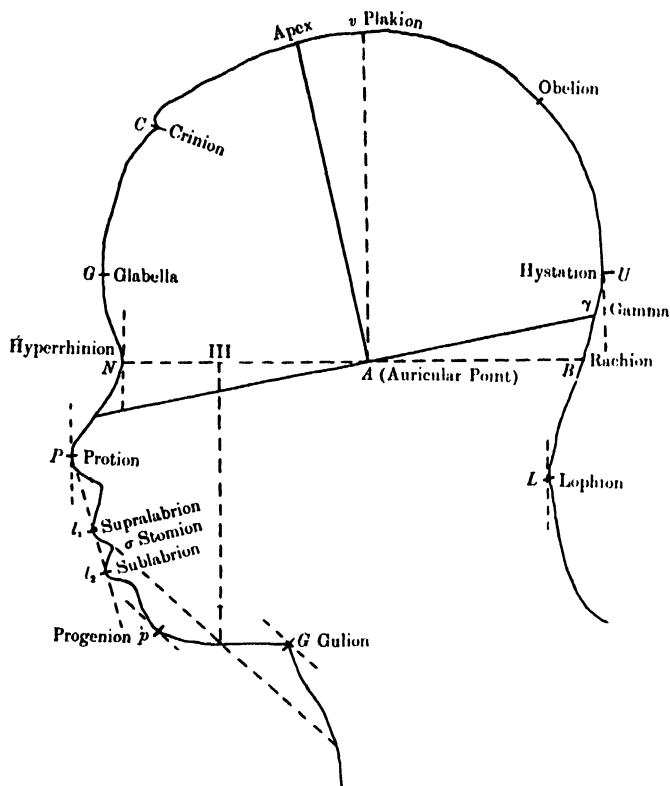


Diagram of Cephalic Points on Silhouette.

Frankfurt Horizontal was taken parallel to the bottom edge of the plate. The original type contour constructed to life-size was reduced to half-scale in reproduction: see Plate I. We regret that the draughtsman—Miss Ida McLearn (Mrs Fraser Larmor)—by an oversight was not instructed to add a collar to the neck as in the case of previous silhouettes. It would have rendered comparison merely by sight less disturbing. Fig. 2 gives the mean values of the coordinates of the Transylvanian Roumanians; and Plate II is the corresponding half-scale silhouette, with the Frankfurt Horizontal again made horizontal. The angle between the auricular-hyperrhinion line and the Frankfurt Horizontal is here slightly less, namely $12^{\circ}2$.

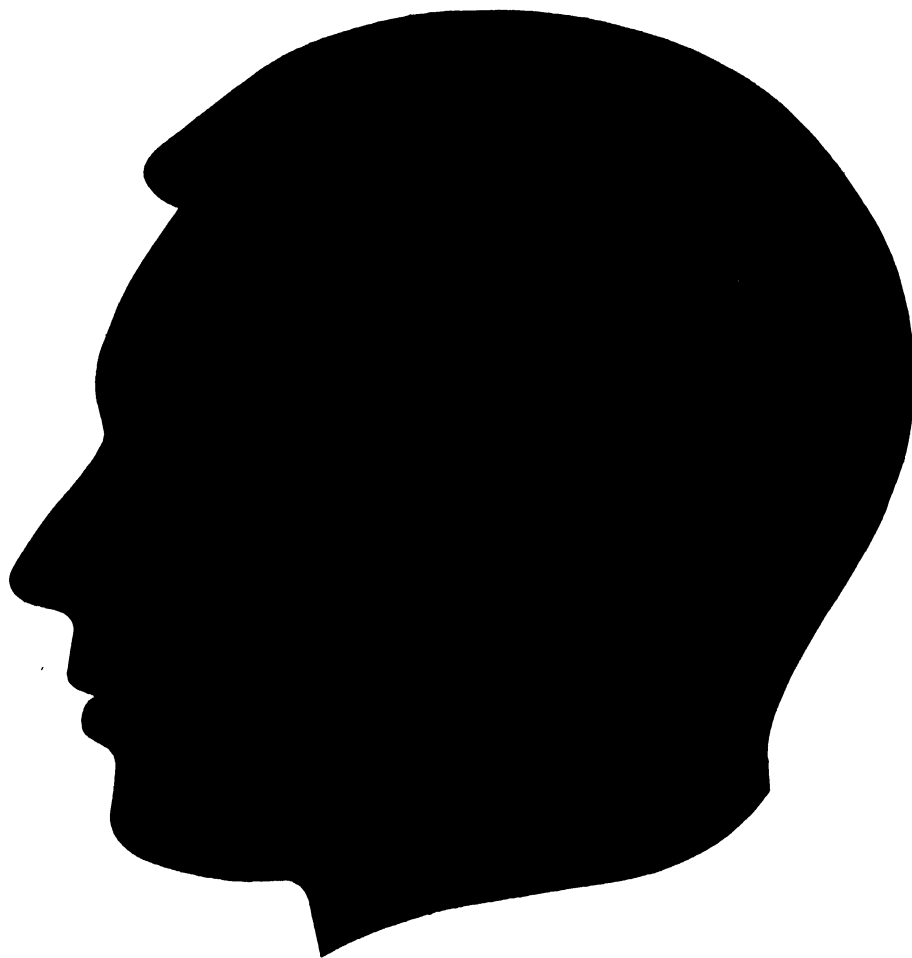
We will now make the few comparisons that it is possible to make at present with the Roumanian silhouettes. We may note that no method of comparison has at present been used as standard; there is not sufficient material available to justify a choice of method. But five methods naturally suggest themselves for trial:

(a) To place auricular point on auricular point and bring the AN lines into contact.

(b) To superpose the hyperrhinions, N , and bring the AN lines into contact.

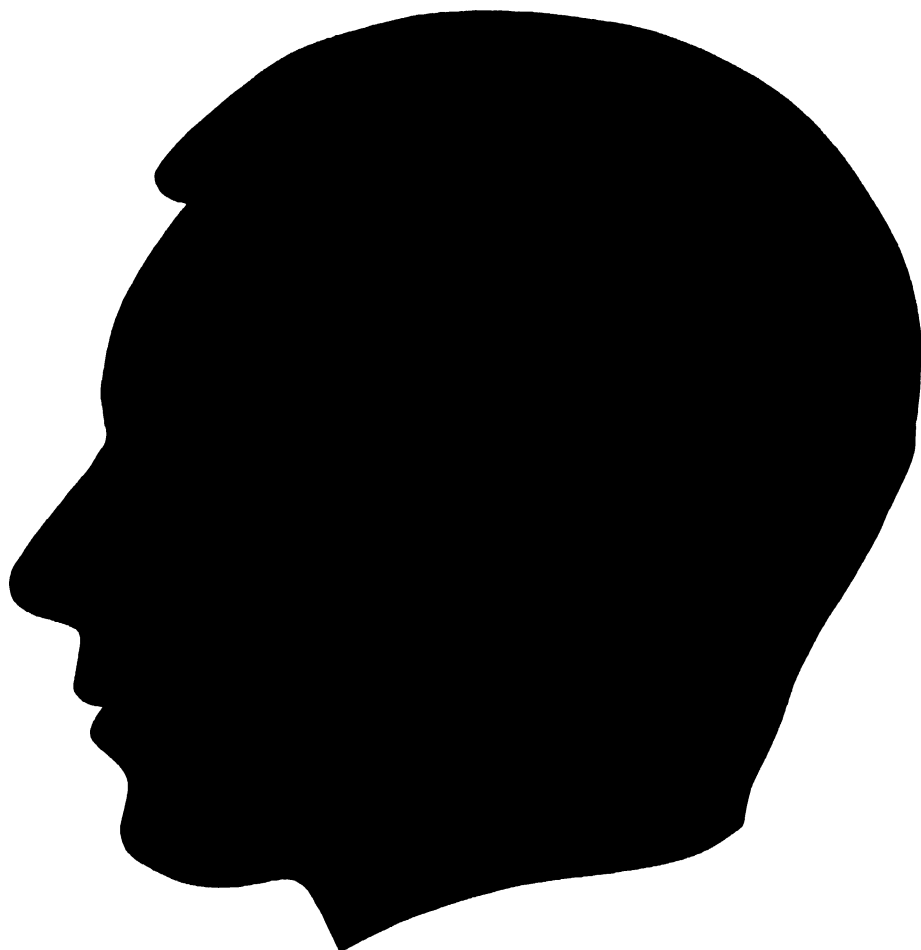
(c) To bring the Frankfurt Horizontals into contact and superpose the A 's.

M. Pertia: *The Roumanian Silhouette*



Type Silhouette, based on fifty male students from Old Roumania.
Reduction Half-Linear.

M. Pertia: *The Roumanian Silhouette*



Type Silhouette, based on fifty male students from Transylvania.

Reduction Half-Linear.

(d) To bring the Frankfurt Horizontals into contact and superpose the nasal terminals of these lines.

(e) To adjust as closely as possible the two faces by superposing the hyperrhinions and turning one tracing round until the maximum closeness is reached, possibly with a slight shift of the point of rotation. This is not difficult in the case of most European groups, because the facial resemblance is great, and it enables us to compare advantageously the relative positions with regard to the face and the relative sizes of the brain-boxes.

All these methods of comparison are easily made by means of the tissues in the pockets at the end of the volumes of *Biometrika* containing racial type silhouettes.

(i) *Comparison of the two local Roumanian races with one another.*

(a) and (b), (c) and (d) give very little difference in effect. They make the forehead of the Old Roumanian slightly recedent and the chin somewhat protuberant in relation to the Transylvanian Roumanian. The divergence might be produced by a slightly more vertical position of the face, possibly by a more upright holding of the body. The shape of the chin, owing to the muscles of the neck, is influenced by the 'angle' at which the head is set to the trunk.

(e) Turning the Old Roumanian type silhouette round the hyperrhinion until the faces almost exactly fit, we find scarcely a difference in the features of the two faces. The ear as judged by the relative positions of the two auricular points is *slightly* higher in the case of the Old Roumanian. The back of the head, however, of the Transylvanian projects from the plakion at the vertex to the lophion at the back of the neck beyond that of the Old Roumanian. Unless the Transylvanians wear longer hair at the back*—and this would hardly apply to the part about the lophion—this seems the only really distinguishing racial difference.

To draw a conclusion from this comparison: We have been unable to discover any marked racial difference in the face between the Old and the Transylvanian Roumanian types. There is a possibility, however, that the latter has a somewhat larger brain-box. The two groups are clearly close "local races."

(ii) *Comparison of the Northern Albanian with the Roumanian Silhouettes.*

(a) With the Old Roumanian type.

There is only a slight difference in angle between the auricular-hyperrhinion line and the Frankfurt Horizontal. Further the auricular-hyperrhinion distance is almost exactly the same for the two races, so that there is again little differentiation to be made between the methods (a), (b), (c) and (d).

(a) and (b) show the receding forehead of the Old Roumanian type and a protruding nose, and the back of the Old Roumanian head protrudes from obelion to rachion (B), but recedes from plakion to obelion. If we try (c) it agrees practically with (a).

* To judge by the hair at the crinion (i.e. the start of the hair on the forehead), the Old Roumanians are the more hirsute.

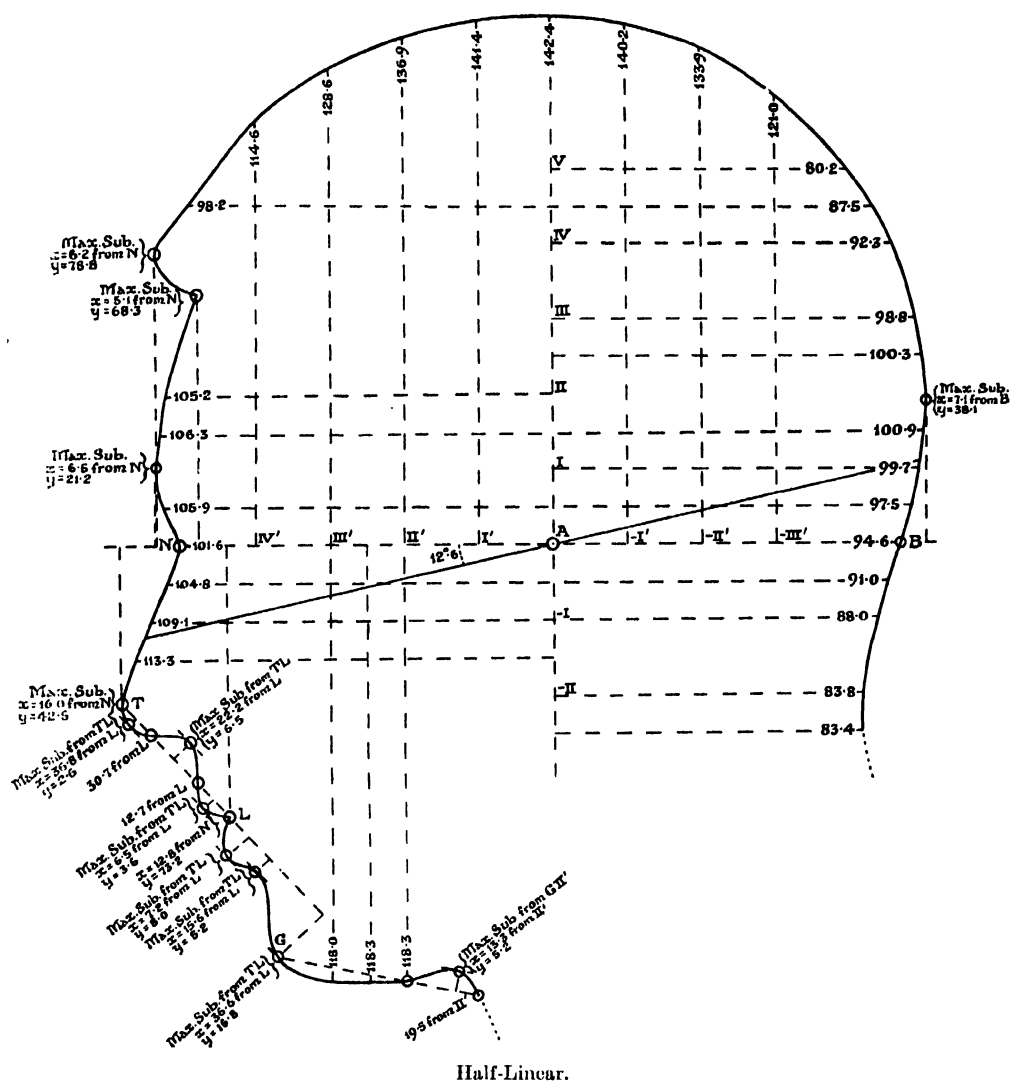
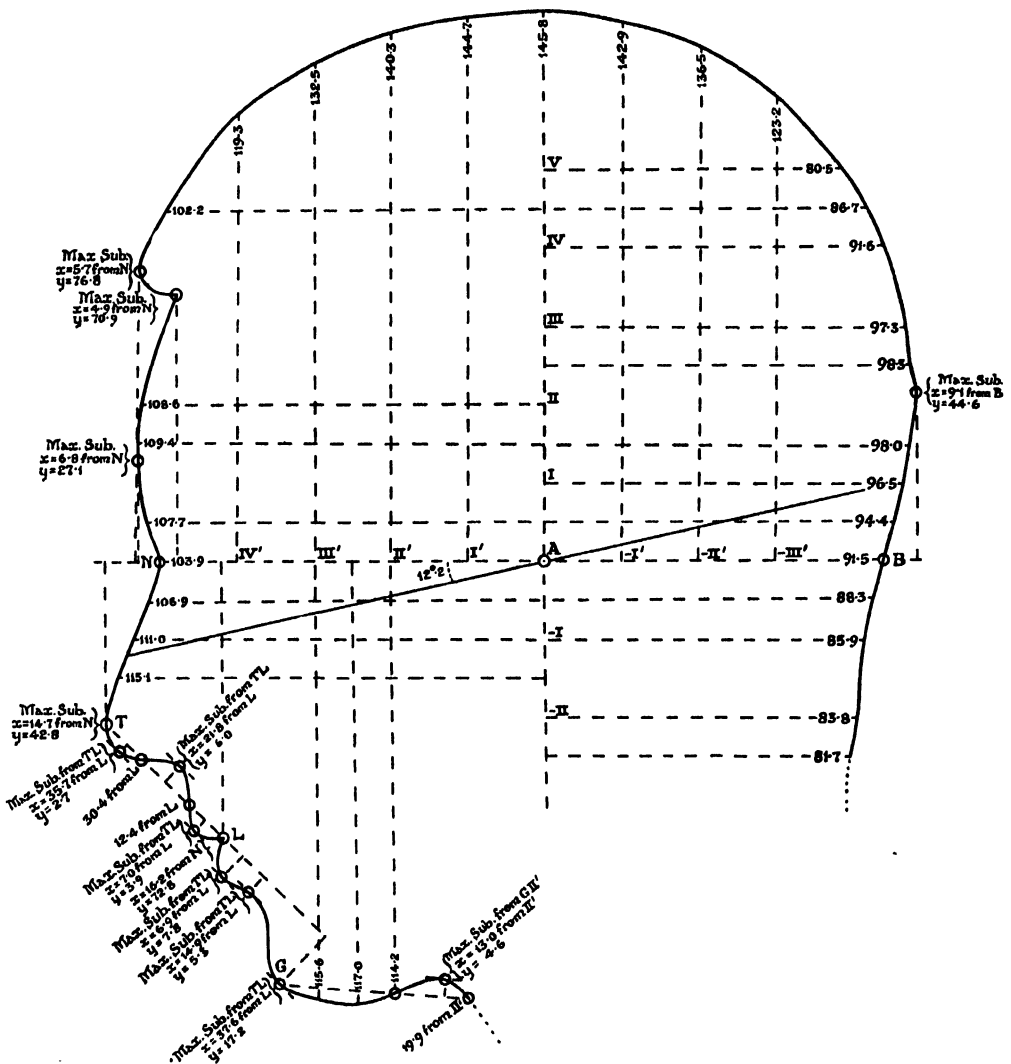


Fig. 1. Type Roumanian α silhouette based on fifty students from Old Roumania,—showing the network of mean coordinates from which it was plotted, and the Frankfurt Horizontal, as indicated by the continuous line through the auricular point A .



Half-Linear.

Fig. 2. Type Roumanian σ silhouette based on fifty students from Transylvania,—showing the network of mean coordinates from which it was plotted, and the Frankfurt Horizontal, as indicated by the continuous line through the auricular point A.

(d) shows us an emphasised recedence of forehead and a lessened but still marked protuberance of lower face from the subnasale to the gulion, while the Old Roumanian brain-box is now in excess from obelion right away to lophion.

(e) Bringing the faces into close contact, we find the facial contours from glabella to subnasale fit each other excellently. The upper lip of the Old Roumanian is longer and the lower lip is also longer and more protuberant. The vertical walls of the chins fit well, but the horizontal base of the mandible from progenion to gulion is higher in the Albanian silhouette. But the most marked racial differences are to be found above the hyperrhinion. The forehead of the Northern Albanian is more nearly vertical (i.e. more infantile) in character, and sensibly higher, the crinion being well above that of the Old Roumanian type. Notwithstanding, from crinion over the top of the head through plakion and obelion right round to rachion, the head of the Old Roumanian is in very considerable excess of the Northern Albanian. Does this connote that the brain-box of the Northern Albanian is less than that of the Old Roumanian? Not at all; for, if starting from the best facial fit in (e), we shift the Albanian tracing up *vertically* so that the two auricular points coincide, there is practically no differentiation in size of the two brain-boxes. If there be a difference in the skull capacity of the two races it must be due to parietal breadth rather than to glabella-occipital length or auricular height.

There are, we judge, sensible racial differences between the Old Roumanian and the Northern Albanian, which might be crudely expressed by saying that while the features of their faces are very similar, the faces themselves appear set at different angles to their brain-boxes.

(β) With the Transylvanian Roumanian type.

(a) The Northern Albanian silhouette lies almost entirely inside the Transylvanian. The forehead of the former being somewhat more vertical does indeed pass outside the latter for the upper third, just below the crinion. There is internal contact with the Transylvanian between plakion and obelion, but otherwise the Northern Albanian falls well inside the Transylvanian from gulion right round to lophion.

(b) The Transylvanian forehead recedes, and from hyperrhinion downwards through protion to gulion, nose, lips and chin protrude. The glabella and the district just above the crinion have contact, but above the bregma to contact just behind the plakion and so onward to the rachion, *B*, the Transylvanian contour shows a larger brain-box. (c) shows little difference except that the bounding lines of the two silhouettes now show no contact above the crinion, and little contact beyond the plakion till we reach the rachion. (d) The forehead of the Transylvanians is now more recedent on the Northern Albanians and the region below the subnasale less so, but the Albanian contour from the vertical through IV' now lies well inside the Transylvanian right round the head to the lophion.

(e) The nose, the forehead halfway up and the upper lip fit reasonably well. The stomion is, however, higher in the Northern Albanians and the chin more

protuberant, the progenions again coinciding. If we slip the tissue up vertically from the (e) position so that the auricular points are on the same level the glabellas practically coincide, but the lower part of the face of the Northern Albanians is shorter than that of the Transylvanians from protion to gulion. Nor is the excess of brain-box in the latter removed.

We conclude that the Northern Albanians differ more widely from the Transylvanians than from the Old Roumanians, having a smaller brain-box in addition to a sensibly smaller lower face. They are indeed more removed from the Northern Albanians than the Old Roumanians are, and this at least is in accordance with geographical position. It is a pity that we have not Serb or Bulgar profiles, to see whether they would provide any intermediate stages. It would be of interest to discover whether the small difference between Old and Transylvanian Roumanian types is due to any admixture of Slav blood with the former or Magyar blood with the latter.

(iii) *Comparison of the Southern Albanian with the Roumanian Silhouettes.*

(a) With the Old Roumanian type.

(a) Here we have a much more marked racial difference. With the exception of the small region above the glabella to the crinion where the vertical forehead of the Albanians crosses anteriorly to the Old Roumanians the entire Albanian contour lies inside the Old Roumanian! There is no doubt that the Old Roumanian type has a much larger head. (b) If we superpose the hyperrhinions and the *NA* lines, the vertical forehead of the Albanians carries the Albanian contour outside the Roumanian from glabella to a point about halfway between the verticals through III' and IV', but the Old Roumanian face from hyperrhinion to gulion is protuberant, and the excess of the Old Roumanians over the Southern Albanians is most remarkable.

(c) Making the Frankfurt Horizontals and the auricular points to coincide, then except for a very small frontal length near the crinion, the Southern Albanian lies entirely inside the Old Roumanian, and in the position (d) this is still nearly true.

(e) Making the closest facial fit, which is very nearly achieved by superposing hyperrhinions and progenions, we see that the faces from hyperrhinion to progenion are in close accord. The vertical forehead of the Albanians again shows a slight frontal excess below the crinion, but from that point right round the head to the lophion the contour of the Old Roumanians stands extravagantly outside that of the Southern Albanians. The upward shift of the auricular point of the latter to the level of that of the former, owing to the very vertical back of the head in the Southern Albanians, while it throws out the correspondence in the facial contours, does practically nothing to reduce the excessive occipital difference.

(β) With the Transylvanian Roumanian type.

(a) With the exception of less than a centimetre at the crinion the whole of the Southern Albanian silhouette lies inside the Transylvanian from gulion right round to lophion. (b) From hyperrhinion to gulion the lower face of the Roumanian protrudes; the glabellas are in contact, but from that point to the crinion the

vertical forehead of the Albanian stands out beyond the Roumanian; thence after contact for about two centimetres of contour, over the vertex round the occipital region to the lophion we have the same remarkable excess of the Transylvanian head. (c) gives the same results as (a), except that it lessens the short distance near the crinion in which the Southern Albanian protrudes. (d) leads to no further information of any importance.

(e) gives a moderately good fitting of the facial contour of the two races from glabella to sublabrion; it shows the projection of the South Albanian chin and its vertical forehead above the glabella, and emphasises the large excess in brain-box of the Transylvanians, if we pay no regard to the parietal breadths.

We may conclude then that both the Transylvanian as well as the Old Roumanian types show a marked racial divergence from the Southern Albanian type which is especially marked by the rounded forehead of the former and the vertical forehead of the latter, and further by the very much larger midsagittal section of the Roumanian brain-boxes. On the other hand the Old Roumanians, while differing racially in some respects from the Northern Albanians, show a greater resemblance to them than the latter do to the Transylvanian Roumanians. The two Roumanian contours are certainly far closer together than the two Albanians are to each other.

It is regrettable that so few comparisons can at present be made with other races. The superposition of the West African negro silhouette with the Roumanian silhouettes would serve no useful purpose. We may, however, compare that of the English students with those of the Roumanian students.

(iv) *Comparison of the English Student Silhouette with those of the Roumanian Students.*

(a) With the Old Roumanian type.

(a) From the vertex to the hystation the contours are in close agreement. From hystation to lopian the Old Roumanian is more protruding. From apex to crinion (both of which are on the same level) onward past nasion, protion down to supralabrion, the English silhouette protrudes considerably. The mouth and chin of the two silhouettes show contact to progenion after which to gulion the Old Roumanian chin drops below the English. (b) Sliding the English A backwards horizontally until the hyperrhinions are superposed, it will be seen that from protion to glabella there is now close contact and even the crinions are but slightly apart; from protion to gulion, however, the Old Roumanian lower face protrudes. But now from crinion to IV' the English protrudes, from IV' to plakion the Old Roumanian, and beyond the English right round to lophion. (c) In this case the entire English face protrudes except from supralabrion to gulion, where the Old Roumanian is slightly in front. There is a small protrusion of the Old Roumanian on the occipital side. On the other hand if we take (d) as our criterion, there is very little divergence in the occipital region. But the English frontal protrudes from glabella to well above the crinion, the nose is longer, but from subnasale to gulion the Old Roumanian is protuberant.

(e) If we make a close facial fit, which may be done fairly well by superposing glabellas and progenions, it will be seen that the English forehead is more vertical than the Old Roumanian, the nose and upper lip are longer, the stomion lower and thus the chin shorter than among the Old Roumanians. The English contour projects at the occipital, and is much less high from the vertical at IV' to the obelion.

On the whole it seems to us that (d) gives the best comparative position, and in this case the longer nose, the recedent lips and chin, associated with occipital excess are the characters which distinguish the English from the Old Roumanians.

We see, however, that the differentiation of English from Old Roumanians is scarcely greater than that between the Old and Transylvanian Roumanians, while distinctly less than that between the Old Roumanians and the Southern Albanians.

(β) With the Transylvanian Roumanian type.

(a) The English silhouette is very close to the Transylvanian especially in the facial portion, but the latter exceeds the former in height of head from the vertical through IV' even to the hystation. (b) The facial correspondence is still further improved, but the English are seen to have a somewhat longer nose. The greater height of the Roumanians is maintained but from hystation to loption the English contour protrudes, reaching a maximum at the rachion. (c) The English facial contour protrudes, the Roumanian still indicates the greater height. (d) Almost as good a fit facially as from (e), Roumanian excess at crown and smaller English excess at rachion maintained, but slightly longer English nose and protrusion at chin.

(e) Superpose stomion and hyperrhinion for the two silhouettes, the facial fit from glabella to near the progenion is excellent. The English brow projects a little below and above the crinion, the excess of height of the Transylvanian Roumanian type is slightly reduced and the English excess at the rachion has practically vanished. The base of the English chin from progenion to gulion lies inside the Roumanian.

It would be difficult to obtain a closer relation even between two admittedly allied races*. Indeed the English type appears almost as close to the Transylvanian Roumanian as the latter is to the Old Roumanian type.

Such a result may seem to discredit the silhouette analysis of racial differences, but what that analysis is bringing out is the close facial resemblance, at any rate in the midsagittal plane, of most members of the European family.

On the basis of this we suggest that the method (e) of comparison is better than (a)—(d), although (d) often gives results closely akin to (e). It would of course be of great advantage to have type silhouettes of non-European races, so that we might judge of how far the facial contour can vary from racial group to racial group. The negro silhouette shows the possibility in this direction, but we need silhouettes of American Indians, Chinese and Australian natives, etc., before

* An investigation of the relationship of the English skull to the Transylvanian Roumanian skull would be of great interest.

we can appreciate the racial value of the silhouette. With the slender material at present available, it seems to us that the shape of the midsagittal section of the brain-box has more discriminative value within the racial family than the facial midsagittal contour. But here again we are met by the difficulties attaching to the growth and fashion of the hair*.

So far as the data of the present paper are concerned we may conclude that:

The Old Roumanians and the Transylvanian Roumanians are closely allied, but hardly identical stocks.

The Old Roumanians are closer to the Northern Albanians than the Transylvanian Roumanians are; closer indeed than the Northern Albanians are to the Southern Albanians. The Transylvanian Roumanians are singularly close to the English, as far as their silhouettes are concerned. Thus we should conclude that the Old Roumanian type differs from the Transylvanian Roumanian in a manner which causes the former to approach the Northern Albanian, and the latter the English silhouette. Little further can be asserted until Magyar, Serb, Bulgar, and Turk silhouettes are available. The great influence the Turk has had on the modern Greek may find itself to some extent paralleled in the case of the Old Roumanian.

* The short-cut hair of soldiers has advantage in this respect, but the student class today throughout Europe tends to adopt a close crop.

ON A NEW METHOD OF DETERMINING "GOODNESS OF FIT."

BY KARL PEARSON.

1. THIS paper is devoted to an account of a very simple method for testing "goodness of fit," that is to say of finding a measure of the probability that a given sample of size N classified in k categories $n_1, n_2, \dots, n_s, \dots, n_k$ can be reasonably supposed to have been taken at random from an indefinitely large population in which the probability of drawing an individual from the s th category has the known value p_s .

The method has certain advantages over my (χ^2, P_{χ^2}) method, in that the demonstration is very simple, and the assumptions made are fewer, and can be very briefly stated. There is no assumption that binomials can be replaced by normal curves, and it can consequently be applied when the frequency in any category is quite small; that is to say there is no need to club together small terminal frequencies.

2. Let us assume continuous frequency with any variate x for the parent population. Let its frequency curve be $y = F(x)$. Then the probability integral of the value x , if the total range be a to b , would be

$$p_x = \int_a^x F(x) dx / \int_a^b F(x) dx.$$

Now let the parent population be divided into k categories with ascending value of x . If we take the frequency in these categories to be in the parent population

$$m_1 + m_2 + \dots + m_s + \dots + m_k = M,$$

and write

$$p_s = m_s / M,$$

then on the hypothesis that these categories are indefinitely large (or that each individual be returned to his category before a second draw is made), we can obtain the probability integral of an individual $x_{s,t}$ falling into the s th category.

Let us suppose the area of the parental frequency curve between the vertical through $x_{s,t}$ and the boundary between m_s and m_{s+1} to be $M\alpha_{s,t}$, then the probability integral of $x_{s,t}$ will be

$$p_{s,t} = \alpha_{s,t} + p_{s+1} + p_{s+2} + \dots + p_k,$$

and denoting the n_s individuals in the s th category who may occur in a sample of N taken from the above parent population as

$$x_{s,1}, x_{s,2}, \dots, x_{s,t}, \dots, x_{s,n_s},$$

we have for the product of their n_s probability integrals, i.e.

$$\prod_{t=1}^{t=n_s} (p_{s,1} p_{s,2} \dots p_{s,t} \dots p_{s,n_s}),$$

the expression $(\alpha_{s,1} + q_s)(\alpha_{s,2} + q_s) \dots (\alpha_{s,t} + q_s) \dots (\alpha_{s,n_s} + q_s)$,

where $q_s = p_{s+1} + p_{s+2} + \dots + p_{s+t} + \dots + p_k$.

We shall make here an assumption, which proved very profitable when Halley suggested it to De Moivre for solving another problem in the theory of probability: Let us replace the geometrical mean by the arithmetical mean, or take

$$\text{Mean } \alpha_{s,t} + q_s = n_s \sqrt{(\alpha_{s,1} + q_s)(\alpha_{s,2} + q_s) \dots (\alpha_{s,t} + q_s) \dots (\alpha_{s,n_s} + q_s)}.$$

Now the most reasonable thing would be to divide the m_s area into n_s equal areas and place an $\alpha_{s,t}$ at the median of each of these sub-areas. We should then have

$$\begin{aligned} \alpha_{s,1} &= p_s \left(1 - \frac{1}{2n_s}\right), \quad \alpha_{s,2} = p_s \left(1 - \frac{3}{2n_s}\right) - \dots \alpha_{s,t} = p_s \left(1 - \frac{2t-1}{2n_s}\right), \\ &\dots \alpha_{s,n_s} = p_s \left(1 - \frac{2n_s-1}{2n_s}\right). \end{aligned}$$

These give a mean $\alpha_{s,t} = p_s(1 - \frac{1}{2}) = \frac{1}{2}p_s$, or

$$\left(\frac{1}{2}p_s + q_s\right)^{n_s} = (\alpha_{s,1} + q_s)(\alpha_{s,2} + q_s) \dots (\alpha_{s,t} + q_s) \dots (\alpha_{s,n_s} + q_s).$$

Thus our assumptions lead us to the same result as if we had placed all the individuals at the median value of the area in the s th category.

With these assumptions, we have, if λ_N be the product of all the probability integrals of the N individuals in the sample,

$$\begin{aligned} \log_{10} \lambda_N &= n_1 \log_{10} \left(\frac{1}{2}p_1 + p_2 + \dots + p_k\right) \\ &+ n_2 \log_{10} \left(\frac{1}{2}p_2 + p_3 + \dots + p_k\right) \\ &+ n_3 \log_{10} \left(\frac{1}{2}p_3 + p_4 + \dots + p_k\right) \\ &+ n_k \log_{10} \left(\frac{1}{2}p_k\right), \end{aligned}$$

and the probability of a sample occurring with a greater probability than the observed sample will be (*Biometrika*, Vol. xxv. pp. 379—410)

$$P_{\lambda_N} = I \left(N-1, -\frac{\log_{10} \lambda_N}{\sqrt{N \log_e 10}} \right).$$

In a similar manner if we take $p'_s = 1 - p_s$, i.e. measure our probability integrals from the other end of the curve of frequency, we have

$$\begin{aligned} \log_{10} \lambda_{n'} &= n_1 \log_{10} \left(\frac{1}{2}p_1\right) \\ &+ n_2 \log_{10} \left(\frac{1}{2}p_2 + p_1\right) \\ &+ n_3 \log_{10} \left(\frac{1}{2}p_3 + p_2 + p_1\right) \\ &+ n_k \log_{10} \left(\frac{1}{2}p_k + p_{k-1} + \dots + p_1\right), \end{aligned}$$

and

$$P_{\lambda_{n'}} = I \left(N-1, -\frac{\log_{10} \lambda_{n'}}{\sqrt{N \log_e 10}} \right),$$

where in both cases $I(q, u)$ is the incomplete Γ -function ratio

$$= \int_0^{u\sqrt{q+1}} x^q e^{-x} dx / \int_0^\infty x^q e^{-x} dx.$$

The value of this function for all values of u up to $q = 50$ is provided in the *Tables of the Incomplete Γ -Function* *. When $q = 100$, we can replace the curve $y = y_0 x^q e^{-x}$ by a normal curve with some degree of accuracy, but when we get beyond small samples, i.e. $N > 50$ up to $N = 100$, the normal curve is only roughly approximate, and if great accuracy is required either one or other of the methods described on pp. xvi—xix of the *Introduction* to the above work or the method of continued fractions must be used (see *Biometrika*, Vol. xxii, pp. 285—297).

Given the curve $y = y_0 x^q e^{-x}$, its

$$\text{Mean} = q + 1, \text{ Mode} = q, \text{ and Standard Deviation} = \sqrt{q + 1}.$$

Now P_{λ_n} will, when $N \geq 100$, be equal nearly to the area of a normal curve up to $q + 1 - \xi_0$ from the mean (i.e. $= N - \log_{10} \lambda_N / \log_{10} e$) $= \frac{1}{2} (1 - \alpha)$ if N be $> \log_{10} \lambda_N / \log_{10} e$, and equal to $\frac{1}{2} (1 + \alpha)$ if N be $< \log_{10} \lambda_N / \log_{10} e$. Here we are supposing our probability integrals measured from left to right with increasing x . On the other hand, if we use λ_n' we are measuring our probability integrals from right to left, and if $\log_{10} \lambda_N' / \log_{10} e$ is $< N$ we require $\frac{1}{2} (1 - \alpha)$, but if it is greater than N we require $\frac{1}{2} (1 + \alpha)$ to determine $P_{\lambda_n'}$.

If P_{λ_n} be large, most samples will be more probable than the observed one; if P_{λ_n} be small, few samples will be more probable than the observed one. In other words $Q_{\lambda_n} = 1 - P_{\lambda_n}$ measures the proportion of samples less probable than the observed one. Similarly $Q_{\lambda_n'}$ will measure for the other series of probability integrals the proportion of samples less probable than the observed one. We have thus two series of probability integrals associated with any sample. If one of these leads to a Q very small and much smaller than the other, it should be chosen, because if one characteristic of a sample shows that the sample is very improbable, it will not be made a probable sample by another characteristic being far less improbable. If the form of the nasal base renders it highly improbable that a skull should be that of a European, the fact that its orbital index might reasonably be that of a European will not cause us to assert that it is the cranium of a European. Accordingly we must judge by the characteristic of the sample which is the more stringent, i.e. renders the hypothesis we are testing the more improbable. In other words we select the smaller Q_{λ_n} as our criterion.

At present therefore in using this test we must determine both λ_n and λ_n' ; it is only in small samples that we can ascertain *a priori* by mere inspection which will be the larger. It would certainly save much labour if a method could be devised of selecting at once the appropriate set of probability integrals, so that double working might be avoided. We will now proceed to illustrations and comparisons of the (λ_n , P_{λ_n}) and (χ^2 , P_{χ^2}) methods.

* Reissue *Biometrika* Office, University College, London, W.C. 1. Price 42s. net.

3. *Illustration* (i). The following data for Whooping Cough cases are given by Dr E. S. Martin*:

Age	Observed Frequency	Frequency with Type I Curve
0—1	212	212·5
1—2	427	419·3
2—3	344	363·8
3—4	270	267·9
4—5	210	182·0
5—6	101	117·1
6—7	68	72·1
7—8	43	42·8
8—9	26	24·6
9—10	15	13·6
10—15	14	14·3
Totals	1730	1730

The computed frequencies have been adjusted by ·1 in two places to give the total 1730.

Dr Martin finds for 11 groups $\chi^2 = 8·21$ and $P_{\chi^2} = ·61$, a good fit.

In the following Table, column (a) gives the observations, (b) the theoretical group chances, p_s , found from the Type I curve, (c) the halves of these sub-group chances, (d) the probability integrals taken from *right to left*, (e) the logarithms of these integrals, and (f) their values when multiplied by the observational frequencies in (a):

TABLE Ia.
Right to Left Probability Integrals.

(a)	(b)	(c)	(d)	(e)	(f)
212	·122,8324	·061,4162	·061,4162	— 1·211,7171	— 256·884,0252
427	·242,3699	·121,1849 ^b	·244,0173 ^b	— ·612,5793	— 261·571,3611
344	·210,2890	·105,1445	·470,3468	— ·327,5819	— 112·688,1736
270	·154,8555	·077,4277 ^b	·652,9190 ^b	— ·185,1403	— 49·987,8810
210	·105,2023	·052,6011 ^b	·782,9479 ^b	— ·106,2671	— 22·316,0910
101	·067,6879	·033,8439 ^b	·869,3930 ^b	— ·060,7839	— 6·139,1739
68	·041,6763	·020,8381 ^b	·924,0751 ^b	— ·034,2926	— 2·331,8968
43	·024,7399	·012,3699 ^b	·957,2832 ^b	— ·018,9595	— ·815,2585
26	·014,2196	·007,1098	·976,7630	— ·010,2108	— ·265,4808
15	·007,8613	·003,9306 ^b	·987,8034 ^b	— ·005,3295	— ·079,9425
14	·008,2659	·004,1329 ^b	·995,8670 ^b	— ·001,7987	— ·025,1818
1730	1·000,0000	·500,0000	7·922,8323	—	— 713·104,4662

* *Biometrika*, Vol. xxvi. pp. 35 *et seq.*

The numbers at the foot of each column are the totals and these serve to check the accuracy of (a), (b) and (c) at sight. The sum of the (d) column

$$= k' + \frac{1}{2} - (p_1 + 2p_2 + 3p_3 + \dots + kp_k) = \text{in our case } 11.5 - 3.577,1617 = 7.922,8323,$$

thus checking (d), the value of $\sum_1^k (sp_s)$ being found by a continuous operation on the first column, exactly like finding its mean.

The value in the (c) column of $\frac{1}{2}p_s$ enables us by a continuous operation to ascertain the probability integrals in (d). We place the first number in (c) on the machine and this records the first number in (d). We turn the handle again and then add the second number in (c), this gives the second number in (d). We turn the handle once more and then add the third number in (c) and so obtain the third number in (d). This process is repeated till all the numbers in (d) are obtained. The process is very rapid.

Column (e) gives the logarithms of the numbers in (d); they are here presented as entirely negative, i.e. with the positive characteristics subtracted from the negative mantissae.

Column (f) is column (e) multiplied by the corresponding figures in (a), and summed at the foot. The numbers in it would not in practice be entered, the whole multiplications and additions being one process on the machine.

We have now $\log_{10}\lambda_n = -713.104,4662$. Dividing by $.434,2945$, we have $-\log_e\lambda_n = 1641.1984$, and our answer is

$$P_{\lambda_n} = I\left(1729, \frac{1641.1984}{\sqrt{1730}}\right).$$

But there is no table of the incomplete Γ -functions which extends to 1730! Accordingly we must replace the Γ -function curve by a normal curve.

The distance of the mean from the start of the Γ -function curve

$$= p + 1 = n - 1 + 1 = 1730.$$

Had we taken the mode it would be $p = 1729$, but this makes no difference for present purposes. The standard deviation about the mean

$$= \sqrt{p + 1} = \sqrt{n} = 41.593,2686.$$

The distance at which we finish the required integral is 1641.1984, or 88.8016 from the mean of the auxiliary normal curve towards the start of the Γ -curve. What we need accordingly is the area of a normal curve of standard deviation 41.5933 from $-\infty$ to -88.8016 from its mean. The ratio of 88.8016 to 41.5933 is 2.135, and the tables of the normal probability integral give

$$P_{\lambda_n} = P = \frac{1}{2}(1 - \alpha) = .0164.$$

Hence $Q_{\lambda_n} = .9836$ = proportion of samples less favourable than the observed one

to the hypothesis that they were drawn at random from the given Type I curve. It is clear that judged by this criterion there was a marked "goodness of fit."

For comparison we have

$$P_{\chi^2} = .609, \text{ and } Q_{\lambda_n} = .984.$$

It is clear that, while our general conclusion would be the same, i.e. that the observations may be reasonably described by the given Type I curve, the divergence between the final results is very considerable. The source of this divergence may be due to:

(a) taking the probability integrals from right to left; we may possibly get a lesser value when we take them from left to right;

(b) the assumption made in the (χ^2, P_{χ^2}) method of replacing a binomial by a normal curve. This is clearly a very weak assumption in the case of the last three or four sub-frequencies;

(c) the assumption made in the P_{λ_n} method that the individuals in a sub-group may be distributed in equal areas upon the sub-range at their median points.

If none of the three alternatives will account for the divergence found, then we must suppose that the two methods do not distribute their probabilities in at all a similar manner along the range 0 to 1, which is quite possible.

We can test (a) directly. We start from a new column (d') obtained by subtracting the numbers in (d) from unity.

TABLE I^b.
Left to Right Probability Integrals.

(a)	(d')	(c')	(f')
212	.938,5838	- .027,5269	- 5.835,7028
427	.755,9826 ⁵	- .121,4882	- 51.875,4614
344	.529,6532	- .276,0092	- 94.947,1648
270	.347,0809 ⁶	- .459,5692	- 124.083,6840
210	.217,0520 ⁶	- .663,4361	- 139.321,5810
101	.130,6069 ⁶	- .884,0337	- 89.287,4037
68	.075,9248 ⁶	- 1.119,6160	- 76.133,8880
43	.042,7167 ⁶	- 1.369,4018	- 58.884,6644
26	.023,2370	- 1.633,8199	- 42.479,3174
15	.012,1965 ⁶	- 1.913,7629	- 28.706,4435
14	.004,1329 ⁶	- 2.383,7398	- 33.372,3572
1730	3.077,1677	—	- 744.927,6682

$$\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{744.927,6682}{.434,2945} = 1715.25927.$$

Again,

$$1730 - 1715.25927 = 14.74073,$$

and
$$\frac{14.74073}{41.5933} = .3544,$$

which gives $P_{\lambda_n} = .3615$ and $Q_{\lambda_n} = .6385$.

Thus $Q_{\lambda_n} = .64$ is well in accordance with $P_{\chi^2} = .61$, and we need not search further than (a) for the divergence of the two criteria. In fact the final divergence = .03, is in my opinion hardly enough to account for error introduced by (b), and it appears that (b) and (c) may be acting in opposite or corrective senses.

Illustration (ii). We will now take an example from Dr Pretorius' paper on "Skew Bivariate Frequency Surfaces"*, which is of interest as giving an example of a very good and very bad fit on the same material. He deals with 14615 observations of the barometric height in the summer season at Greenwich. He fits them with a Pearson Type III curve and a Charlier Type Aa curve, dividing the observations into 17 groups. The data are as follows:

Barometric Height	Observed Frequency	Theoretical from Type III	Theoretical from Type Aa
30.45 and above	10	10.5	11.2
30.35	81	79.0	102.6
30.25	338	329.9	311.3
30.15	830	862.2	825.4
30.05	1576	1580.3	1513.1
29.95	2256	2196.3	2182.3
29.85	2127	2446.4	2516.2
29.75	2279	2274.1	2353.8
29.65	1812	1818.4	1827.2
29.55	1236	1280.4	1226.1
29.45	822	808.7	756.4
29.35	484	461.8	452.9
29.25	252	246.1	262.7
29.15	122	121.2	140.0
29.05	47	56.0	64.8
28.95	27	24.4	25.2
28.85 and below	16	16.3	10.8
	14615	14615.0	14615.0

Applying the (χ^2 , P_{χ^2}) method Pretorius obtains

for Type III: $\chi^2 = 7.73$, and $P_{\chi^2} = .956$,

for Type Aa: $\chi^2 = 33.76$, and $P_{\chi^2} = .006$.

As he is making a comparison of the "goodness of fit" of two theoretical distributions, and keeping his fitting moments the same for the two theoretical distributions, i.e. not considering the sample as one out of many samples from one

* *Biometrika*, Vol. xxii. p. 215, Table XIII.

or other of these distributions, I am inclined to think he should have looked out his P_{χ^2} under $n' = 15$, not $n' = 17$. In this case he would have found

for Type III: $\chi^2 = 7.73$, and $P_{\chi^2} = .902$,

for Type Aa: $\chi^2 = 33.76$, and $P_{\chi^2} = .005$.

The numbers of course make no difference in our judgement of good and bad fitting.

TABLE II^a.

Left to Right Probability Integrals for Type Aa.

(a)	(b) p_*	(c) $\frac{1}{2}p_*$	(d) P. I.'s	(e) \log_{10} P. I.'s	(f) = (a) \times (e)
10	.0007,6634	.0003,8317	.9996,1683	— .000,1664	— .001,6640
81	.0070,2018	.0035,1009	.9957,2357	— .001,8612 ⁵	— .150,7612 ⁵
338	.0235,5799	.0117,7899 ⁵	.9804,3448 ⁵	— .008,5814	— 2.900,5132
830	.0564,7622	.0282,3811	.9404,1738	— .026,6794	— 22.143,9020
1576	.1035,3062	.0517,6531	.8604,1396	— .065,2926	— 102.901,1376
2256	.1493,1919	.0746,5959 ⁵	.7339,8905 ⁵	— .134,3105	— 303.004,4880
2427	.1721,6558	.0860,8279	.5732,4667	— .241,6584	— 586.504,9368
2279	.1610,5371	.0805,2685 ⁵	.4066,3702 ⁵	— .390,7931	— 890.617,4749
1812	.1250,2224	.0625,1112	.2635,9905	— .579,0561	— 1049.249,6532
1236	.0838,9326	.0419,4663	.1591,4130	— .798,2171	— 986.596,3356
822	.0517,5505	.0258,7752 ⁵	.0913,1714 ⁵	— 1.039,4477	— 854.426,0094
484	.0309,8871	.0154,9435 ⁵	.0499,4526 ⁵	— 1.301,5057	— 629.928,7588
252	.0179,7468	.0089,8734	.0254,6357	— 1.594,0806	— 401.708,3112
122	.0095,7920	.0047,8960	.0116,8663	— 1.932,3106	— 235.741,8932
47	.0044,3380	.0022,1690	.0046,8013	— 2.329,7421	— 109.197,8787
27	.0017,2426	.0008,6213	.0016,0110	— 2.795,5815	— 75.480,7005
16	.0007,3897	.0003,6948 ⁵	.0003,6948 ⁵	— 3.432,4032	— 54.918,4512
14615	1.0000,0000	.5000,0000	7.0982,8265	—	6305.772,8695 ⁵

Now $k + \frac{1}{2} - (p_1 + 2p_2 + \dots + kp_k) = 17.5 - 10.4017,1735 = 7.0982,8265$,

hence column (d) is checked

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{6305.772,8695^5}{.434,2945} = 14519.578004,$$

and this stretches from the start of the Γ -function curve to a distance 295.421,996 short of the mean. Divided by the standard deviation $\sqrt{N} = 120,8925$, this gives .7893128. Looking up this value in the tables of the normal probability integral we find

$$P_{\lambda_n} = .7850,$$

and Q_{λ_n} = probability of a worse result = .2150.

We must next proceed to consider whether a more or a less probable result will be obtained when we read our probability integrals the other way about. We thus have Table II^b.

TABLE II^b.*Right to Left Probability Integrals for Type Aa.*

(a)	(d') = 1 - (d)	(e')	(f)
10	·0003,8317	4·583,3915	- 40 + 5·833,9150
81	·0042,7643	3·631,0814	- 243 + 51·117,5934
338	·0195,6551 ⁶	2·291,4913	- 676 + 98·524,0594
830	·0595,8262	2·775,1196 ⁵	- 1660 + 643·349,3095
1576	·1395,8604	1·144,8420	- 1576 + 228·270,9920
2256	·2660,1094 ⁵	1·424,8995 ⁶	- 2256 + 958·573,3848
2427	·4267,5333	1·630,1769	- 2427 + 1529·439,3363
2279	·5933,6297 ⁵	1·773,3205	- 2279 + 1762·397,4195
1812	·7364,0095	1·867,1144	- 1812 + 1571·211,2928
1236	·8408,5870	1·924,7231	- 1236 + 1142·957,7516
822	·9086,8285 ⁵	1·958,4134	- 822 + 787·815,8148
484	·9500,5473 ⁵	1·977,7486	- 484 + 473·230,3224
252	·9745,3643	1·988,7981	- 252 + 249·177,1212
122	·9883,1337	1·994,9847	- 122 + 121·388,1331
47	·9953,1987	1·997,9627	- 47 + 16·904,2469
27	·9983,9890	1·999,3040	- 27 + 26·981,2080
16	·9996,3051 ⁵	1·999,8395	- 16 + 15·997,4320
14615	9·9017,1735		- 15975 + 9713·169,3330 = - 6261·830,6670

To test (d'), its sum should be $k + \frac{1}{2} - (kp_1 + (k-1)p_2 + \dots + p_k)$, but

$$kp_1 + (k-1)p_2 + \dots + p_k = k + 1 - (p_1 + 2p_2 + \dots + kp_k).$$

Thus $S(d') = k - S(d) = 17 - 7\cdot0982,8268 = 9\cdot9017,1735$.

It will be observed that in column (e') we have not subtracted the mantissae from the characteristics.

The reader can select which method he prefers, but if the present method be chosen there will be two multiplications to be performed by continuous operation on the machine.

$$\text{We now have } -\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{6261\cdot830,6670}{434,2945} = 14418\cdot397,348,$$

and the difference of the mean 14615 and this = 196·602,652, and this divided by the standard deviation, 120·8925, gives 1·62626. This by the tables of the probability integral gives

$$P_{\lambda_n} = \cdot9481,$$

and accordingly

$$Q_{\lambda_n} = \cdot0519.$$

This is a lower probability than the left to right probability integrals provide and accordingly, as more stringent, is to be taken as our measure.

Thus the probability of a worse sample by the (χ^2, P_{χ^2}) method is ·006 and by the $(\lambda_n, P_{\lambda_n})$ method is ·052. It might be argued that the (χ^2, P_{χ^2}) method giving the more stringent result should be adopted, and this would be correct, if the

assumptions made were equally valid. But the reader must exercise his judgment as to whether it is more reasonable to represent curves of the form $y = y_0 x^{1.4615} e^{-x}$ or binomials of the type $(.97 + .03)^{1.4615}$, $(.996 + .004)^{1.4615}$ and $(.9993 + .0007)^{1.4615}$ by the normal curve.

It may be doubted whether a skew-binomial of greater asymmetry than $(.9 + .1)^N$ can legitimately be represented by a normal curve for the purpose of computing probability integrals. The relative legitimacy of representing the binomial and the curve $y = y_0 x^{n-1} e^{-x}$ by normal curves would be well worth a special study*.

By either method, however, we can conclude that the Type Aa curve gives a poor fit.

We now turn to the Type III curve where Pretorius, using (χ^2, P_{χ^2}) , finds a good fit.

TABLE III^a.
Left to Right Probability Integrals for Type III.

(a)	(b) p_n	(c) $\frac{1}{2}p_n$	(d) P. I.'s	(e) \log_{10} P. I.'s
10	.0007,1844	.0003,5922	.9996,4078	— .000,1561
81	.0054,0541	.0027,0270 ^b	.9965,7885 ^b	— .001,4884
338	.0225,7270	.0112,8635	.9825,8980	— .007,6277 ^b
830	.0589,9418	.0294,9709	.9418,0636	— .026,0383
1576	.1081,2863	.0540,6431 ^b	.8582,4495 ^b	— .066,3887
2256	.1502,7712	.0751,3856	.7290,4208	— .137,2474
2427	.1673,8967	.0836,9483 ^b	.5702,0868 ^b	— .243,9662
2279	.1556,0041	.0778,0020 ^b	.4087,1364 ^b	— .388,5808
1812	.1244,2012	.0622,1006	.2687,0338	— .570,7268
1236	.0876,0862	.0438,0431	.1626,8901	— .788,6417
822	.0553,3356	.0276,6678	.0912,1792	— 1.039,9198
484	.0318,0294	.0159,0147	.0476,4967	— 1.321,9401
252	.0168,3886	.0084,1943	.0233,2877	— 1.632,1081
122	.0082,9285	.0041,4642 ^b	.0107,6291 ^b	— 1.968,0701
47	.0038,3168	.0019,1584	.0047,0065	— 2.327,8421
27	.0016,6952	.0008,3476	.0019,5005	— 2.709,9543
16	.0011,1529	.0005,5764 ^b	.0005,5764 ^b	— 3.253,6422
11615	1.0000,0000	.5000,0000	7.0983,8517	— 6305.835,0432 = (e) × (a)

Column (d) checks since

$$S(d) = 17.5 - (kp_1 + (k-1)p_2 + \dots + p_k) = 17.5 - 10.4016,1483 = 7.0983,8517$$

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = -\frac{6305.835,0263}{.434,2945} = 14519.721,1254.$$

* Values of the Incomplete Γ -Function as found for the normal curve might be tested against the true values by aid of Muller's method (see *Biometrika*, Vol. xxii, pp. 284—297). It is known, however, that the normal curve does not give very accurate results for such values at the point where the *Tables of the Incomplete Γ -Function* cease, i.e. $n=50$.

This is less than the mean by 95.278,8746 which, divided by the standard deviation 120.8925, gives .788,1289, and hence

$$P_{\lambda_n} = \frac{1}{2} (1 - \alpha) = .2153,$$

leading to $Q_{\lambda_n} = .7847$, a high degree of probability.

It remains to consider the probability integrals taken in the reverse direction.

TABLE III^b.

Right to Left Probability Integrals for Type III.

(a)	(d') P.I.'s	(e') log ₁₀ P.I.'s	(e') × (a)
10	.0003,5922	- 3.444,6395	- 34.446,3950
81	.0034,2114 ⁶	- 2.465,8285	- 199.732,1085
338	.0174,1020	- 1.759,1965	- 594.608,4170
830	.0581,9364	- 1.235,1215	- 1025.153,3350
1576	.1417,5504 ⁵	- .848,4615	- 1337.175,3240
2256	.2709,5792	- .567,0980	- 1279.373,0880
2427	.4297,9131 ⁵	- .366,7424	- 890.083,8048
2279	.5912,8635 ⁵	- .228,2021	- 520.072,5859
1812	.7312,9662	- .135,9065	- 246.262,5780
1236	.8373,1099	- .077,1132	- 95.311,9152
822	.9087,8208	- .041,5402	- 34.146,0444
484	.9523,5033	- .021,2032 ⁵	- 10.262,2373
252	.9766,7123	- .010,2517	- 2.583,4284
122	.9892,3708 ⁵	- .004,6096	- .573,3512
47	.9952,9935	- .002,0463 ⁵	- .096,1784 ⁵
27	.9980,4995	- .000,8478	- .022,8906
16	.9994,4235 ⁵	- .000,2422 ⁵	- .003,8760
14615	9.9016,1483	-	- 6269.907,5577 ⁵

The (d') column checks since $17 - 7.0983,8517 = 9.9016,1483$, which accords with Table III^a, column (d).

Again,
$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{6269.907,5577^5}{4.34,2945} = 14436.995,075.$$

Subtracted from the mean we have 178.004,925 and this, divided by the standard deviation, gives us 1.4724, whence from the normal probability integral tables

$$P_{\lambda_n} = \frac{1}{2} (1 - \alpha) = .0705, \text{ and } Q_{\lambda_n} = .9295.$$

This is nearer to the P_{χ^2} result (i.e. .956 or .902), but as we feel compelled to take that distribution for probability integrals which gives the lowest Q_{λ_n} , we have to select the Table III^a value. Thus we conclude for this material :

	(χ^2 , P_{χ^2}) Test	(λ_n , P_{λ_n}) Test
Type Aa Curve	.006 (.005)	.0519
Type III Curve	.956 (.902)	.7847

Thus, although the (χ^2, P_{χ^2}) test makes a greater distinction between the two curves, the $(\lambda_n, P_{\lambda_n})$ test very adequately measures the relative goodness of fit.

Illustration (iii). In *Biometrika*, Vol. xxiv. pp. 384 and 395, the observed frequencies of 500 samples taken from a normal population of correlation 0.9 are compared with the results obtained by Fisher's transformation of the true curve.

Value of r	Fisher's Curve	Observations	p_s	$\frac{1}{2}p_s$
.95	26.41	34	.05282	.02641
.90	103.71	97	.20742	.10371
.85	108.62	92	.21724	.10862
.80	83.59	86	.16718	.08359
.75	58.53	46	.11706	.05853
.70	39.64	56	.07928	.03964
.65	26.51	23	.05302	.02651
.60	17.67	21	.03534	.01767
.55	11.78	15	.02356	.01178
.50	7.87	15	.01574	.00787
.45	5.25	4	.01050	.00525
Under .45	10.42	11	.02084	.01042
	500.00	500	1.00000	.50000

Here the (χ^2, P_{χ^2}) test gives $P_{\chi^2} = .024$.

TABLE IV^a.
Probability Integrals, Left to Right.

(a)	(d) P.I.'s	(c) \log_{10} P.I.'s	(a) \times (c)
34	.97359	— .011,6239	— .395,2126
97	.84347	— .073,9304	— 7.171,2488
92	.63114	— .199,8743	— 18.388,4356
86	.43893	— .357,6047	— 30.754,0042
46	.29681	— .527,5215	— 24.265,9890
56	.19864	— .701,9333	— 39.308,2648
23	.13249	— .877,8169	— 20.189,7887
21	.08831	— 1.053,9901	— 22.133,7921
15	.05886	— 1.230,1797	— 18.452,6955
15	.03921	— 1.406,6032	— 21.099,0480
4	.02609	— 1.583,5259	— 6.334,1036
11	.01042	1.982,1323	— 21.803,6753
500	3.73796		— 230.296,2582

Taking the p_s 's from the bottom as we have taken the probability integrals, we have

$$\begin{aligned}
 k + \frac{1}{2} - (kp_1 + (k-1)p_2 + \dots + 2p_{k-1} + k) \\
 = 12.5 - 8.76204 = 3.73796, \text{ which checks.}
 \end{aligned}$$

Again,
$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{230.296,2582}{.434,2945} = 530.276,709.$$

Mean = 500, or the bounding point is 30.276,709 *beyond* mean. This, divided by the standard deviation $\sqrt{500} = 22.360,680$, gives 1.3540, and area up to this point = $P_{\lambda_n} = .9121$.

Hence
$$Q_n = .0879.$$

We have yet to examine the value of Q_n with the probability integrals read from right to left.

TABLE IV^b.
Probability Integrals, Right to Left.

(a)	(d') P. I.'s	(e') \log_{10} P. I.'s	(a) \times (e')
34	.02641	-1.578,2316	-53.659,8744
97	.15653	- .805,4024	- 78.124,0328
92	.36886	.433,1384	- 39.848,7328
86	.56107	- .250,9830	- 21.584,5380
46	.70319	- .152,9273	- 7.031,6558
56	.80136	- .096,1723	5.385,6488
23	.86751	- .061,7255	- 1.419,9165
21	.91169	- .040,1528	- .843,2088
15	.94114	- .026,3458	- .395,1870
15	.96079	- .017,3715	- .260,5725
4	.97391	- .011,4812	- .045,9248
11	.98958	- .004,5491	- .050,0401
500	8.26201	—	- 208.652,3323

Column (d') is checked since $8.26204 + 3.73796 = 12$,

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{208.652,3323}{.434,2945} = 480.439,72995.$$

This is below mean 500, by 19.560,270, and dividing by the standard deviation 22.360,680, we have for the ratio .87476 and for P_{λ_n} the value .1909, giving

$$Q_{\lambda_n} = .8091.$$

It is clear that the right to left probability integrals give a very reasonable system, in 81 per cent. of drawings we should get a less probable system. But the system as measured the other way is more stringent and must be taken as our criterion. The Q_n in that case being about .09 is more favourable to Fisher's approximation than the (χ^2, P_{χ^2}) test with its .02. But we cannot at present determine whether the (χ^2, P_{χ^2}) test on account of its greater stringency is therefore to be accepted. It is clear that there are at least seven, if not eight, values of p_r in

the table on p. 436 which give binomials, which it is very risky to replace by normal curves as the (χ^2 , P_{χ^2}) test requires. We must conclude that the goodness of fit of the correlation coefficients in this example, if not very probable on the basis of Fisher's approximation, is far from so improbable as to tell heavily against it as a practical method. More experimental comparisons are desirable.

Illustration (iv). The following data are given by Dr J. M. de Roux in *Biometrika*, Vol. XXIII. p. 168. They consist of the variances of 500 samples of 10 drawn from a Type III curve and fitted respectively with the curve of the frequency of variances as drawn from a normal curve, and with a Type VI curve as the appropriate distribution of variances for samples from a Type III curve. The object of this investigation was to test how far the normal curve distribution of variances could be applied to a skew distribution. Dr de Roux found with 19 groups that the normal curve distribution gave a $\chi^2 = 35.441$ and $P_{\chi^2} = .008$, while the Type VI curve gave with 20 groups $\chi^2 = 19.301$, $P_{\chi^2} = .438$. It was thus clear that for this sample the normal theory distribution was inadequate, and the Type VI distribution was adequate. Would any difference of judgment have resulted had the (λ_n , P_{λ_n}) test been applied?

TABLE V^a.

Observations and the two p_s 's.

(a) Observations	(b) Normal Theory	(c) p_s	(d) $\frac{1}{2}p_s$	(a') Observations	(b') Type VI	(c') p_s	(d') $\frac{1}{2}p_s$
20	20	.0400	.0200	20	27.2	.0544	.0272
37	20.5	.0410	.0205	37	28.1	.0562	.0281
27.5	28.0	.0560	.0280	27.5	35.2	.0704	.0352
40.5	34.1	.0682	.0341	40.5	39.1	.0782	.0391
36	38.1	.0762	.0381	36	40.3	.0806	.0403
53	40.1	.0802	.0401	53	39.4	.0788	.0394
35.5	40.2	.0804	.0402	35.5	37.2	.0744	.0372
28.5	38.7	.0774	.0387	28.5	34.1	.0682	.0341
35	36.1	.0722	.0361	35	30.7	.0614	.0307
27	32.8	.0656	.0328	27	27.2	.0544	.0272
33	29.1	.0582	.0291	33	23.8	.0476	.0238
17	25.4	.0508	.0254	17	20.6	.0412	.0206
15.5	21.7	.0434	.0217	15.5	17.7	.0354	.0177
16.5	18.4	.0368	.0184	16.5	15.2	.0304	.0152
11	15.3	.0306	.0153	11	12.9	.0258	.0129
13	12.5	.0250	.0125	13	11.0	.0220	.0110
17	18.6	.0372	.0186	17	17.2	.0344	.0172
14	16.2	.0324	.0162	14	17	.0340	.0170
23	14.2	.0284	.0142	11	12.6	.0252	.0126
—	—	—	—	12	13.5	.0270	.0135
500	500	1.0000	.5000	500	500	1.0000	.5000

TABLE V^b.*Probability Integrals, Normal Theory Variances.*

(a) Observations	Left to Right			Right to Left		
	(d) P.I.'s	(e) log ₁₀ P.I.'s	(a) × (e)	(d') P.I.'s	(e') log ₁₀ P.I.'s	(a) × (e')
20	·9800	— ·008,7739	— ·175,4780	·0200	— 1·698,9700	— 33·979,4000
37	·9395	— ·027,1032	— 1·002,8184	·0605	— 1·218,2446	— 45·075,0502
27·5	·8910	— ·050,1223	— 1·378,3632 ⁵	·1090	— ·962,5735	— 26·470,7712 ⁶
40·5	·8289	— ·081,4979	— 3·300,6649 ⁵	·1711	— ·766,7500	— 31·053,3750
36	·7567	— ·121,0763	— 4·358,7468	·2433	— ·613,8579	— 22·098,8844
53	·6785	— ·168,4501	— 8·927,8553	·3215	— ·492,8190	— 26·119,4070
35·5	·5982	— ·223,1536	— 7·921,9528	·4018	— ·395,9901	— 14·057,6485 ⁵
28·5	·5193	— ·284,5817	— 8·110,5784 ⁵	·4807	— ·318,1259	— 9·066,5881 ⁵
35	·4415	— ·352,1282	— 12·324,4870	·5555	— ·255,3159	— 8·936,0565
27	·3756	— ·425,2744	— 11·482,4088	·6244	— ·204,5371	— 5·522,5017
33	·3137	— ·503,4855	— 16·615,0215	·6863	— ·163,4860	— 5·395,0380
17	·2592	— ·586,3650	— 9·968,2050	·7408	— ·130,2990	— 2·215,0830
15·5	·2121	— ·673,4593	— 10·438,6191 ⁵	·7879	— ·103,5289	— 1·604,6979 ⁵
16·5	·1720	— ·764,4716	— 12·613,7814	·8280	— ·081,9697	— 1·352,5000 ⁵
11	·1383	— ·859,1778	— 9·450,9558	·8617	— ·064,6439	— ·711,0829
13	·1105	— ·956,6377	— 12·436,2901	·8895	— ·050,8510	— ·661,1020
17	·0794	— 1·100,1795	— 18·703,0515	·9206	— ·035,9290	— ·610,7930
14	·0416	— 1·350,6651	— 18·909,3114	·9554	— ·019,8148	— ·277,4072
23	·0142	— 1·847,7117	— 42·497,3691	·9858	— ·006,2112	— ·142,8576
500	8·3562	—	— 210·615,9587	10·6438	—	— 235·350,2444 ⁵

Table V^b gives the probability integrals both ways on the normal curve theory distribution of the variances and Table V^c the same quantities on the Type VI hypothesis. Further:

$$k + \frac{1}{2} - (p_k + 2p_{k-1} + \dots + kp_1) = 19 \cdot 5 - 11 \cdot 1438 = 8 \cdot 3562,$$

and $19 - 8 \cdot 3562 = 10 \cdot 6438$, thus columns (d) and (d') are both checked. Further, for the left to right probability integrals,

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{210 \cdot 615,9587}{\cdot 434,2945} = 484 \cdot 961,1068,$$

and this gives a distance from the mean, 500, of 15·038,8932, which has a ratio to the standard deviation, 22·360,680, of ·67256 corresponding to a value of

$$P_{\lambda_n} = \cdot 2506 \text{ and } Q_{\lambda_n} = \cdot 7494.$$

For the right to left probability integrals we have

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{235 \cdot 350,2444^5}{\cdot 434,2945} = 541 \cdot 913,9419,$$

which gives a distance *beyond* the mean of 41·913,9419, and a ratio to the standard deviation of 1·87445. This gives $P_{\lambda_n} = \cdot 9696$ and $Q_{\lambda_n} = \cdot 0304$. Clearly, as the more stringent, this is the Q_{λ_n} we must select.

We now turn to the other series of probability integrals in Table V^c.

TABLE V^c.
Probability Integrals, Type VI.

(a) Observations	Left to Right			Right to Left		
	(d) P.I.'s	(e) \log_{10} P.I.'s	(a) \times (e)	(d') P.I.'s	(e') \log_{10} P.I.'s	(a) \times (e')
20	.9728	— .011,9764	— .239,5280	.0272	— 1.565,4311	— 31.308,6220
37	.9175	— .037,3939	— 1.383,5743	.0825	— 1.083,5461	— 40.091,2057
27.5	.8542	— .068,4404	— 1.882,1110	.1458	— .836,2425	— 22.996,6687 ⁵
40.5	.7799	— .107,9611	— 4.372,4245 ⁵	.2201	— .657,3800	— 26.623,8900
36	.7005	— .154,5919	— 5.565,3084	.2995	— .523,6032	— 18.849,7152
53	.6208	— .207,0483	— 10.973,5599	.3792	— .421,1317	— 22.319,9801
35.5	.5442	— .264,2415	— 9.380,5732 ⁶	.4558	— .341,2259	— 12.113,5123 ⁸
28.5	.4729	— .325,2307	— 9.269,0749 ⁵	.5271	— .278,1070	— 7.926,0495
35	.4081	— .389,2334	— 13.623,1690	.5919	— .227,7517	— 7.971,3095
27	.3502	— .455,6839	— 12.303,4653	.6498	— .187,2203	— 5.054,9481
33	.2992	— .524,0384	— 12.052,8832	.7008	— .154,4059	— 3.551,3357
17	.2548	— .593,8006	— 10.094,6102	.7452	— .127,7272	— 2.171,3624
15.5	.2165	— .664,5421	— 10.300,4025 ⁵	.7835	— .105,9610	— 1.642,3955
16.5	.1836	— .736,1273	— 12.146,1004 ⁶	.8164	— .088,0970	— 1.453,6005
11	.1555	— .808,2696	— 8.890,9656	.8445	— .073,4003	— .807,4033
13	.1316	— .880,7441	— 11.449,6733	.8684	— .061,2802	— .796,6426
17	.1034	— .985,4795	— 16.753,1515	.8966	— .047,4013	— .805,8221
14	.0692	— 1.159,8939	— 16.238,5146	.9308	— .031,1436	— .436,0104
11	.0396	— 1.402,3048	— 15.425,3528	.9604	— .017,5478	— .193,0258
12	.0135	— 1.869,6662	— 22.436,0004	.9865	— .005,9029	— .070,8348
500	8.0880	—	— 204.780,4432 ⁵	11.9120	—	— 207.184,3343

Here $k + \frac{1}{2} - (p_k + 2p_{k-1} + 3p_{k-2} + \dots + kp_1) = 20.5 - 12.4120 = 8.0880$,

and $20 - 8.0880 = 11.9120$, and thus columns (d) and (d') are checked. Of course each corresponding P.I. in (d) and (d') should have a sum equal to unity, which will be seen to be fulfilled.

For the left to right probability integrals we find

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{204.780,4432^5}{.434,2945} = 471.524,3763,$$

and accordingly the distance before the mean, 500, is 28.475,6237, and dividing this by the standard deviation 22.360,680, we have for the ratio the value 1.27346, giving us $P_{\lambda_n} = \frac{1}{2}(1 - \alpha) = .1014$ and $Q_{\lambda_n} = .8986$.

Turning now to the right to left probability integrals we have

$$-\frac{\log_{10} \lambda_n}{\log_{10} e} = \frac{207.184,3343}{.434,2945} = 477.059,5398,$$

or the limit is at a distance 22.940,4602 before the mean. The ratio of this to the standard deviation 22.360,680 is 1.02593, which gives from the table of the normal

probability integral $P_{\lambda_n} = .1525$ and $Q_{\lambda_n} = .8475$. This latter value being less than the former is to be selected. We accordingly conclude that by the $(\lambda_n, P_{\lambda_n})$ test that in random sampling from the normal theory of variance, a more probable distribution of probability integrals than the observed one would occur in 97 per cent. of cases, while in random sampling from a Type VI distribution of variances 15 per cent. of samples would have a more probable distribution of probability integrals. The general conclusion may be said to be the same as that deduced from the (χ^2, P_{χ^2}) test, but the improbability of the sample being attributable to the normal theory of variances is lessened, and the probability of the sample arising from the Type VI theory is considerably increased.

Conclusions. We have taken six cases of observations of fairly diverse character, and the results of the two tests are given in Table VI.

TABLE VI.

Nature of Observations and Theory applied	(χ^2, P_{χ^2}) Test P_{χ^2}	$(\lambda_n, P_{\lambda_n})$ Test $Q_{\lambda_n} = 1 - P_{\lambda_n}$
Whooping Cough: Type I Curve609	.639
Barometric Heights: Charlier Type Aa005	.052
Pearson Type III902	.785
Samples of correlation coefficient from a normal } bivariate population: Fisher's transformation }	.024	.088
Distribution of Variances from a non-normal population:		
(i) Normal Distribution Theory008	.030
(ii) Type VI Theory438	.848

It is clear from these illustrations that the two tests in their general sense agree, but there is considerable differences in the values of the probabilities they provide. Nor is it easy to see which test provides the "better" series of probabilities. We have noted, however, that in most of the above cases sub-groups have been taken in which it is far from legitimate to replace the binomial $(p_s + (1 - p_s))^n$ by a normal curve; to do so must influence the resulting probability. On the other hand the $(\lambda_n, P_{\lambda_n})$ test, while avoiding such an assumption, has been based on an assumption of its own as to a "reasonable" distribution of individuals in the subrange frequency. We may possibly be able to get over this hindrance to comparative applications by considering the results of applying both tests to discrete observations, such as theoretical tossing or dice returns; but again this might be handicapping the (χ^2, P_{χ^2}) test, if it were applied to all categories without clubbing groups together, which it must be if we are to get rid of the distribution assumption in the $(\lambda_n, P_{\lambda_n})$ test.

Two difficulties have presented themselves in the course of applying the $(\lambda_n, P_{\lambda_n})$ test to the present illustrations. First we have used the normal curve as adequately giving the area of the Γ -curve, when the power of the variate is 500

or more. The areas of the curve $y = y_0 x^{n-1} e^{-x}$ are so important for statistical purposes, that it is needful to have ready methods for determining them for $n > 50$. I propose therefore to investigate for a series of values of n , at stated multiples of the standard deviation, the errors made in using the probability integral of the normal curve to replace the incomplete Γ -function beyond the range of the published tables of the latter.

The second difficulty arises from the labour of determining the value of P_{λ_n} from *two* sets of probability integrals. It is true that the work is only duplicated from the point at which the first series have been found, for the second series are the q_s 's = $1 - p_s$'s of the first series, but the double series of logarithms has to be found. I have failed on the basis of this actually small series of illustrations to obtain a criterion for the *a priori* discovery of whether $\lambda_n = P_1^{n_1} P_2^{n_2} \dots P_k^{n_k}$ is greater or less than $\lambda_n' = (1 - P_1)^{n_1} (1 - P_2)^{n_2} \dots (1 - P_k)^{n_k}$, where

$$P_s = \frac{1}{2} p_s + p_{s+1} + p_{s+2} + \dots + p_k,$$

and

$$1 - P_s = p_1 + p_2 + \dots + p_{s-1} + \frac{1}{2} p_s.$$

Until such a criterion is forthcoming it seems very needful to find both λ_n and λ_n' .

It may be argued that as we have two sets of k probability integrals, both of which may be looked upon as random selections from a rectangular population, we should treat them as a combined system of $2k$ probability integrals, but this overlooks the fact that they are not *independent* random samples, which is a requisite of the theory. The additional labour is far from prohibitive, and the assumptions made in the $(\lambda_n, P_{\lambda_n})$ test seem to me more reasonable than those of the (χ^2, P_{χ^2}) test, especially in the case of small categories which do not appear to need any limitation in the case of the $(\lambda_n, P_{\lambda_n})$ test.

A STATISTICAL STUDY OF THE *DAUCUS CAROTA* L.

(SECOND ARTICLE.)

By WILLIAM DOWELL BATEN, PH.D., University of Michigan.

THE object of this article is to compare statistically two samples, each of one thousand, of the *Daucus carota* L. taken from the roadsides in Michigan and Indiana; and then to compare two samples grown from seeds taken from these two plots, but grown under the same conditions in the Botanical Gardens of the University of Michigan.

The first article, which appeared in *Biometrika**, presented a comparison between a sample from Michigan during the summer of 1930 and a sample from Indiana during the summer of 1931. In the beginning of that article will be found a picture of cross-sections of inflorescences of the *Daucus carota* L. and also a short description of the plant and flowers.

In order to prevent the influences of seasonal changes from entering into the problem, samples were gathered from the two localities during the same summer. Seeds from these plots were obtained from the plants maturing in 1931 and planted under the same conditions during the spring of 1932.

I. COMPARISON OF THE SAMPLES TAKEN FROM THE ROADSIDES IN MICHIGAN AND INDIANA DURING THE SUMMER OF 1931.

The sample from Michigan was taken from the Saline road about three miles from Ann Arbor, while that from Indiana was taken about three miles west of Terre Haute in a country lane. The following presents the chief characteristics of the distributions of the number of bracts from both samples and also those for the distributions of the number of rays or flower arms. The significance of the difference of the means is given, together with the linear correlation between the number of bracts and the number of rays.

(a) *Chief Characteristics of the Distributions of Bracts.*

The table on p. 444 gives the frequency distributions of the number of bracts for the 1931 samples.

The two distributions differ in several ways. The range for the sample from Michigan is 12 bracts, while the range for the sample from Indiana is 8. There were no flower clusters on the Indiana plants which had less than 8 bracts, while there were 113 plants from Michigan which had less than 8 bracts per cluster.

* W. D. Baten, "A Statistical Study of the *Daucus carota* L." *Biometrika*, Vol. xxv. May 1933, pp. 185—195.

Number of bracts per cluster	Michigan Frequencies	Indiana Frequencies
5	10	0
6	9	0
7	94	0
8	318	98
9	253	143
10	153	159
11	92	205
12	40	201
13	26	189
14	4	3
15	0	2
16	1	0
Totals	1000	1000

	Michigan	Indiana
Mean	9.015 bracts	10.857 bracts
Standard deviation	1.54298 „	1.6151 „
Skewness6576	-.2150

$$\text{Significance of Means} = \frac{\text{Difference of Means}}{\text{Probable Error of Difference of Means}} = 38.66.$$

Most of the clusters from Michigan had 9 or less bracts per cluster, while most of those from Indiana had 11 or more bracts; yet there was one cluster from Michigan which had 16 bracts, while the highest number from the other sample was 15 bracts. 83.7 per cent. from Michigan had 10 or *less* bracts per cluster, while 75.9 per cent. from Indiana had 10 or *more* bracts per cluster. 16.3 per cent. from Michigan had 11 or more, while 60 per cent. from Indiana had 11 or more. 31 clusters from Michigan had 13 or more bracts per cluster, while 194 from Indiana had 13 or more bracts per cluster.

Another difference between these distributions is manifested by skewness; that for the Michigan sample is .6576, while that for Indiana is -.2150. One distribution is skew to the right and the other is skew to the left.

The frequency polygons in Fig. 1 help the eye to distinguish these differences to some extent.

The significance of the means shows that the two samples were not the result of random sampling. The probability that one mean would differ from the other so much suggests that it is almost impossible for these samples to have been taken from the same parent population at random. The significance is nearly 39 probable errors of the difference of the means, which shows clearly that the samples are not consistent with random sampling. The vertical lines in the figures indicate the means.

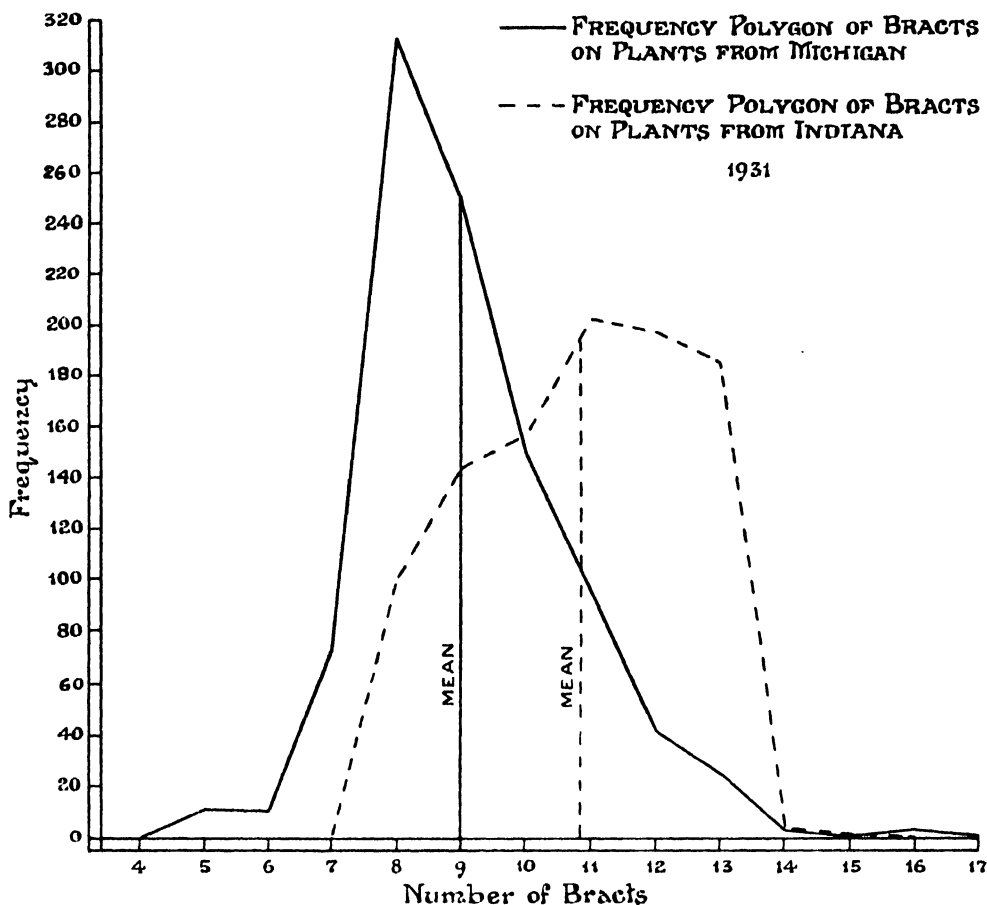


Fig. 1. (Wild plants, 1931.)

(b) Characteristics of the Distribution of the Rays.

The table on p. 446 gives the frequencies relating to the flower arms or rays in clusters of the flowers of the Wild Carrot.

There were no flower clusters from Indiana that had 27 or less rays, while there were 71 from the Michigan sample. 15 clusters from Michigan had 70 or more rays, while 63 from Indiana had 70 or more. More than two-thirds of the distribution from Michigan had 47 or *less* rays per cluster, while more than seven-tenths of the distribution from Indiana had *more* than 47 rays. More than 84 per cent. of the distribution from Michigan lies below the mean of the Indiana sample, while more than 84 per cent. of the distribution from Indiana lies above the mean of the distribution from Michigan.

The frequency polygons in Fig. 2 show clearly how the distributions differ, as to range, means, and the nature of the distributions at the ends. The large frequencies

Number of rays	Michigan Frequency	Indiana Frequency	Number of rays	Michigan Frequency	Indiana Frequency
13	1	0	53	22	36
14	1	0	54	17	58
15	2	0	55	14	26
16	2	0	56	17	50
18	3	0	57	14	37
19	3	0	58	16	31
20	3	0	59	8	20
21	6	0	60	12	41
22	3	0	61	4	21
23	7	0	62	9	28
24	4	0	63	5	12
25	8	0	64	4	17
26	13	0	65	3	8
27	15	0	66	6	22
28	13	1	67	6	13
29	20	0	68	6	17
30	24	4	69	1	14
31	18	0	70	2	9
32	17	4	71	0	5
33	21	4	72	1	8
34	26	7	73	1	4
35	23	3	74	1	7
36	51	5	75	2	2
37	34	5	76	1	6
38	49	22	77	1	2
39	42	9	78	1	2
40	44	28	79	1	1
41	32	13	80	2	4
42	44	34	81	0	2
43	33	23	82	0	5
44	40	35	83	1	2
45	25	17	84	0	2
46	35	36	85	0	0
47	24	22	86	0	0
48	39	53	87	0	0
49	28	24	88	0	0
50	33	44	89	0	1
51	14	44	92	1	0
52	21	49	105	0	1
			Totals	1000	1000

Mean Michigan Indiana
 42·777 rays 53·509 rays

of the Michigan sample correspond in a measure to the small frequencies for the other sample and *vice versa*. The vertical lines indicate the means and help the eye to see at a glance how one distribution is located with respect to the mean of the other.

The frequency polygons in Fig. 2 show that the frequencies for even numbers of rays are greater than those for odd numbers. This was true for the clusters from Michigan and Indiana. The table on p. 447 is a portion of that given above on this page.

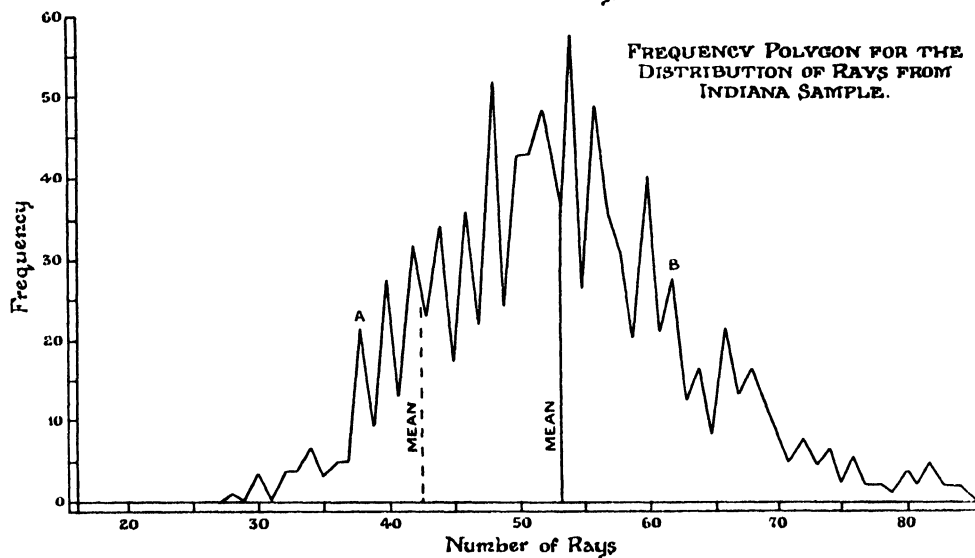
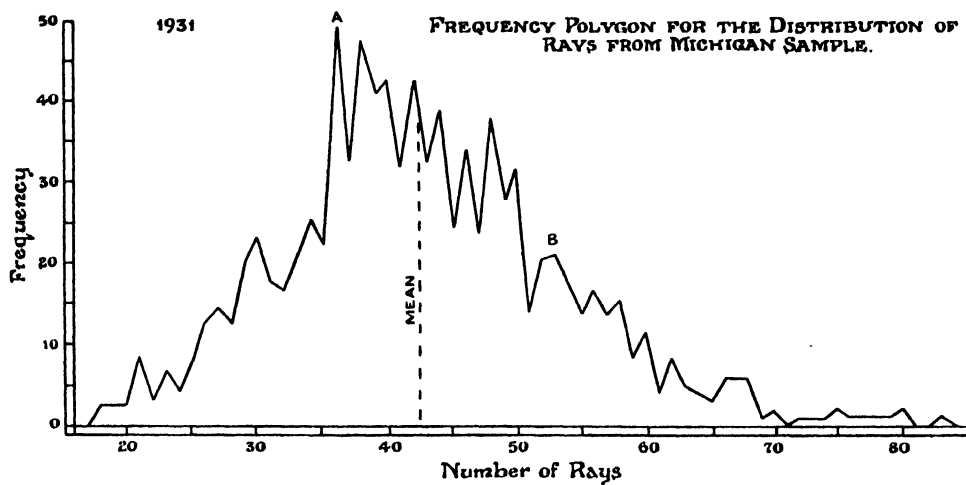


Fig. 2. (Wild plants, 1931.)

Number of rays	Michigan Frequency	Indiana Frequency	Number of rays	Michigan Frequency	Indiana Frequency
33	21	4	42	44	34
34	26	7	43	33	23
35	23	3	44	40	35
36	51	5	45	25	17
37	34	5	46	35	36
38	49	22	47	24	22
39	42	9	48	39	53
40	44	88	49	28	24
41	32	13	50	33	44

This rise at the even numbers and fall at the odd numbers can be easily seen by allowing the eye to move along the polygons on p. 447 from *A* to *B*. For the Indiana sample the sum of the frequencies for the even numbers is 610, while the sum for the odds is 390. Just why there are larger frequencies for the even numbers of rays I cannot say.

Since there were so many irregularities caused by the predominance of the plants with even numbers of rays, it was thought best to arrange the data in groups of fives. This smooths out the data. The following table gives the frequency distributions of the numbers of rays for the two samples in groups of fives.

Classes Rays	Michigan Frequency	Indiana Frequency	Classes Rays	Michigan Frequency	Indiana Frequency
10-14	2	0	55-59	69	164
15-19	10	0	60-64	34	119
20-24	23	0	65-69	22	74
25-29	69	1	70-74	5	34
30-34	106	19	75-79	6	13
35-39	199	44	80-84	3	15
40-44	193	133	85-89	0	1
45-49	151	152	90-94	1	0
50-54	107	230	105-109	0	1
			Totals	1000	1000

	Michigan	Indiana
Mean	42.76 rays	53.51 rays
Standard deviation	11.9278 „	10.1526 „
Skewness1746	.5097

Significance of the Means—

Differences of Means

$$= 32.18.$$

The standard deviation and skewness for the grouped data were obtained by using corrections similar to Shepperd's corrections for continuous variates. Since the variates, which are grouped, are discrete variates the following formula* was used:

$$\begin{aligned} \mu_n = & V_n - \binom{n}{2} \frac{1 - 1/k^2}{12} V_{n-2} + \binom{n}{4} \frac{(1 - 1/k^2)(7 - 3/k^2)}{240} V_{n-4} \\ & - \binom{n}{6} \frac{(1 - 1/k^2)(31 - 18/k^2 + 3/k^4)}{1344} V_{n-6} + \dots, \end{aligned}$$

where k is the number of groups combined into a single group. See *Annals of Mathematical Statistics*, Vol. iv (1933), pp. 243—277.

Histograms of these distributions appear in Fig. 3 and show the nature of the two samples of the rays when grouped in fives. The two vertical lines again point out how much of one distribution lies to the left or right of the mean of the other. A smooth curve has been drawn in with free hand to aid the eye.

* H. C. Carver, "Editorial," *The Annals of Mathematical Statistics*, Vol. i. No. 1, Feb. 1930, p. 111, and J. R. Abernethy, "On the Elimination of Systematic Errors due to Grouping," *Ibid.* Vol. iv, 1933, pp. 263—277.

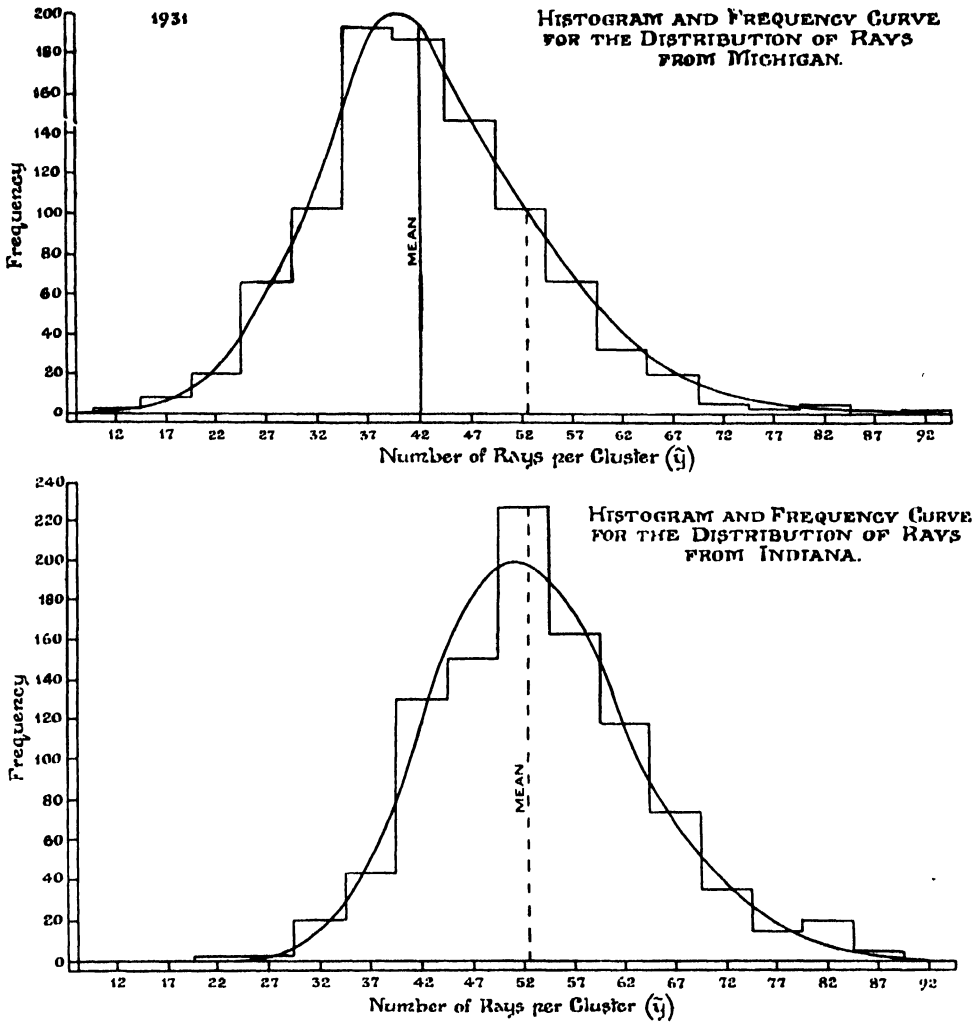


Fig. 3. (Wild plants, 1931.)

The significance of the means is 32.18 times the probable error of the difference of the means, and again shows that the two samples were not drawn at random from the same parent population.

(c) *Correlation Coefficient between Numbers of Bracts and Rays.*

While counting the bracts and rays, clusters were found with 8 bracts and 29 rays, one with 8 bracts and 41 rays, also some with 11 bracts and 71 rays, and others with 11 bracts and 54 rays. The natural question was whether clusters with a large number of bracts had also a large number of rays, and those with a small number of bracts had a small number of rays. The linear correlation coefficient was considered to be the answer to this question for both samples. The

linear correlation coefficients between the number of bracts and the number of rays for the two samples are:

Michigan .5495,
Indiana .6297.

These coefficients show that there is a rather definite relation between the number of bracts and the number of rays per cluster. This denotes that on the average clusters with a small number of bracts will also have a small number of rays. On examining the data from Indiana it was seen that there were only three clusters with 8 bracts which had more than 54 rays, while there were two which had 15 bracts and these had 70 rays or more. Most of the clusters with 8 bracts contained 44 or less rays, while most of those with 13 bracts contained 60 or more rays. The following two tables show how the clusters were distributed for the number of bracts.

Number of Rays per Cluster of the *Daucus carota* L. from Michigan, 1931

Number of bracts per cluster	Classes	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94	Totals
16								1											1
15																			0
14										1		1			1				4
13							1		3	5	10	5			1				26
12					1	1	2	5	4	7	4	4	6	1	2	2		1	40
11							5	15	23	25	10	10	3	1					92
10						5	21	34	47	24	15	3	4						153
9					7	21	51	68	41	29	20	7	7	1	1				253
8	1	2	7	38	54	95	60	29	15	10	3	1	1	1	1	1			318
7		3	9	21	23	23	10	3	1			1							94
6			4	2	2	1													9
5	1	5	3						1										10
Totals		2	10	23	69	106	199	193	151	107	69	34	22	5	6	3	—	1	1000

Number of Rays per Cluster of the *Daucus carota* L. from Indiana, 1931

Number of bracts per cluster	Classes	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94	95-99	100-104	105-109	Totals
15									1		1						2
14										1							3
13							2		20	9	8						189
12		1	1	5	21	41	38	40	25	8	3	1				1	201
11			3	19	29	66	48	28	8	2	6						205
10		3	13	31	29	49	22	10	1	1							159
9		4	14	39	42	32	7	1	2	2							143
8	1	9	13	37	23	12	2	1									98
Totals		1	19	44	133	152	230	164	119	74	34	13	15	1	—	1	1000

The following equations, obtained by the method of least squares, give the straight lines which fit the data best:

$$\text{Michigan} \quad \tilde{y} = 4.464 + 4.248x^*,$$

$$\text{Indiana} \quad \tilde{y} = 10.533 + 3.959x^\dagger.$$

In these equations x represents the number of bracts per inflorescence, while \tilde{y} represents the average number of rays per inflorescence. These lines are plotted in Fig. 4. The point C on Fig. 4 is the actual average number of rays for the

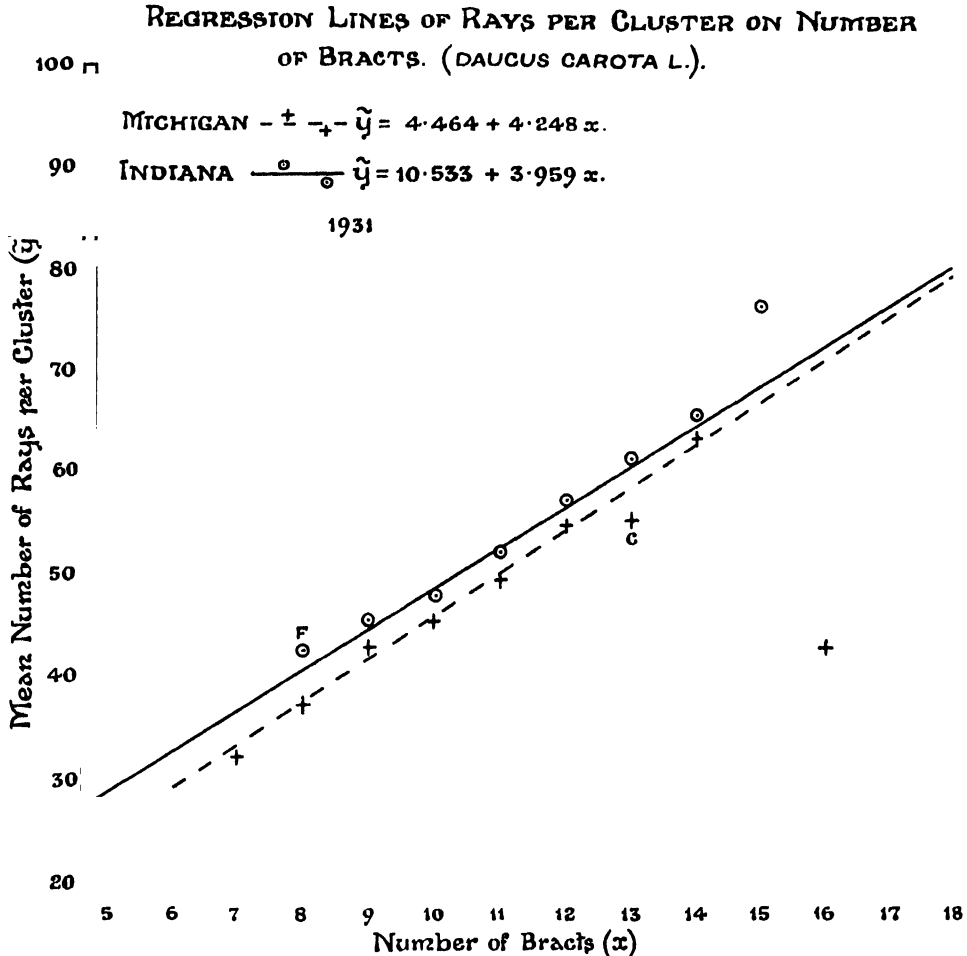


Fig. 4. (Wild plants, 1931.)

clusters with 13 bracts from the Michigan sample, while the circle at point F represents the actual average number of rays per cluster for the clusters with 8 bracts from Indiana. The crosses and circles represent the averages for the

* For number of bracts on flower arms the regression line is $\tilde{x} = 5.975 + .0711y$.

† For number of bracts on flower arms the regression line is $\tilde{x} = 5.579 +$.

different number of bracts, and show how the average number of rays for the corresponding bracts are related for the different samples. In every case the circle is above the cross, showing that the plants from Indiana on the average produced more rays per cluster than those from Michigan. The Table below shows what is revealed in Fig. 4 with regard to the averages of the arrays of the y 's on the x 's.

The circles lie rather close to the heavy line, showing that the fit is very good for the averages of the rays for Indiana clusters. The dotted line does not fit the crosses so well.

The following table gives the average number of rays for the various number of bracts.

Bracts	Michigan	Indiana
5	4.10	--
6	4.50	---
7	4.71	--
8	4.78	5.15
9	4.93	5.18
10	4.67	4.19
11	4.61	4.84
12	4.68	4.87
13	4.35	4.78
14	4.55	4.79
15	--	5.13
16	2.69	—

The following two figures give the average number of rays for the entire distributions:

Michigan 4.74 rays,

Indiana 4.93 rays.

On the average the Indiana clusters had more rays per bract than the clusters from Michigan.

The above detailed study of the two samples shows clearly that the Indiana plants produce larger flower clusters, with a greater number of bracts and a greater number of rays. From all appearances, as one passes these on the road, they seem alike, but the present type of analysis reveals the vast differences between the two. The question arises as to whether these differences are due to environment or heredity.

II. COMPARISON OF THE SAMPLES GROWN FROM SEEDS UNDER THE SAME CONDITIONS.

Seeds were gathered from the plots from which the samples discussed in Part I were taken. These seeds were obtained during the early part of April of 1932 and planted later under the same conditions at the Botanical Gardens of the University of Michigan. 38 per cent. of the plants which came up from the seeds from Michigan bloomed, while 31 per cent. bloomed from the seeds gathered from Indiana. This indicates that the *Daucus carota* L. is an annual as well as a biennial.

(a) *Chief Characteristics of the Distributions of Bracts.*

The following table gives the distribution of the bracts taken from plants which grew from seeds from Michigan and Indiana.

Number of bracts	Michigan seeds Frequency	Indiana seeds Frequency
6	2	5
7	29	28
8	156	185
9	227	224
10	216	208
11	178	154
12	100	123
13	73	65
14	16	8
15	3	---
Totals	1000	1000

	Michigan	Indiana
Mean	10.042 bracts	9.929 bracts
Standard Deviation	1.64567 "	1.62789 "
Skewness3636	.2758

Significance of Means = 2.29.

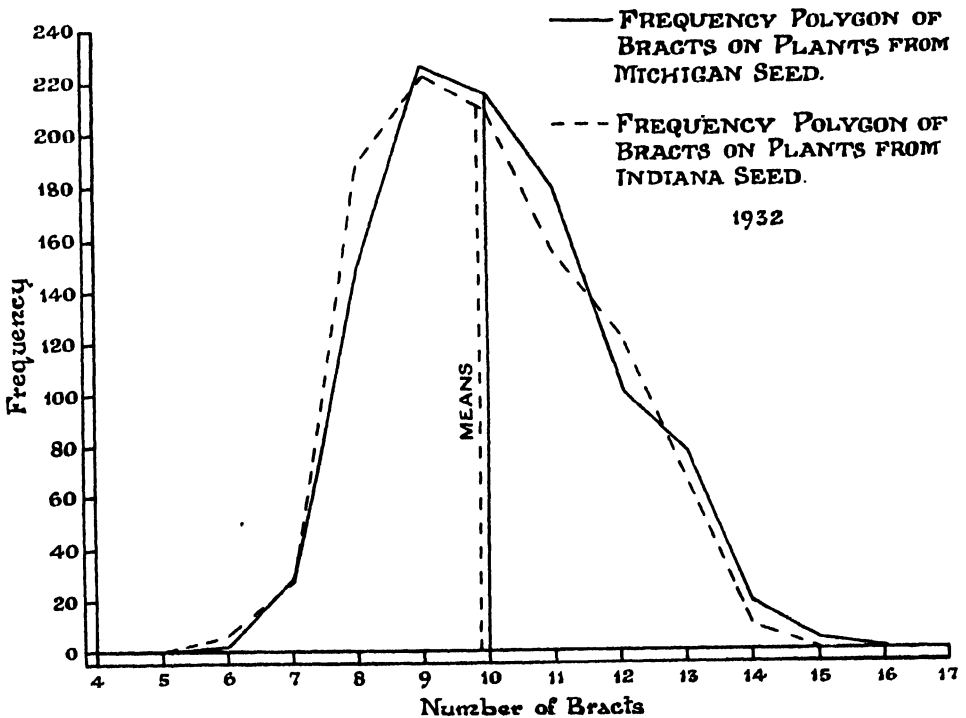


Fig. 5. (Annual cultivated plants, 1932.)

On Fig. 5 (p. 453) frequency polygons are given which exhibit these two distributions more clearly. The ranges are 9 and 8 respectively and the means are almost the same. The two vertical lines in Fig. 5 reveal how little the averages of the bracts per cluster really differ. The significance of the means shows that the two distributions might have been the result of random sampling from the same parent population, for it is only 2.29—well within the range of fluctuation due to random sampling. To appreciate just how near the means are to each other compare the distance between the vertical lines in Fig. 1 with those in Fig. 5. The frequency polygons of the two samples now have the same general shape and differ

Number of rays per cluster	Michigan Frequency	Indiana Frequency	Number of rays per cluster	Michigan Frequency	Indiana Frequency
16	0	1	58	25	38
17	0	1	59	16	27
19	0	3	60	21	25
22	1	1	61	23	15
23	0	2	62	18	23
24	2	3	63	15	22
25	0	5	64	12	24
26	5	3	65	10	15
27	5	6	66	17	20
28	4	9	67	15	13
29	6	7	68	8	15
30	4	8	69	12	15
31	7	16	70	7	14
32	14	11	71	10	12
33	11	22	72	6	14
34	8	9	73	4	8
35	12	11	74	6	10
36	16	14	75	1	4
37	19	11	76	2	9
38	18	17	77	3	5
39	18	19	78	2	8
40	34	21	79	3	6
41	26	20	80	0	3
42	34	19	81	1	4
43	41	18	82	4	2
44	32	17	83	1	3
45	40	21	84	1	1
46	41	22	85	0	3
47	35	20	86	1	0
48	30	25	88	0	1
49	38	25	89	1	0
50	31	27	90	1	1
51	40	30	93	1	1
52	38	34	94	0	1
53	32	31	95	0	1
54	34	32	96	0	2
55	33	30	102	1	2
56	17	32	108	0	1
57	26	29			
			Totals	1000	1000

very little in skewness. It appears from the distributions of the bracts that environment is an important factor in determining the characteristics of the cluster in the Wild Carrot and that the plants from Indiana are not new species and are similar to those from Michigan when grown under the same conditions.

(b) *Characteristics of the Distribution of Rays.*

The table on p. 454 gives the distribution of the rays of the *Daucus carota* L. grown from seed taken from Michigan and Indiana and grown under the same conditions.

Fig. 6 shows the frequency polygons which still have many peaks, but they are not all at the even numbers as before. The peaks are not as high as those in

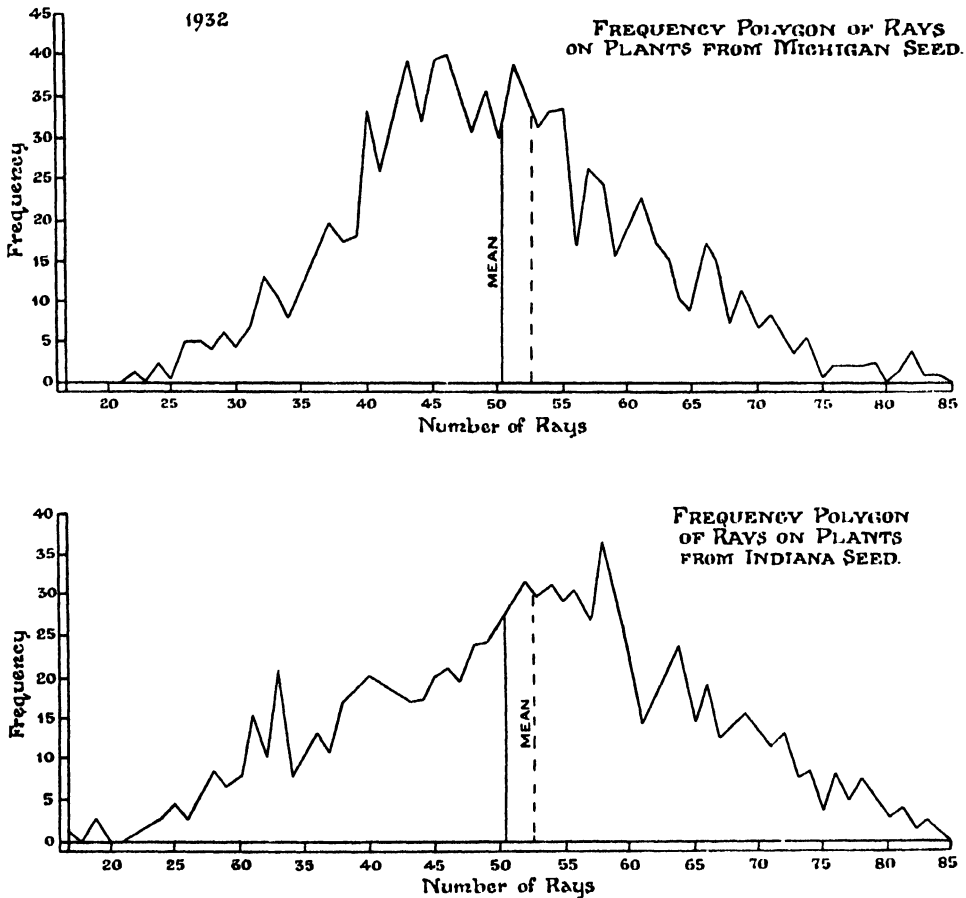


Fig. 6. (Annual cultivated plants, 1932.)

Part I; also the difference between adjacent peaks is not as large as that found in Part I. The ranges are about the same for these two distributions which are, however, spread out more than those in Fig. 2.

The following table gives the frequency distributions of the number of rays for the two samples when thrown into groups of fives:

Classes Rays	Michigan Frequency	Indiana Frequency	Classes Rays	Michigan Frequency	Indiana Frequency
15—19	0	5	65—69	61	78
20—24	3	6	70—74	33	58
25—29	21	30	75—79	11	32
30—34	44	66	80—84	7	13
35—39	84	72	85—89	2	4
40—44	166	95	90—94	2	3
45—49	180	113	95—99	0	3
50—54	177	154	100—104	1	2
55—59	120	156	105—109	0	1
60—64	88	109			
			Totals	1000	1000

	Michigan	Indiana
Mean	50·435 rays	52·92 rays
Standard deviation	11·359 „	13·907 „
Skewness	·4401	·1839 (see page 448 for corrections made)

Significance of the Means = 6·488.

Fig. 7 shows the frequency distributions with the vertical lines indicating the means. These two distributions are not as much alike as those for the bracts (see Fig. 5), yet the means are not separated by such a great distance as those in Part I. The significance was 32·3 before and now it is only 6·488. This significance of 6·488 is too large for the samples to have been chosen at random from the same parent population*, yet it seems reasonable to assert that growing the plants under the same conditions has caused the means to be nearer each other by a vast amount, and has produced distributions which do not differ so widely and in so many respects as before. Compare Figs. 2 and 5 with regard to the distances between the vertical lines, and also as to how much of one distribution lies to the right or left of the mean of the other.

(c) *Correlation Coefficient between Bracts and Rays.*

The tables on p. 458 give the frequencies for the respective number of bracts, and enable the eye to observe more details of the two distributions.

The correlation coefficients between the number of bracts and rays for the two samples grown from seeds are given by the following figures:

Michigan	·4506,
Indiana	·6613.

* [* The probability of the two series being samples from the same parent population for 6·488 is $P=000,006$. Ed.]

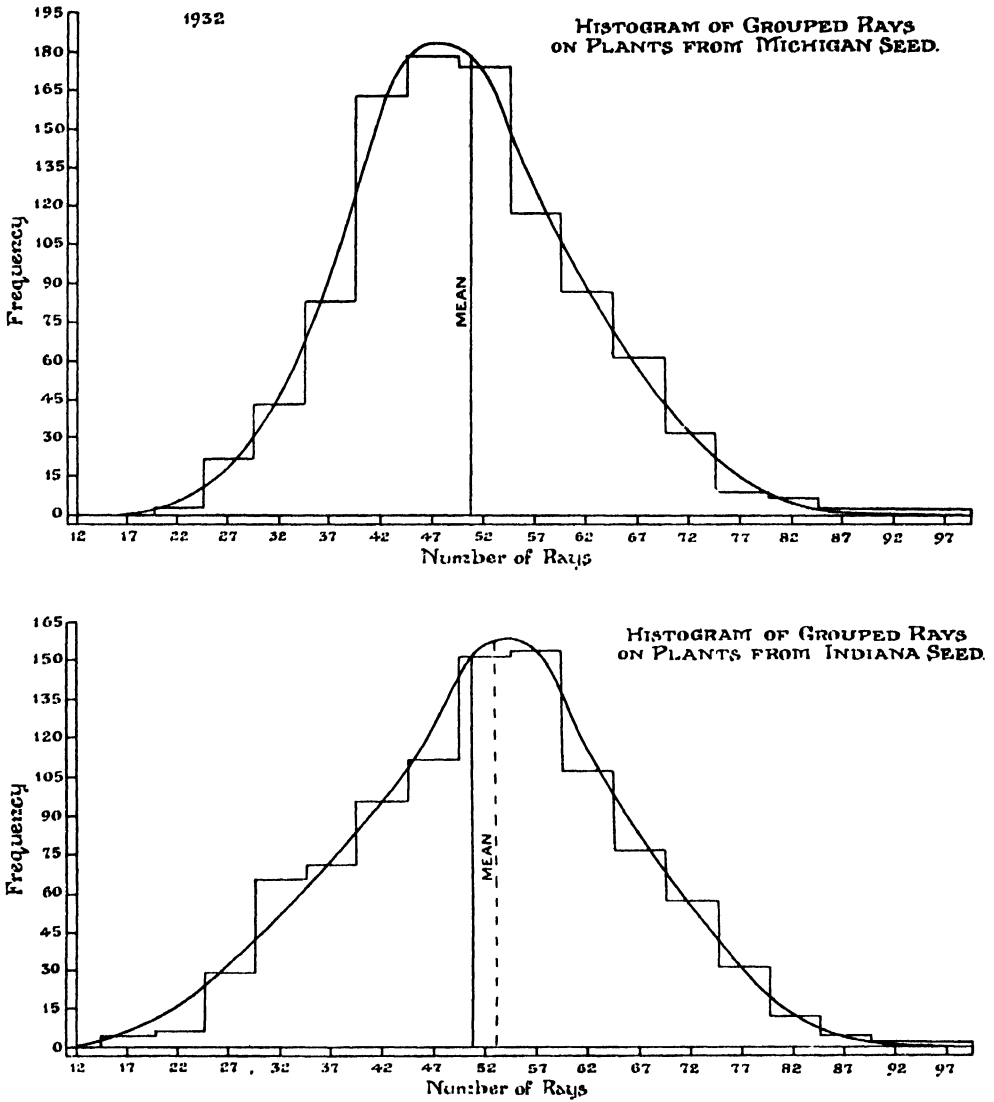


Fig. 7. (Annual cultivated plants, 1932.)

Number of rays per cluster of the *Daucus carota* L. from Michigan seed, 1932

Number of bracts per cluster	Classes	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94	95-99	100-104	105-109	Totals
	15	—	—	—	—	—	—	1	—	—	—	—	2	—	—	—	—	—	—	3
14	—	—	1	—	—	—	1	2	3	4	2	2	1	—	—	—	—	—	—	16
13	—	—	—	—	—	3	17	13	17	9	7	6	—	1	—	—	—	—	—	73
12	—	—	—	2	3	9	16	15	11	17	15	6	4	1	—	—	—	1	—	100
11	—	—	—	5	4	21	30	36	34	19	14	8	3	2	1	1	—	—	—	178
10	1	4	6	9	43	37	50	23	17	12	8	8	1	3	1	1	—	—	—	216
9	—	3	9	32	49	44	43	17	20	8	2	2	—	—	—	—	—	—	—	227
8	—	8	16	31	36	32	17	12	1	2	1	—	—	—	—	—	—	—	—	166
7	2	4	6	5	4	3	—	3	1	1	—	—	—	—	—	—	—	—	—	29
6	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals		3	21	44	84	166	180	177	120	88	61	33	11	7	2	2	—	1	—	1000

Number of rays per cluster of the *Daucus carota* L. from Indiana seed, 1932

Classes	Number of bracts per cluster																			Totals
	15—19	20—24	25—29	30—34	35—39	40—44	45—49	50—54	55—59	60—64	65—69	70—74	75—79	80—84	85—89	90—94	95—99	100—104	105—109	
14	—	—	—	—	—	—	—	1	—	—	3	1	1	1	—	1	—	—	—	8
13	—	—	—	—	—	—	2	2	9	11	15	8	12	1	2	—	—	—	—	65
12	—	—	—	—	3	7	3	13	32	25	16	10	11	5	1	1	1	1	1	123
11	—	—	—	—	2	13	22	33	26	18	23	4	4	5	1	—	—	—	—	154
10	—	—	3	7	9	24	29	44	41	19	17	9	4	1	—	1	—	—	—	208
9	—	2	3	17	18	33	41	50	27	22	7	4	—	—	—	—	—	—	—	224
8	—	2	12	34	38	31	23	21	13	6	2	3	—	—	—	—	—	—	—	185
7	3	11	8	2	—	—	2	1	1	—	—	—	—	—	—	—	—	—	—	28
6	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
Totals	5	6	30	66	72	95	113	154	156	109	78	58	32	13	4	3	3	2	1	1000

The correlation coefficient between bracts and rays for clusters from Indiana seed increased, while it decreased for plants from Michigan seeds.

The following equations, obtained by the method of least squares, give the regression straight lines of number of rays on bracts:

$$\text{Michigan } \bar{y} = 19.18 + 3.108x^*,$$

$$\text{Indiana } \bar{y} = -3.171 + 5.649x^\dagger.$$

In these equations x represents the number of bracts per cluster, while \bar{y} represents the average number of rays. These lines are drawn in Fig. 8 (p. 459). The actual averages of rays for the various numbers of bracts are represented for the Indiana sample by circles and for the other sample by crosses. The crosses fit the dotted line much better than the circles fit the heavy line. It appears as though the

* For the number of bracts on rays the regression line is $\bar{x} = 6.867 + .06295y$.

† For the number of bracts on rays the regression line is $\bar{x} = 5.883 + .0774y$.

REGRESSION LINES OF RAYS PER CLUSTER ON NUMBER OF BRACTS. (*DAUCUS CAROTA* L.).

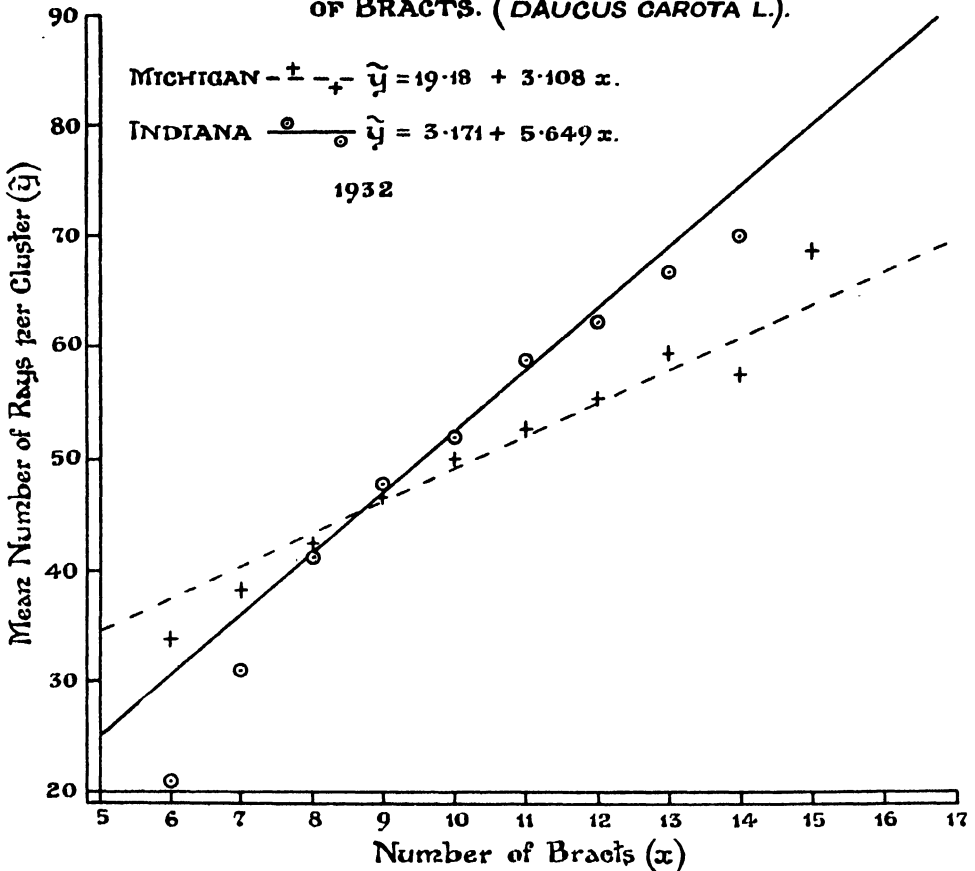


Fig. 8. (Annual cultivated plants.)

circles would fit a second degree better than a first degree equation. As in Part I, the circles are in most cases above the crosses. The differences between the crosses and circles are larger than the differences in Fig. 4.

The following table gives the average number of rays per bract for various numbers of bracts:

Bracts	Michigan	Indiana
6	5.75	3.47
7	5.63	4.49
8	5.37	5.27
9	5.25	5.41
10	5.11	5.33
11	4.93	5.14
12	4.76	5.32
13	4.34	5.28
14	4.28	5.11
15	4.67	—

From the above table (p. 459) it is seen that the number of rays per bract for the clusters which had ten bracts was 5.11 rays, for the clusters with 13 bracts was 4.34 rays, etc., as is seen in the second column.

The following are the average numbers of rays for the entire distributions in Michigan and Indiana:

Michigan	5.00 rays,
Indiana	5.03 rays.

Comparing the table on p. 452 with that on the previous page it is seen at once that the averages for the most part are larger in the latter table. The averages for the entire distributions are also much greater. One would be safe in asserting that the average number of rays per bract for the *Daucus carota* L. from the last two samples is 5.0 rays.

The above study has shown that the *Daucus carota* L. changes its organic structure under different environmental conditions, but has a tendency to throw off these differences when grown under the same conditions.

A similar study will now be made with the biennial blossoms of the plants which flowered during the past summer, comparing their distributions with those under dissimilar conditions and with the annual blossoms.

III. A STUDY OF THE BIENNIAL INFLORESCENCES.

About 43 per cent. (100 plants out of 230) of the plants which came from Michigan seeds bloomed the first year, while about 31 per cent. (75 out of 240) from Indiana seeds bloomed the first year. The plants which did not bloom the first year were in fine condition before the winter set in. The spring of 1933 was unusually wet, while the summer was unusually dry. On the plot containing plants from Indiana seeds only 11 plants survived, while 22 plants survived on the plot containing plants from Michigan seeds. To allow each plant to enter into the results with the same weight, fifty inflorescences were taken at random from each plant on the two plots.

(a) *Characteristics of the Distributions of the Bracts.*

The following table (p. 461) gives the frequency distributions of the number of bracts per inflorescence for the second year crop of blooms, or the biennial blooms.

Fig. 9 presents frequency polygons for the above distributions in percentages. The polygons are not as much alike as those for the annual inflorescences, but the means are much nearer each other, as the vertical lines indicate. The main object of this study was to compare the means for the various samples. The vertical lines in Fig. 9 show that the means for the number of bracts on the biennial clusters are almost the same. The significance of the means is only .7247 probable errors of the difference of the means, and hence shows that the small difference in the means may be due to fluctuations in random sampling from the same parent population.

Number of bracts per cluster	Plants from Michigan seed. Frequency	Plants from Indiana seed. Frequency
7	9	3
8	220	77
9	233	119
10	219	138
11	171	109
12	131	73
13	106	27
14	10	4
15	0	0
16	1	0
Totals	1100	550

		Michigan	Indiana
Mean	...	10.088 bracts	10.127 bracts
Standard deviation	...	1.65983 „	1.44498 „
Skewness3771	.2251

Significance of the Means = .7247.

**FREQUENCY OF BRACTS. (*DAUCUS CAROTA*).
BIENNIAL CLUSTERS.**

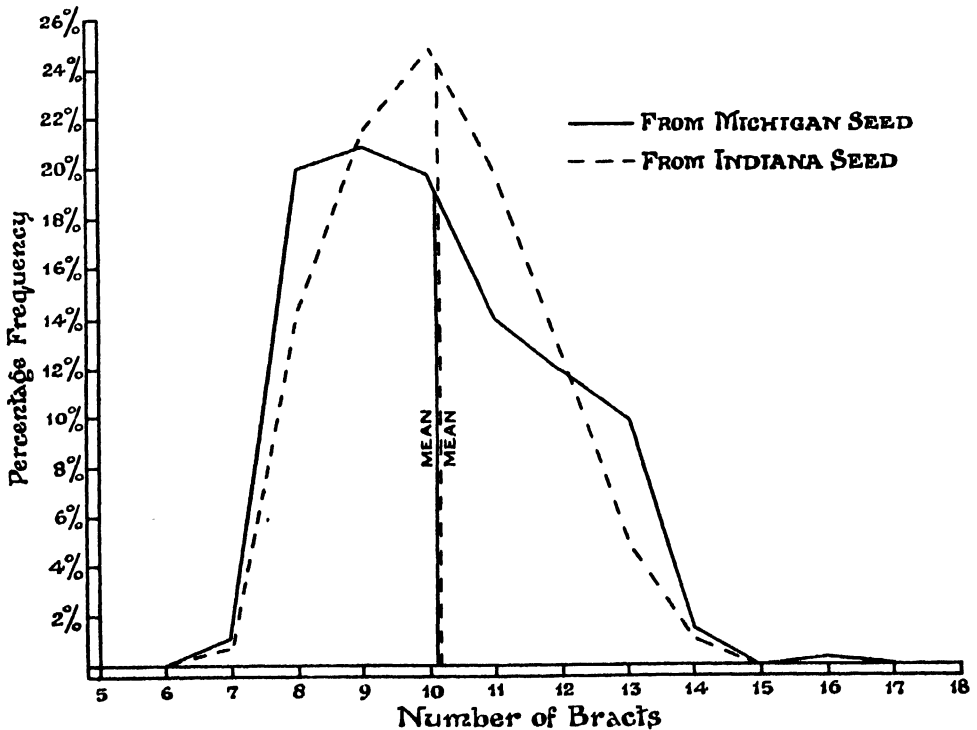


Fig. 9. (Biennial cultivated plants, 1933.)

Environment played a large rôle in creating the average number of bracts per cluster, as can be seen by examining the averages for the three different years.

(b) Characteristics of the Distributions of the Rays.

The following table gives the distribution of the rays for the biennial inflorescences of *Daucus carota* L.

Number of rays per cluster	Plants from Michigan seeds. Frequency	Plants from Indiana seeds. Frequency	Number of rays per cluster	Plants from Michigan seeds. Frequency	Plants from Indiana seeds. Frequency
19	1	0	56	21	16
22	2	2	57	24	10
23	2	2	58	21	9
24	1	0	59	41	16
25	2	2	60	14	5
26	4	2	61	19	8
27	7	5	62	14	12
28	2	3	63	14	12
29	12	4	64	10	8
30	11	5	65	6	7
31	19	10	66	9	5
32	18	8	67	9	6
33	29	12	68	9	6
34	24	9	69	11	1
35	40	9	70	5	4
36	27	6	71	7	1
37	10	13	72	1	2
38	33	9	73	3	2
39	47	15	74	4	1
40	26	10	75	2	3
41	52	24	76	1	3
42	40	13	77	3	3
43	49	23	78	2	0
44	32	13	79	4	1
45	34	24	80	1	1
46	21	17	81	2	0
47	27	27	82	1	0
48	23	12	83	1	1
49	52	21	84	0	1
50	16	15	85	1	0
51	33	25	91	1	0
52	26	14	92	1	0
53	24	22	93	2	0
54	20	15	94	1	1
55	32	14	97	1	0
			Totals	1100	550

The large frequencies are now for the odd number of rays, whereas before the large frequencies were for the even number of rays. Compare with reference given at the beginning. I do not know how to account for this as the same person counted the rays for the annual and biennial inflorescences. This difference may be in the counting, but it seems strange that the same person would one time have a tendency to stop counting on an even number and then one year from that time

have a tendency to stop on the odd numbers. The averages are closer to each other than for the preceding crops.

The following table gives the frequency distribution of the number of rays for the biennial clusters when thrown into groups of fives.

Classes Rays	Michigan Frequency	Indiana Frequency	Classes Rays	Michigan Frequency	Indiana Frequency
15--19	1	0	60--64	71	45
20--24	5	4	65--69	44	25
25--29	27	16	70--74	20	10
30--34	101	44	75--79	15	10
35--39	187	52	80--84	5	3
40--44	199	83	85--89	1	0
45--49	160	101	90--94	5	1
50--54	119	91	95--99	1	0
55--59	139	65			
			Totals	1100	550

		Michigan	Indiana
Mean	47.3882 rays	48.773 rays
Standard Deviation	11.9693 "	11.5705 "
Skewness7047	.3373
Significance of the Means=3.3596.			

These constants were obtained from the table on page 462.

Histograms for the grouped data are exhibited in Figure 10 (p. 464). The ranges of the distributions are about the same, while the heavy vertical lines near the middle of the figure which represent means are nearer together than before. Frequency curves have been drawn in freehand to aid the eye. The significance of the means is 3.36 probable errors of the difference of the means. This is not significant, that is this could be due to fluctuations in random sampling. Thus, as far as the means of the rays are concerned we cannot say that the plants which grew in Indiana during the summer of 1931 came from one species while the plants which grew in Michigan came from another. The results of planting the seeds under the same conditions have shown that the biennial inflorescences from Michigan seeds are not different from the inflorescence from Indiana seeds as to average number of bracts and average number of rays. This study certainly has shown that environmental conditions play an important rôle in creating the average number of bracts and the average number of rays per cluster. Important physical features of the clusters vary significantly for different environments, but do not vary significantly when the seeds are grown under the same conditions.

(c) *Correlation between Bracts and Rays.*

Correlation tables are given on p. 465. These tables show that there is higher correlation between bracts and rays for clusters which came from Michigan seeds than for those which came from Indiana seeds.

FREQUENCY OF RAYS (DAUCUS CAROTA). BIENNIAL CLUSTERS.

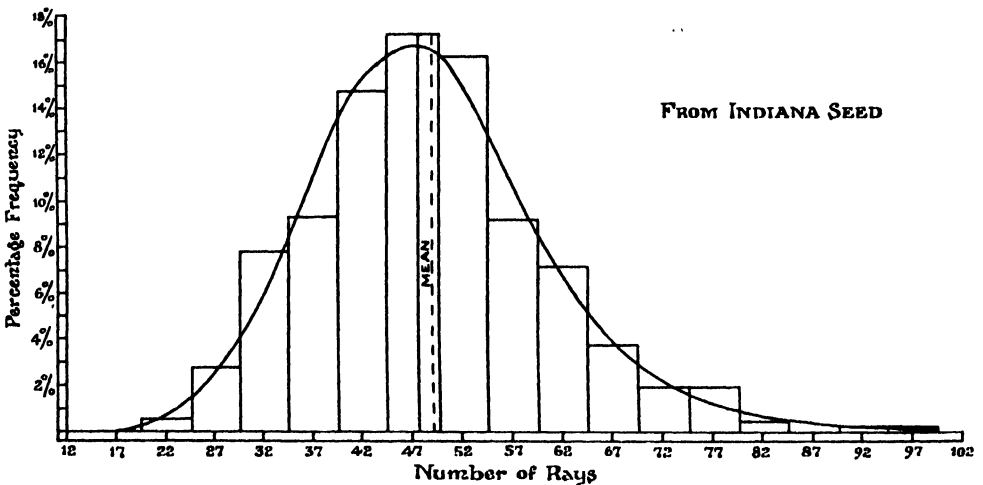
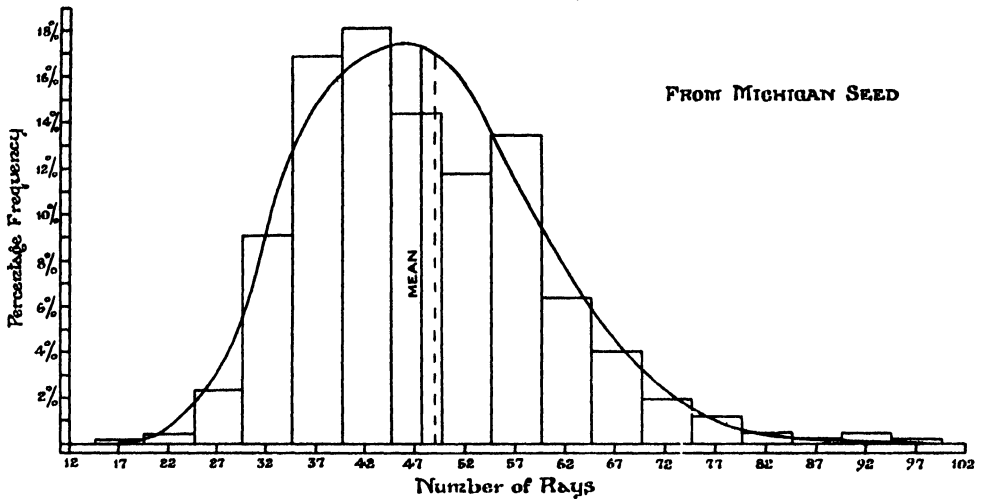


Fig. 10. (Biennial cultivated plants, 1933.)

The correlation coefficient for the biennial clusters from Michigan plants is .70 while that for the biennial clusters from Indiana plants is .49. The large correlation coefficient for the plants from Michigan seeds is unexpected as the coefficients for the two previous crops were not so large. A decrease in the coefficient for the Indiana plants is also surprising as the correlation coefficients for the other crops were much larger and about the same size. There is significance between the coefficients for the two plots for the annual and biennial inflorescences.

Planting the seeds under the same environmental conditions no doubt caused the means of the bracts and the means of the rays to differ slightly while it seemed

Correlation Table. Rays and Bracts. (Biennials, 1933.)

Number of rays per inflorescence from plants from Michigan seed

Number of bracts	Classes	15-19	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94	95-99	Totals
16	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
14	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	10
13	—	—	—	—	—	—	—	—	1	9	17	22	15	20	8	6	—	—	106
12	—	—	—	—	—	1	15	21	18	42	20	8	2	2	2	1	—	—	131
11	—	—	—	—	1	8	27	40	35	37	12	4	5	2	—	—	—	—	171
10	1	—	1	7	27	55	48	29	22	13	11	4	—	—	—	—	—	—	219
9	—	2	2	31	71	59	32	15	12	9	—	—	—	—	—	—	—	—	233
8	—	3	22	59	76	42	10	5	2	1	—	—	—	—	—	—	—	—	220
7	—	—	2	3	4	—	—	—	—	—	—	—	—	—	—	—	—	—	9
Totals		1	5	27	101	187	199	160	119	139	71	14	20	15	5	1	5	1	1100

A glance at the correlation table shows a rather definitely high correlation between bracts and rays per cluster.

Correlation Table. Rays and Bracts. (Biennials, 1933.)

Number of rays per inflorescence from plants from Indiana seed

Number of bracts	Classes	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85-89	90-94	95-99	Totals
14	—	—	—	—	—	—	1	2	—	1	—	—	—	—	—	—	—	4
13	—	—	—	—	—	1	4	6	5	4	2	—	3	1	—	—	—	27
12	—	—	—	2	3	3	11	14	14	8	6	6	2	1	—	—	—	73
11	—	5	6	6	3	8	21	15	13	18	14	3	2	1	—	—	—	109
10	1	4	7	7	10	15	28	36	20	12	3	1	1	—	—	—	—	138
9	2	5	12	15	34	24	15	10	1	—	—	—	1	—	—	—	—	119
8	1	1	17	20	22	9	3	2	2	1	—	—	1	—	—	—	—	77
7	—	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	3
Totals		4	16	44	52	83	101	91	65	45	25	10	10	3	0	1	0	550

to cause the correlation coefficients to differ more widely. In every case there seems to be correlation between rays and bracts. The number of rays per inflorescence appears to be a function of the number of bracts.

The following figures give the correlation coefficients for the samples taken from the crops for the three years.

			Michigan	Indiana
1931—First year samples55	.62
1932—Annual inflorescences43	.66
1933—Biennial inflorescences70	.49

These figures show that there is a considerable amount of variation in the size of the correlation coefficient.

On Fig. 11 are plotted regression lines for predicting the average number of rays for a given number of bracts. These lines show that clusters with a small number of bracts have a small number of rays, while clusters with a large number of bracts have also on the average a large number of rays, etc.

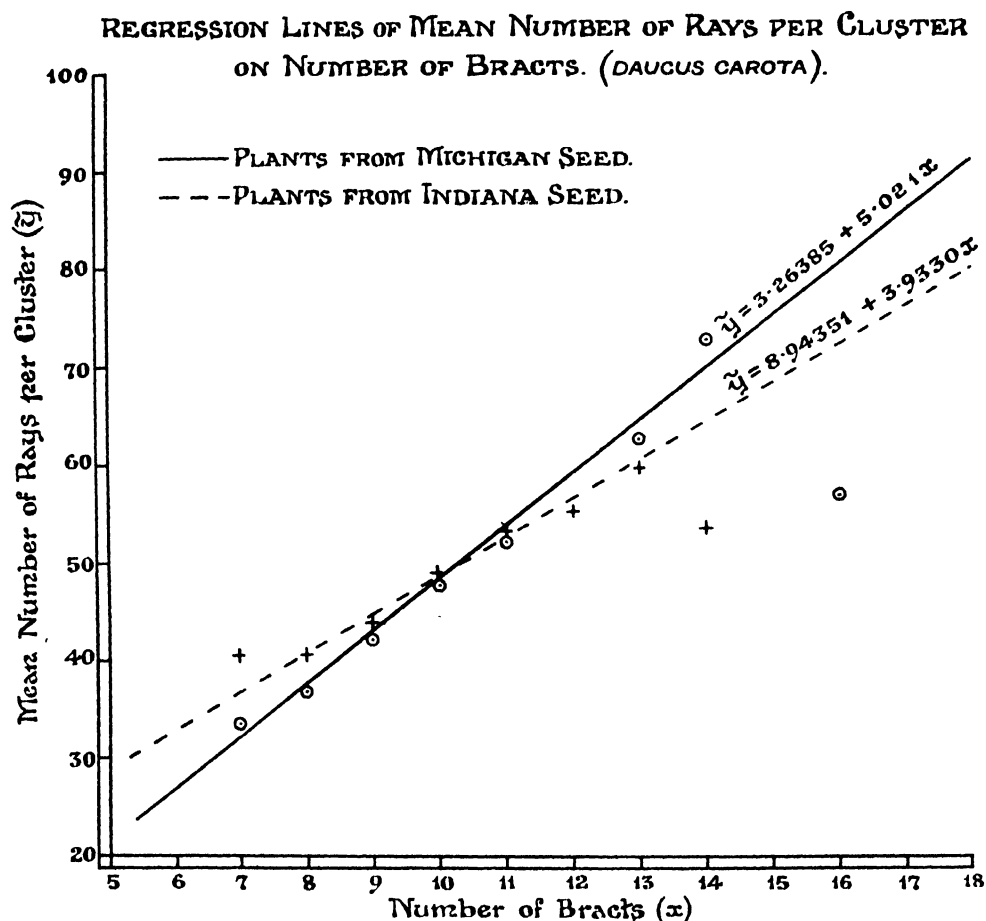


Fig. 11. (Biennial cultivated plants, 1933.)

The regression equations for predicting the average number of bracts for a given number of rays are as follows:

Michigan	Indiana
$\bar{x} = 5.49136 + .097 y$	$\bar{x} = 7.11527 + .06175 y$

The following figures give the average number of rays per bract for the samples for the three years.

Average Number of Rays per Cluster

Year	Michigan	Indiana
1931	4.74	4.93
1932	5.02	5.03
1933	4.70	4.82

It appears from these figures that there are about five rays for each bract.

IV. SUMMARY.

This statistical study of *Daucus carota* L., which has extended over a period of three years, has shown the following.

(a) The number of bracts per inflorescence of the Wild Carrot varies with the environment and the average number of rays varies even more so with the environment.

(b) The average number of bracts and the average number of rays per inflorescence are about the same when the plants are grown under the same conditions.

(c) One does not always have the right to conclude that two plants growing in different localities belong to different species when the significance of the means of certain characters is greater than five.

(d) Plants which differed widely when grown in different localities were brought back to similar physical conditions by growing the seeds in the same environment.

Bracts.

Year	Means			Standard Deviation		Skewness		Correlation Coefficient	
	Michigan	Indiana	Significance	Michigan	Indiana	Michigan	Indiana	Michigan	Indiana
1931	9.02	10.86	38.7	1.54	1.62	.66	-.22	.55	.62
1932	10.04	9.93	2.4	1.65	1.63	.36	.28	.43	.66
1933	10.09	10.13	.7	1.66	1.44	.37	.23	.70	.49

Rays.

Year	Means			Standard Deviation		Skewness		Average number of rays per bract	
	Michigan	Indiana	Significance	Michigan	Indiana	Michigan	Indiana	Michigan	Indiana
1931	42.76	53.51	32.3	11.93	10.15	.18	.51	4.74	4.92
1932	50.92	52.92	6.5	11.36	13.91	.44	.18	5.02	5.03
1933	47.39	48.77	3.4	11.96	11.57	.70	.34	4.70	4.82

(e) There is considerable correlation between the number of bracts and the number of rays per inflorescence.

(f) The amount of correlation between bracts and rays varies for different environments.

(g) On the average there are about five rays to each bract. That is a cluster with 12 bracts can be expected to have 60 rays, while one with 9 bracts will have 45 rays, etc.

(h) We do not know whether the plants from Indiana came from one species while those from Michigan came from another. The above results seem to show that the plants from both states came from the same species, and that the differences were due to changes in environment.

The author feels as though this study points in one general direction, yet believes that the evidence might have been more conclusive had there been more plants the second year from which to sample.

The author wishes to express appreciation for the valuable assistance of Mr W. H. Long, who so untiringly made the majority of the counts, for the computations of Mr G. M. Brown and also for the financial assistance rendered by the Committee in charge of the Research Fund at the University of Michigan.

[The Editor has felt some difficulty in publishing the preceding memoir. It is the first occasion on which "smooth curves drawn in with freehand" have been permitted in *Biometrika* to replace properly computed graduation curves of frequency. It is further difficult to believe that in 1931 there could be such a redundancy of even numbers of rays in both series, while this could be reversed in the biennial flowering of seeds from the same plants. Unfortunately the author did not do the counting himself, and cannot explain such a startling change. Again a "significance" of 6.4883 in the case of rays of the Michigan and Indiana annual flowers in the same environment corresponding to a probability of .000,006 would seem to most statisticians evidence against the two groups being samples of the same parent population. Further a "significance" of 3.0142 involving a probability of .0117 in the rays from the biennial flowers of the same groups appears to point in the same direction. It is not clear that the original difference in counts found between plants growing wild in Michigan and in Indiana may not to some extent be due to the Michigan plants having more annual, and the Indiana plants more biennial blossoms. The author does not tell us how biennial were distinguished from annual blossoms, when the plants were in the wild state. Notwithstanding these criticisms, however, there does seem definite evidence in favour of a wide environmental influence, such as the biometricians found many years ago in celandines grown on either side of the same hedgerow. Perhaps a final test might be made by taking the seed of both Michigan and Indiana *biennial* plants and growing them side by side in both Michigan and Indiana and counting both the annual and in the following year the biennial flowers.]

The paper on account of the important problems it raises does, notwithstanding these criticisms, seem worthy of publication in *Biometrika*.]

MISCELLANEA.

Tabellen für alle statistische Zwecke in Wissenschaft und Praxis*.

By PAUL HARZER.

There are two criticisms to be made of this set of tables. The first is that the title, which in full reads "Tabellen für alle statistische Zwecke in Wissenschaft und Praxis, insbesondere zur Ableitung des Korrelations-Koeffizient und seines mittleren Fehlers," is misleading and gives the impression that the Tables cover all scientific problems amenable to statistical analysis, whereas they only refer to those in which a correlation coefficient is concerned.

The second, a far more serious one, is that the tables are not founded on any accepted principles of mathematical statistics. It has also long been known that the skewness of the distribution of the correlation coefficient renders the use of its standard error in many cases unsound; two alternative methods are available, (i) Fisher's logarithmic transformation†, which affords a variate approximately normally distributed, or (ii) the use of the exact frequency distribution, which has been tabled‡.

The tables give the values of two functions denoted by r and $m(r)$ of the observed correlation s for various values of s and n , the number of degrees of freedom. Thus if n = sample size, then $n = n - 1$ or n , according as the sample means or the population means have been used in calculating s . In three previous papers, *Astronomische Nachrichten*, Bd. 219, Nr. 5247; Bd. 232, Nr. 5567 and Bd. 234, Nr. 5605, Harzer obtained these functions as power series in $\frac{1}{n}$, thus the first few terms are (Nr. 5247, S. 252):

$$r = s \left[1 + \frac{1-s^2}{2n} + \frac{(1-s^2)(1+3s^2)}{8n^2} - \frac{(1-s^2)(1+18s^2-27s^4)}{16n^3} + \dots \right] \dots\dots\dots(i),$$

$$m(r) = \frac{1-s^2}{n^{\frac{1}{2}}} \left[1 + \frac{2+s^2}{4n} - \frac{44-316s^2+193s^4}{32n^2} + \dots \right] \dots\dots\dots(ii).$$

Further terms are given in Nr. 5605.

To facilitate tabling the transformations

$$r = \sin \rho,$$

$$s = \sin \sigma$$

are made; the argument σ is in degrees, $\rho - \sigma$ in thousandths of a degree (S. 7—50) and $m(r)$ in units of the 5th decimal (S. 51—91). Subsidiary tables for finding the coefficients of the series (i) and (ii) are also given (S. 3—6).

On S. 251—252 (A.N., Nr. 5247) and, later on, S. 263—264 (Nr. 5605), the moments of s are found; in particular the equation for the mean is solved for the population correlation coefficient r , and when s is substituted for the mean of s the equation (i) is obtained. Thus the function r

* *Abhandlungen der Bayerischen Akademie der Wissenschaften*, Neue Folge, Heft 21, Munich 1933.

† *Metron*, Vol. 1, No. 4 (1921).

‡ *Tables for Statisticians and Biometricians*, Part II, pp. 182—210.

of the observed correlation is quite an arbitrary one; it is not, for example, the "optimum" value of the population correlation coefficient, which is known* to be

$$r = s \left[1 - \frac{1-s^2}{2n} + \frac{(1-s^2)(1-5s^2)}{8n^2} + \dots \right].$$

The function r will have a sampling distribution depending on the true correlation; by using his expressions for the moments of s Harzer found the standard deviation $m(r)$ on S. 265—266 of Nr. 5605, and then expressed it in terms of s by means of the substitution (i), thus obtaining the function (ii). The value of the mean is not given, but we find

$$\text{mean } r = r \left[1 - \frac{3(1-r^2)^2}{2n^2} + \dots \right],$$

where, of course, the population correlation is used on the right-hand side.

It therefore appears that these tables cannot be used for the purpose for which they were apparently designed, because of the confusion between the "mathematically expected" value and the observed value of the correlation coefficient.

F. GARWOOD.

* *Metron*, loc. cit. p. 13. [This result was given four years earlier in *Biometrika*, Vol. xi. p. 356, Formula (lxviii), 1917.]

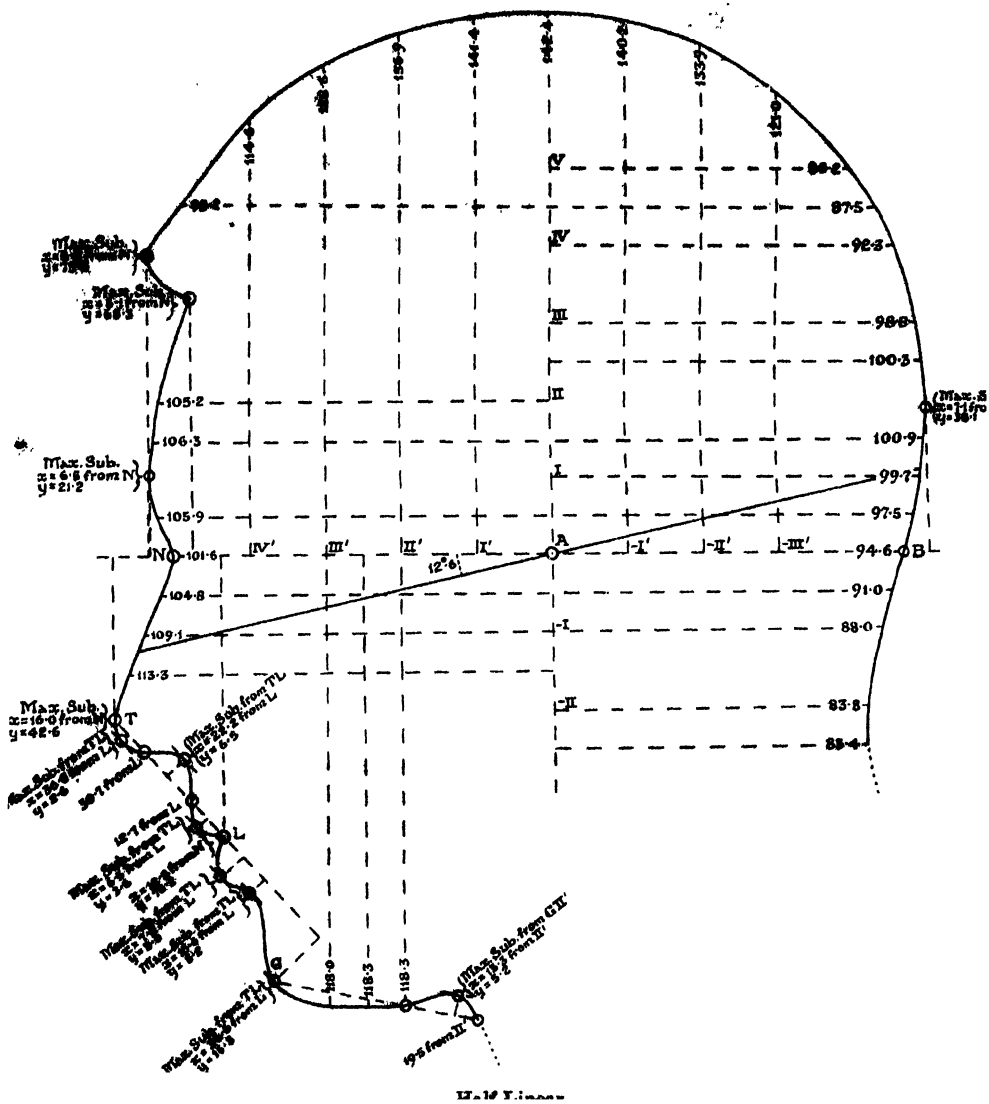
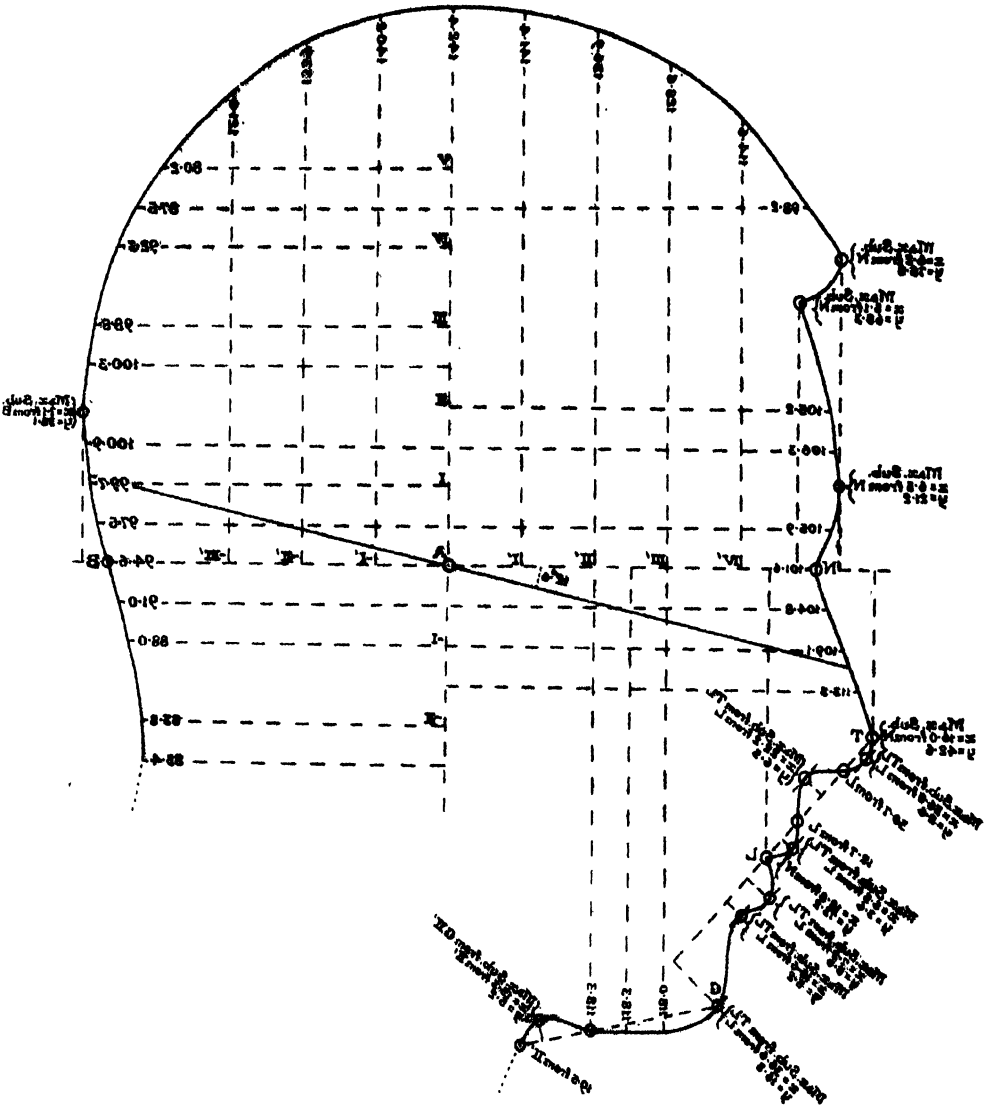


Fig. 1. Type Roumanian σ silhouette based on fifty students from Old Roumania,—showing the network of mean coordinates from which it was plotted, and the Frankfurt Horizontal, as indicated by the continuous line through the auricular point A.

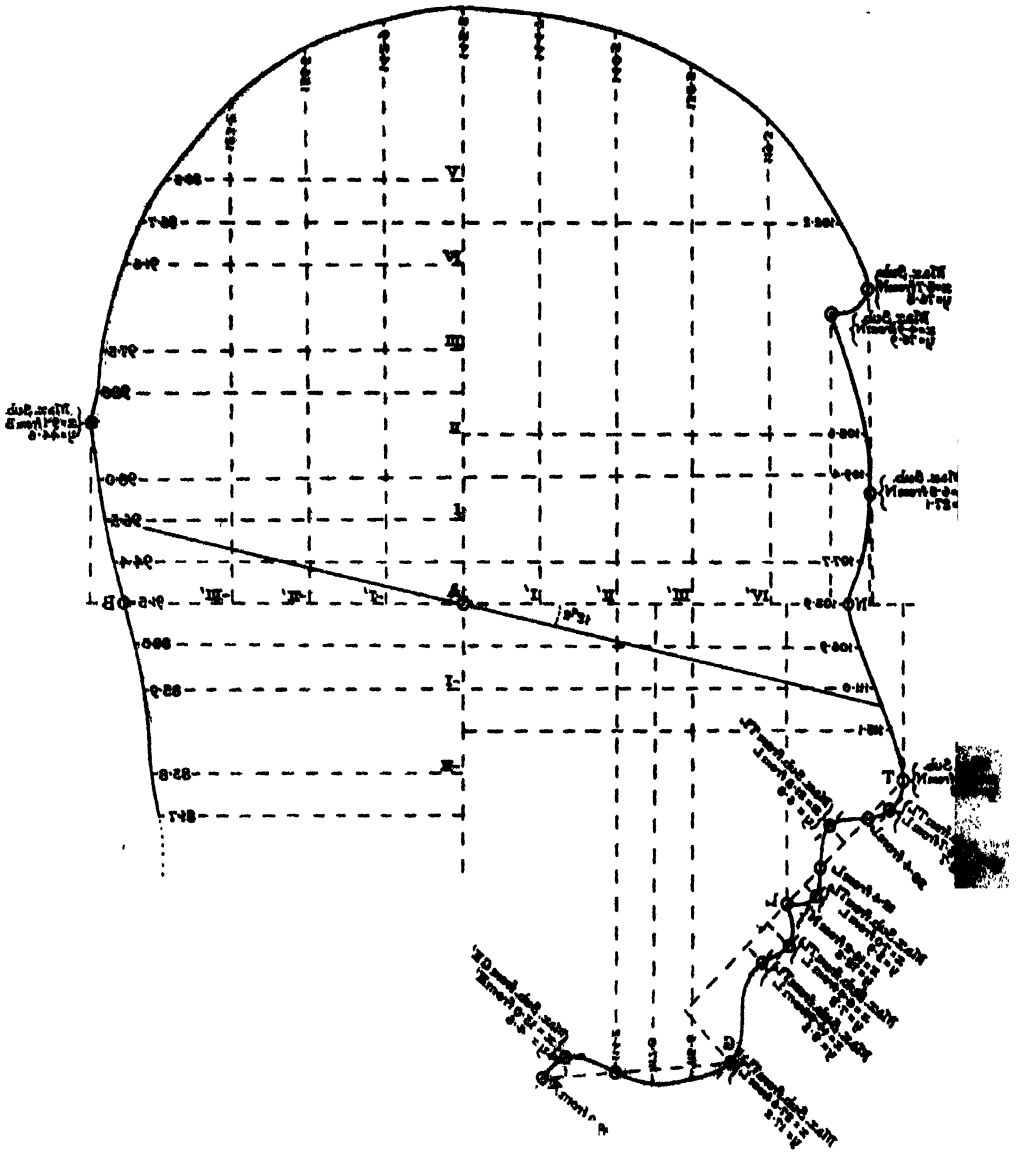
Fig. 1. Type Romanian γ silhouette based on fifty students from Old Roumania, showing the network of mean coordinates from which it was plotted, and the Frankfurt Horizontal, as indicated by the continuous line through the auricular point A.

Half-linear.



continuous line through the articular point A.
 2. Type Romanian & aliphonetic based on fifty students from Transylvania - showing the network of mean coordinates from which it was plotted, and the Frankfurt Horizontal, as indicated by the

Half-linear.



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